An Elementary Proof of the Irrationality of $e$

We present a proof of the irrationality of $e$ that requires (almost) no calculus. Motivated by $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 1 + \frac{1}{2}(1 + \frac{1}{3}(1 + \cdots)))$, let’s define a sequence $\{x_n\}_{n \in \mathbb{N}}$ by

$$x_n = \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \cdots.$$

The following properties are obvious:

1. $e = 1 + x_1 = 1 + \frac{1}{1}(1 + x_2) = 1 + \frac{1}{1}(1 + \frac{1}{2}(1 + x_3)) = \cdots.$
2. $x_n = \frac{1}{n}(1 + x_{n+1}).$
3. $x_1 > x_2 > x_3 > \cdots > 0.$

Assuming that $e$ is rational, so are all $x_n$ because of Property 1. Let $x_n = \frac{p_n}{q_n}$ where $p_n$ and $q_n$ are relatively prime positive integers. From Property 2, $\frac{p_n}{q_n} = \frac{1}{n} \left(1 + \frac{p_{n+1}}{q_{n+1}}\right)$, so $\frac{p_{n+1}}{q_{n+1}} = \frac{q_{n+1}}{q_n} \frac{p_n - q_n}{q_n}$. Thus $q_n \geq q_{n+1}$, i.e., $q_1 \geq q_2 \geq q_3 \geq \cdots$.

Combining this with Property 3, $\frac{p_1}{q_1} > \frac{p_2}{q_2} > \frac{p_3}{q_3} > \cdots > 0$; we conclude that $p_1 > p_2 > p_3 > \cdots > 0$, i.e., $\{p_n\}$ is a strictly decreasing infinite sequence of positive integers. This is impossible.

Remark. Unlike other existing proofs based on the series definition of $e$, this proof does not require any series analysis as long as we accept that $e$ is well-defined. It actually shares the flavor of the continued fraction approach (e.g., [1], [2], and [3, pp. 185–190]).

REFERENCES


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