Subject for this video: Applied Problems

Reading:

- **General:** Section 2.5, Basic Differentiation Properties
- **Closest Example:** Section 2.4 Example 8, bottom of page 139 – bottom of page 140

Homework:

Applied Problems (2.5# 89, 91, 97)

Most Important Concepts in Calculus

Recall what I have called the most important concept of the first month of the course:

When can one cancel terms, and why?

For instance, for the expression

$$\frac{f(x+h)-f(x)}{h}$$

when we substitute $h=0$ we find

$$\frac{f\big(x+(0)\big)-f(x)}{(0)}=\frac{f(x)-f(x)}{0}=\frac{0}{0}\ \ Does\ Not\ Exist$$

The value of this expression does not exist, because one *cannot cancel* $\frac{0}{0}$

But when computing the derivative $f'(x)$ using the *Definition of the Derivative*,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

after some arithmetic steps, one arrives at a step where the expression is of the form

$$\lim_{h\to 0}\left(\frac{h}{h}\cdot something\right)$$

Because this is a *limit* as $h\to 0$, we know that $h\neq 0$, so we *can cancel* $\frac{h}{h}$

That cancellation is the key step in computing the $f'(x)$.

At this point in the course, we turn our attention turns to learning Derivative Rules, which allow us to compute $f'(x)$ without having to use the difficult limit formula from the *Definition of the Derivative*. The issue of when we *can* and *cannot* cancel terms will no longer come up.

For the next month of the course, a different concept will be the most important concept.

Recall what we have learned about the relationship between the derivative of a function, the instantaneous rate of change of a function, and the line that is tangent to the graph of a function.

Three Equal Quantities

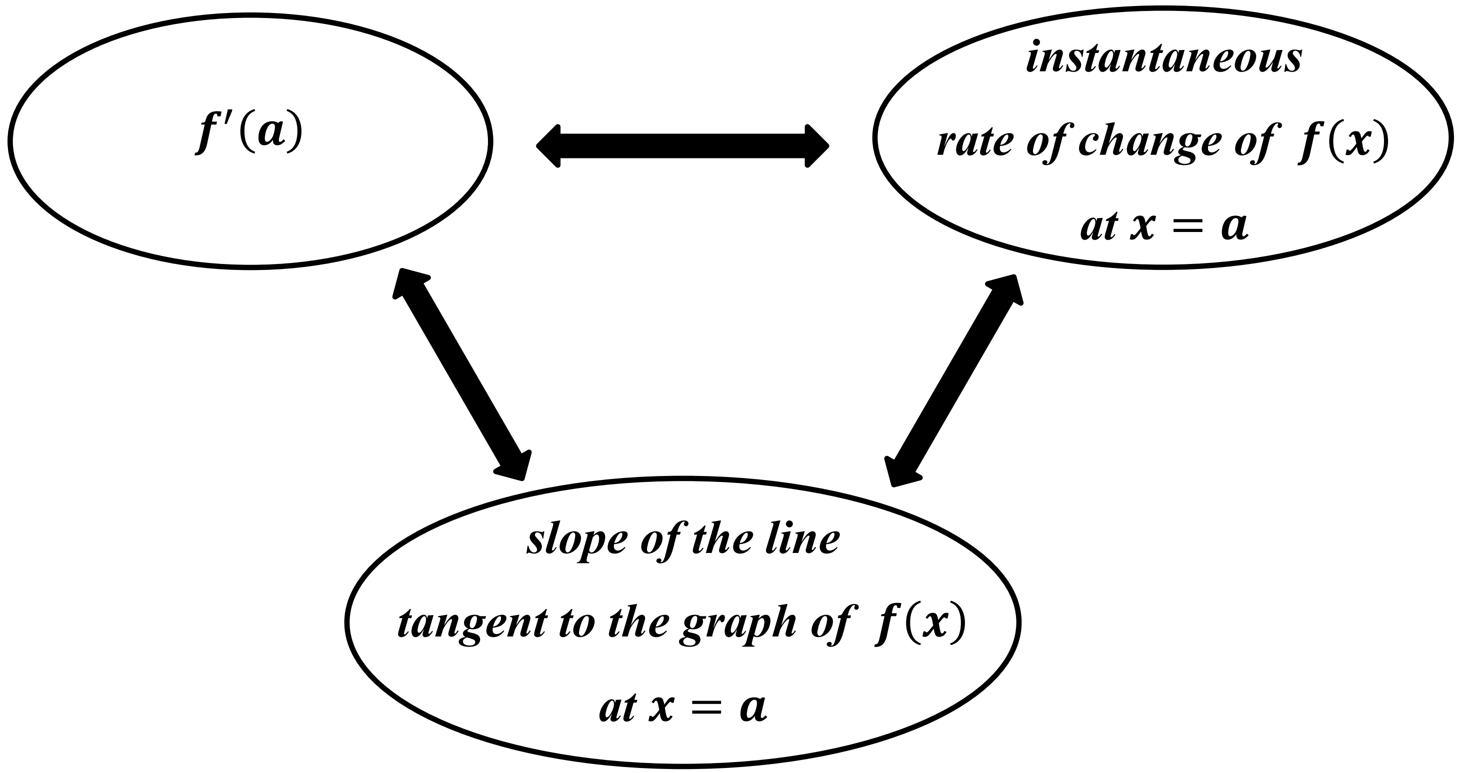
words: the ***instantaneous rate of change of*** $\boldsymbol{f(x)}$ ***at*** $\boldsymbol{x = a}$

words: the ***slope of the line tangent to the graph of*** $\boldsymbol{f(x)}$ ***at*** $\boldsymbol{x = a}$

words: the ***derivative of*** $\boldsymbol{f(x)}$ ***at*** $\boldsymbol{x = a}$

symbol: $\boldsymbol{f'(a)}$

It is helpful to have a diagram that illustrates the equality of these three quantities.



Three Equal Quantities

(the most important concept of the second month of the course)

In this video, we will study three examples involving *Applications*. Remember that that means we will be using functions to model behavior of actual real-world situations. All three of the examples will involve the *rate of change* of some quantity. The concept of the *Three Equal Quantities* will be central, and the *diagram* of the three equal quantities will be helpful for *visualizing* the structure of the problems.

[Example 1] (Similar to 2.5#89) A startup company introduces a new product. Total sales are described by the function

$$S(t) = -0.01t^3 + 0.36t^2 \quad 0 \le t \le 24$$

where t is the time in months since the product was introduced and $S(t)$ is the total sales (in millions of dollars) at time t.

(A) Find $S'(t)$

Solution

$$S'(t) = \frac{d}{dt}\left(-0.01t^3 + 0.36t^2\right)$$

use sum and constant multiple rule

$$= -0.01\underbrace{\left(\frac{d}{dt}t^3\right)}_{\text{power rule with } n=3} + 0.36\underbrace{\left(\frac{d}{dt}t^2\right)}_{\text{power rule with } n=2}$$

$$= -0.01\left(3 \cdot t^{3-1}\right) + 0.36\left(2 \cdot t^{2-1}\right)$$

$$= -0.01(3t^2) + 0.36(2t)$$

$$= \boxed{-0.03t^2 + 0.72t}$$

(B) Find $S(10)$ and $S'(10)$.

Solution

$S(10) = -0.01(10)^3 + 0.36(10)^2$

Sub $t=10$ into $S(t) = -0.01t^3 + .36t^2$

$= -0.01(1000) + 0.36(100)$

$= -10 + 36$

$= 26$

$S'(10) = -0.03(10)^2 + 0.72(10)$

Sub $t=10$ into $S'(t) = -0.03t^2 + 0.72t$

$= -0.03(100) + 0.72(10)$

$= -3 + 7.2$

$= 4.2$

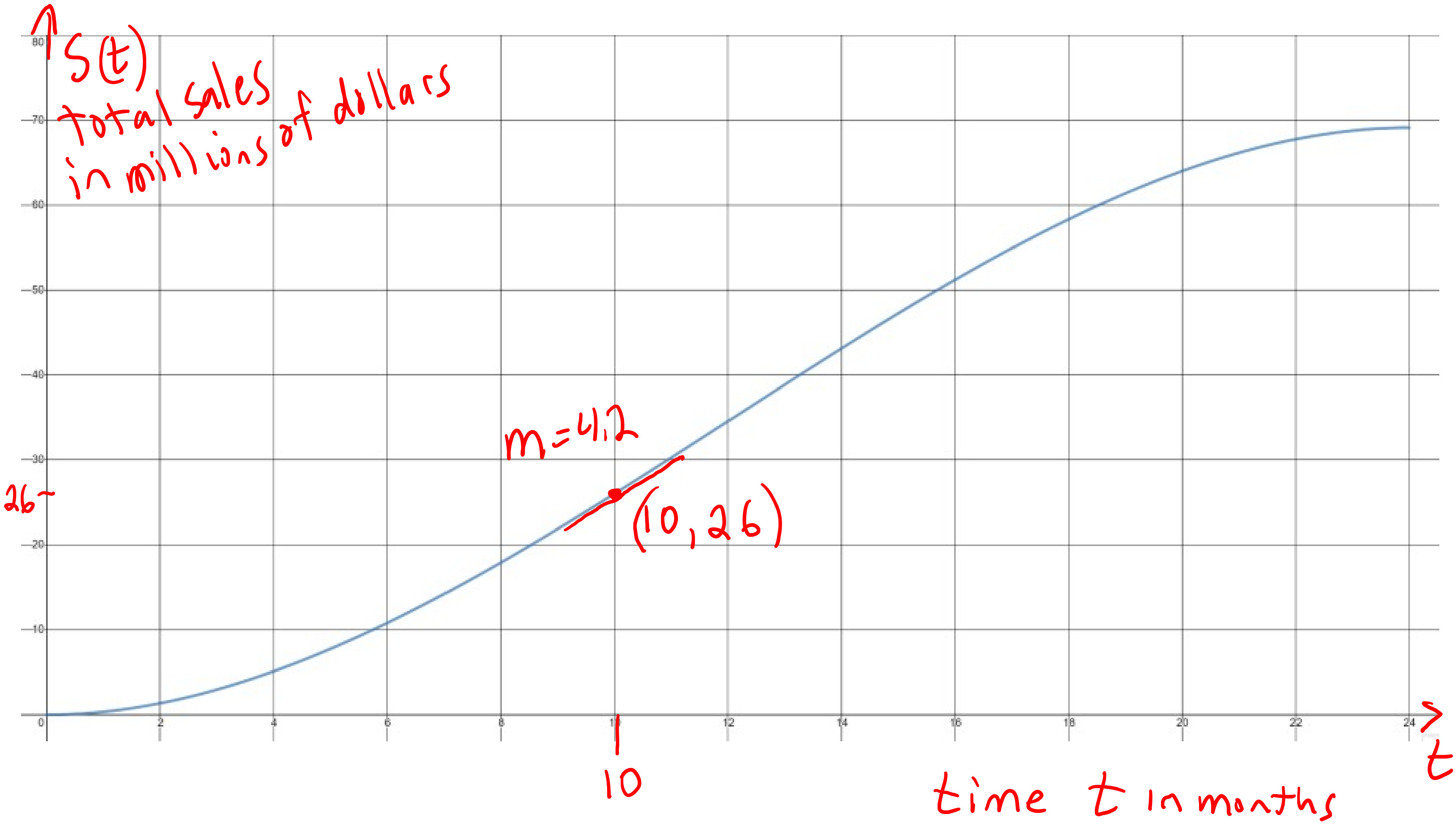
(C) Write a brief verbal interpretation of your results from (B)

We have to explain what the mathematical results from (B) tell us about the sales of the product.

At time $t = 10$ months after the product was introduced, the total sales are 26 million dollars,

and sales are increasing at a rate of 4.2 million dollars per month.

(D) Illustrate your results from (B) on the given graph of $S(t)$.



[Example 2] (Similar to 2.5#91) A phone manufacturer had planned to spend 2 million dollars on an ad campaign for a new phone, but they are considering increasing the advertising budget. Their VP of marketing has a model that predicts that sales of the phone will be described by the function

$$N(x) = 20 - \frac{12}{x} \qquad 1 \le x \le 5$$

where x is the number of millions of dollars spent on the ad campaign and $N(x)$ is the total number of millions of phones that will be sold.

(A) Find $N'(x)$

Solution Observe that N(x) is in positive exponent form
Convert to power function form.

$$N(x) = 20 - \frac{12}{x} = 20 - 12x^{-1}$$

Now find the derivative

constant function rule

Power Rule with $n=-1$

$$N'(x) = \frac{d}{dx}\left(20 - 12x^{-1}\right) = \frac{d}{dx}20 - 12\frac{d}{dx}x^{-1} = 0 - 12\left(-1 \cdot x^{-1-1}\right)$$

use sum and constant multiple rule

$$= \frac{12}{x^2}$$

(B) Find $N'(2)$.

Solution

$$N'(2) = \frac{12}{(2)^2} = \frac{12}{4} = 3$$

Substitute $x=2$ into $N'(x) = \frac{12}{x^2}$

(C) Write a brief verbal interpretation of your result from (B)

In (B), we found that $N'(2) = 3$

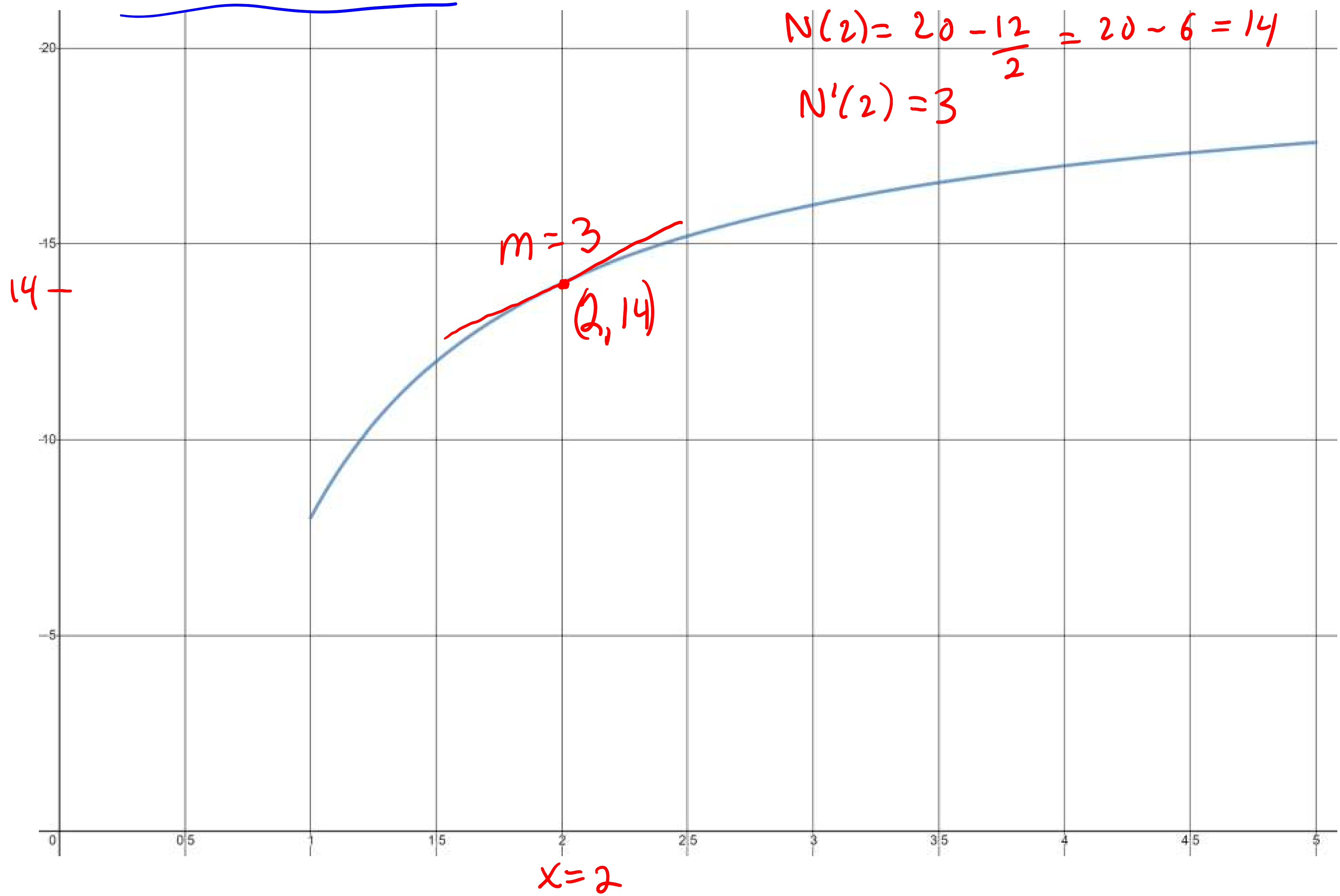
This tells us that when the company spends 2 million dollars on the ad campaign,
the instantaneous rate of change in the number of phones sold is 3 million phones per million dollars of spending on ads.

In other words, every additional dollar spent on buying ads will result in roughly 3 more phones sold.

(D) Illustrate your result from (B) on the given graph of $N(x) = 20 - \frac{12}{x}$.

$N(2) = 20 - \frac{12}{2} = 20 - 6 = 14$

$N'(2) = 3$



[Example 3] (Similar to 2.5#97) A study shows that the number of words that a student studying a new langage can learn is described by the equation

$$y = 6\sqrt{x} \qquad 0 \le x \le 16$$

where ~~y~~ x is the number of hours the student studies and y is the total number of words that the student learns.

(A) Find the rate of learning after 1 hour and after 9 hours.

Solution We are being asked to find $y'(1)$ and $y'(9)$

Strategy
- find $y'(x)$
- substitute $x=1$ to get $y'(1)$
- substitute $x=9$ to get $y'(9)$

The function is $y(x) = \underbrace{6\sqrt{x}}_{\text{radical form}} = \underbrace{6x^{1/2}}_{\text{power function form}}$

$$y'(x) = \frac{d}{dx} 6x^{1/2} \underset{\uparrow}{=} 6\frac{d}{dx} x^{1/2} \underset{\uparrow}{=} 6\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) = \underbrace{3x^{-1/2}}_{\text{power function form}} = \underbrace{\frac{3}{\sqrt{x}}}_{\text{radical form}}$$

constant multiple rule

use Power Rule with $n = 1/2$

Substitute $x = 1$ into $S'(x) = \frac{3}{\sqrt{x}}$

$$S'(1) = \frac{3}{\sqrt{1}} = 3$$

Similarly

$$S'(9) = \frac{3}{\sqrt{9}} = \frac{3}{3} = 1$$

After the student has been studying for 1 hour, their rate of learning new words is 3 words per hour

After the student has been studying for 9 hours, their rate of learning new words is 1 word per hour.

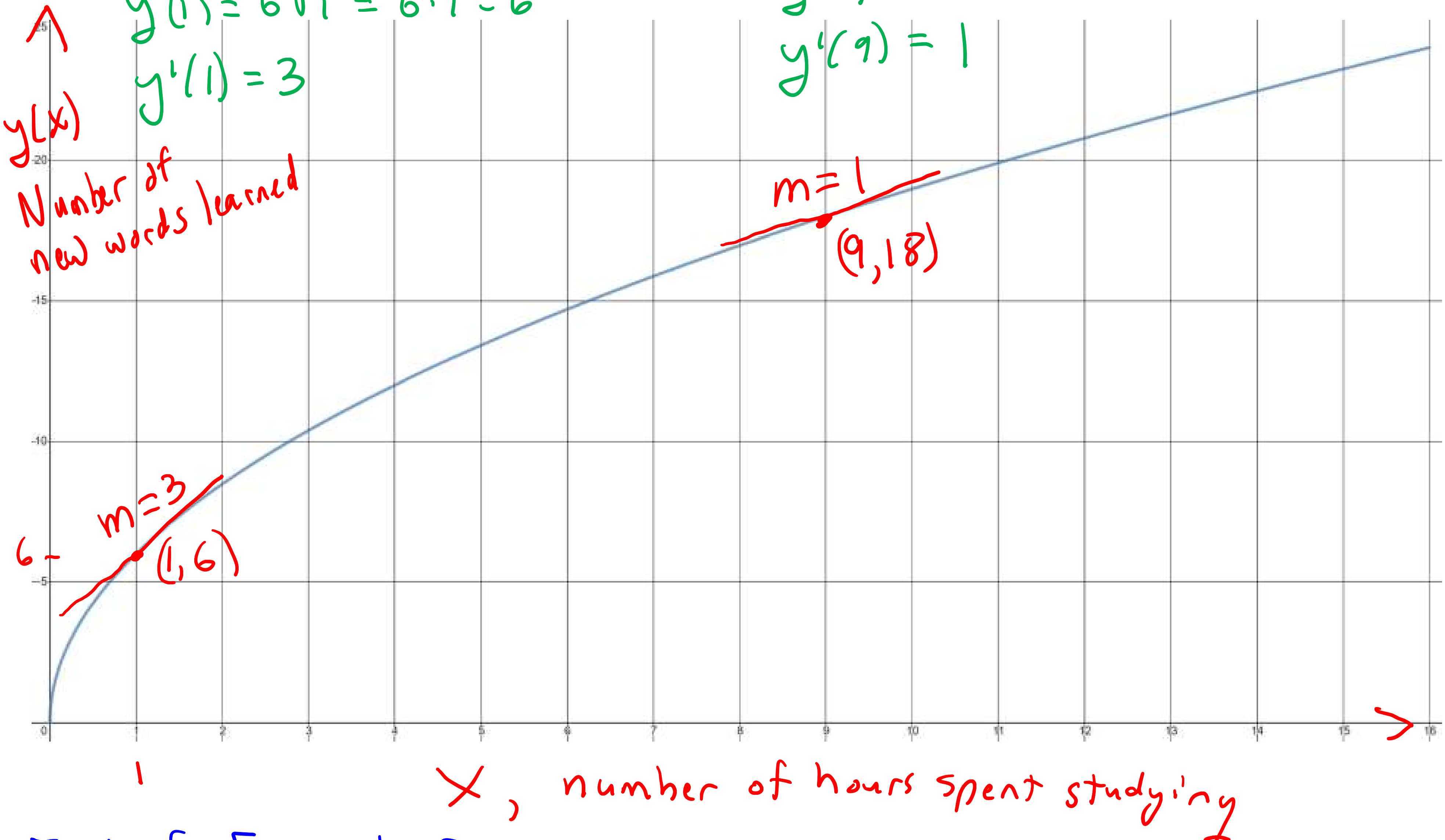
(B) Illustrate your result from (A) on the given graph of $y = 6\sqrt{x}$.

$y(1) = 6\sqrt{1} = 6\cdot 1 = 6$

$y'(1) = 3$

$y(9) = 6\sqrt{9} = 6\cdot 3 = 18$

$y'(9) = 1$



End of Example 3

End of video