Chapter 1

Quantum Dynamics

1.1 Time Evolution

Physical system given by $|\alpha\rangle$ at time t_0. At some later time $t > t_0$ state is given by $|\alpha, t_0; t\rangle$. Since time is a continuous parameter

$$\lim_{t \to t_0} |\alpha, t_0; t\rangle \equiv |\alpha\rangle . \tag{1.1}$$

Two *Kets* at different times can be related by an operator, called the time-evolution operator $u(t, t_0)$.

$$|\alpha, t_0; t\rangle = u(t, t_0) |\alpha, t_0\rangle . \tag{1.2}$$

Properties of the time-evolution operator? First important property is the unitarily requirement, follows from probability conservation. Suppose at t_0 the state *Ket* is expanded in terms of eigenkets of same observable A:

$$|\alpha, t_0\rangle = \sum_{a'} c_{a'} (t_0) |a'\rangle \tag{1.3}$$

then at a later time t

$$|\alpha, t_0; t\rangle \;=\; \sum_{a'} c_{a'}\;(t)\;|a'\rangle\;. \qquad (1.4)$$

In general

$$|c_{a'}\;(t)| \;\neq\; |c_{a'}\;(t_0)| \qquad (1.5)$$

(unless A commutes with H). Probability conservation requires

$$\sum_{a'}\;|c_{a'}\;(t_0)|^2 \;=\; \sum_{a'}\;|c_{a'}\;(t)|^2\;. \qquad (1.6)$$

Stated in a different way, this means if the state Ket is normalized to unity at a time t_0, it must remain normalized to unity at all later times. This property is guaranteed if the time-evolution operator is taken to be unitary

$$u \;\Longrightarrow\; U^+\;(t, t_0)\;U(t, t_0) \;=\; 1\;. \qquad (1.7)$$

Unitarily is often synonymous with probability conservation. A further requirement is the composition property

$$U(t_2, t_0) \;=\; U(t_2, t_1)\;U(t_1, t_0)\;; \quad t_2\;\rangle\;t_2\;\rangle\;t_0\;. \qquad (1.8)$$

Consider an infinitesimal time-evolution at

$$|\alpha, t_0; t_0 \;+\; dt\rangle \;=\; U(t_0 \;+\; dt, t_0)\;|\alpha,\; t_0\rangle\;. \qquad (1.9)$$

Because of continuity, the infinitesimal time-evolution operator must reduce to the identity operator if $dt \;\rightarrow\; 0$

$$\lim_{dt\;\rightarrow\;0}\;U(t_0 \;+\; dt,\; t_0) \;=\; \mathbf{1}\;. \qquad (1.10)$$

We expect the difference between $U(t_0 + dt, t_0)$ and $\mathbf{1}$ to be of first order in dt.

$$U\ (t_0 + dt, t_0)\ =\ \mathbf{1}\ -\ i\Omega\ dt \tag{1.11}$$

where Ω is a Hermitian operator, $\Omega^+\ \Omega\ =\ \mathbf{1}$. Unitarily property can be checked as follows:

$$U^+\ (t_0\ +\ dt,\ t_0)\ U(t_0\ +\ dt,\ t_0)\ =\ (1\ +\ i\Omega\ dt)\ (1\ -\ i\Omega\ dt)\ =\ 1 \tag{1.12}$$

to the extent that terms $\approx\ (dt)^2$ can be neglected.

The operator Ω has the dimension of frequency or inverse time. From Planck-Einstein relation $E\ =\ \hbar\ \omega$, it is natural to relate Ω to the Hamiltonian H

$$\Omega\ =\ \frac{H}{\hbar}\ . \tag{1.13}$$

So, the infinitesimal time-evolution operator can be written as

$$U(t\ +\ dt, t_0)\ =\ \mathbf{1}\ -\ \frac{i}{\hbar}\ H\ dt\ . \tag{1.14}$$

Exploit the composition property to derive an equation of U and consider

$$U(t\ +\ dt, t_0)\ =\ U(t\ +\ dt, t)\ U(t, t_0)\ =\ (\mathbf{1}\ -\ \frac{i}{\hbar}\ H\ dt)\ U(t, t_0) \tag{1.15}$$

where the time difference $t\ -\ t_0$ does not need to be infinitesimal. From (1.15) follows

$$U(t\ +\ dt,\ t_0)\ -\ U\ (t, t_0)\ =\ -\ \frac{i}{\hbar}\ H\ dt\ U\ (t, t_0) \tag{1.16}$$

or equivalently

$$i\ \hbar\ \frac{\partial}{\partial t}\ U\ (t, t_0)\ =\ H\ U\ (t, t_0)\ . \tag{1.17}$$

This is the **Schrödinger equation for the time-evolution operator**.

Applying (1.17) to a state Ket leads immediately to the Schrödinger equation for the state Ket

$$i \hbar \frac{\partial}{\partial t} U\ (t, t_0)\ |\alpha, t_0\rangle \ = \ H\ U\ (t, t_0)\ |\alpha, t_0\rangle\ . \tag{1.18}$$

Since per definition $|\alpha, t_0\rangle$ does not depend on t, this gives with (1.2)

$$i \hbar \frac{\partial}{\partial t}\ |\alpha, t_0; t\rangle \ = \ H\ |\alpha, t_0; t\rangle\ . \tag{1.19}$$

If we are given $U(t, t_0)$ and in addition know how $U(t, t_0)$ acts on the initial state $|\alpha, t_0\rangle$ it is not necessary to bother with the Schrödinger equation for the state Ket (1.19). All one has to do is apply $U(t, t_0)$ to $|\alpha, t_0\rangle$ and obtain the state Ket or any time t. The first task is to derive formal solutions for the time-evolution operator (1.17). There are three cases to be considered.

Case 1: The Hamiltonian operator H is independent of time. This means even when the parameter t changes, the H operator remains unchanged.

Example: Hamiltonian for a spin-magnetic moment interaction with time-independent magnetic field.

In this case, the solution of (1.17) is given by

$$U(t, t_0) \ = \exp\left[-\ \frac{i}{\hbar}\ H(t - t_0)\right]\ . \tag{1.20}$$

To prose this expand the potential

$$\exp\left[-\frac{i}{\hbar}\ H\ (t - t_0)\right] \ = \ \mathbf{1} \ - \ \frac{i}{\hbar}\ H\ (t - t_0) \ + \ \frac{1}{2}\ \left(-\frac{i}{\hbar}\right)^2\ [H\ (t - t_0)]^2 \ + \ \cdots \tag{1.21}$$

The derivation is given by

$$\frac{\partial}{\partial t} \exp\left[-\ \frac{i}{\hbar}\ H(t - t_0)\right] \ = \ -\ \frac{i}{\hbar}\ H \ + \ \frac{1}{2}\ \cdot\ 2\ \left(-\ \frac{i}{\hbar}\right)^2\ H^2\ (t - t_0) \ + \ \cdots \tag{1.22}$$

Comparison with (1.17) shows that (1.20) fulfills the differential equation. For $t \rightarrow t_0$ (1.20) reduces to the identity thus the boundary conditions we fulfilled.

Case 2: The Hamiltonian is time-dependent, but the $H's$ at different times commute.

Example: Spin magnetic moment with magnetic field, whose strength varies with time, but the direction is unchanged.

The formal solution of (1.17) is given by

$$U(t,\ t_0) \ = \exp\left[-\ \frac{i}{\hbar}\ \int_{t_0}^{t}\ dt'\ H\ (t')\right]\ . \tag{1.23}$$

For a proof replace in (1.20) $H(t-t_0)$ with $\int_{t_0}^{t}\ dt'\ H(t')$.

Case 3: The $H's$ at different times do **not** commute. In the example above that would involve a magnetic field whose direction changes with time. Since, e.g., S_x and S_y do not commute, a Hamiltonian with a term $\vec{S}\ \cdot\ \vec{B}$ would fall in this category.

The formal solution in this case is given by

$$U(t,t_0)\ =\ \mathbf{1} + \sum_{n=1}^{\infty}\left(-\ \frac{i}{\hbar}\right)^n \int_{t_0}^{t}\ dt_1\ \int_{t_0}^{t}\ dt_2\ \cdots\ \int_{t_0}^{t}\ dt_n\ H\ (t_1)\ H\ (t_2)\ \cdots\ H\ (t_n) \tag{1.24}$$

which is sometimes knows as **Dyson Series**. Dyson developed a perturbative expansion of this form in quantum field theory.

In usual applications, one considers Case 1. Then the effect of the time-evolution operator is particularly straightforward to obtain, if the basis states are eigenstates to an operator A with

$$[A,\ H]\ =\ 0 \tag{1.25}$$

such that the eigenstates of A are also eigenstates of H with

$$H\ |a'\rangle\ =\ E_{a'}\ |a'\rangle\ . \tag{1.26}$$

Expanding (1.20) in terms of $|a'\rangle\ \langle a'|$ (at $t_0\ =\ 0$ for simplicity) gives

$$\begin{aligned} exp\left(-\frac{i}{\hbar}\ H\ t\right) &= \sum_{a'}\sum_{a'}\ |a'\rangle\ \langle a'|\exp\left(-\frac{i}{\hbar}\ H\ t\right)\ |a'\rangle\ \langle a'| \\ &= \sum_{a'}\ |a'\rangle \exp\left(-\frac{i}{\hbar}\ E_{a'\ t}\right)\ .\ \langle a'| \end{aligned} \tag{1.27}$$

The time-evolution operator written in this form allows to solve any initial value problem once the expansion of the initial Ket in terms of $\{|a'\}$ is known. Suppose

$$|\alpha,\ t_0\ =\ 0\rangle\ =\ \sum_{a'}\ |a'\rangle\ \langle a'|\alpha\rangle\ =\ \sum_{a'}\ c_{a'}|a'\rangle\ . \tag{1.28}$$

We then have

$$|\alpha,\ t_0\ = 0\,;t\rangle\ = \exp\left(-\frac{i}{\hbar}\ H\ t\right)\ |\alpha,\ t_0\ = 0\rangle\ =\ \sum_{a'}\ |a'\rangle\ \langle a'|\alpha\rangle \exp\left(-\frac{i}{\hbar}\ E_{a'}\ t\right)\ . \tag{1.29}$$

In other words, the expansion coefficient changes with time as

$$c_{a'}\ (t\ =\ 0)\ \longrightarrow\ c_{a'\ (t)}\ =\ c_{a'}\ (t\ =\ 0)\exp\left(-\frac{i}{\hbar}\ E_{a'}\ t\right)\ . \tag{1.30}$$

It is modulus unchanged. The relative phases among the various components vary with time since the oscillation frequencies are different. If the initial state $\{|a'\rangle\}$, i.e., $|\alpha,\ t_0\ =\ 0\rangle\ =\ |a'\rangle$ then

$$|\alpha,\ t_0\ =\ 0\ ;\ t\rangle\ =\ |a'\rangle \exp\left(-\frac{i}{\hbar}\ E_{a'}\ t\right)\ , \tag{1.31}$$

thus, if the system is initially a simultaneous eigenstate of A and H, it remains so at all times, within a phase modulation. In this sense the observable A is compatible with H and is a constant of motion.

It is instructive to study how the expectation value of an observable changes as function of time. Suppose at $t = 0$ the initial state is eigenstate of an observable A with $[A,\ H] = 0$, and we look at the expectation value of an observable B, which does not commute with A or H. Because of $|a',\ t_0 = 0\ ;\ t\rangle = U\ (t,\ 0)\ |a'\rangle$ we have

$$\begin{aligned} \langle B\rangle &\equiv \langle a'|U^\dagger\ (t,0)\ B\ U\ (t,0)\ |a'\rangle \\ &= \langle a'|\exp\left(\frac{i}{\hbar}\ E'\ t\right)\ B\exp\left(-\ \frac{i}{\hbar}\ E_{a'}\ t\right)\ |a'\rangle \\ &= \langle a'|\ B\ |a'\rangle \end{aligned} \tag{1.32}$$

which is *independent of* t. For this reason energy eigenstates are called *stationary states.*

The situation is more interesting when the expectation value is taken with respect to a *superposition* of energy eigenstates or a non-stationary state. Suppose the initial state is given by

$$|\alpha,\ t = 0\rangle = \sum_{a'} c_{a'}|a'\ . \tag{1.33}$$

Then

$$\begin{aligned} \langle B\rangle &= \left[\sum_{a'} c_a^*\ \langle a'|\exp\left(\frac{i}{\hbar}\ E_{a'}\ t\right)\right]\ B\ \left[\sum_{a''} c_{a''}\exp\left(-\ \frac{i}{\hbar}\ E_{a''}\ t\right)\ |a''\rangle\right] \\ &= \sum_{a'}\sum_{a''} c_{a'}^*\ c_{a''}\ \langle a'\ |B\ \ a''\rangle\exp\left(-\ \frac{i}{\hbar}\ (E_{a''}\ -\ E_a)\ t\right)\ . \end{aligned} \tag{1.34}$$

Thus the expectation value consists of oscillating terms whose frequencies are determined by

$$\omega_{a''a'} = \frac{(E_{a''}\ -\ E_{a'})}{\hbar}\ . \tag{1.35}$$

1.2 Time-Dependent Wave Equation

Consider the Schrödinger picture and study the time-evolution of $|\alpha, t_0; t\rangle$ in the coordinate representation, i.e., examine the behavior of the wave function

$$\psi\ (\vec{x}',\ t)\ =\ \langle \vec{x}'\ |\alpha, t_0; t\rangle \tag{1.36}$$

as function of time. The Hamiltonian is given by

$$H\ =\ \frac{p^2}{2m}\ +\ V(\vec{x}) \tag{1.37}$$

where $V(\vec{x})$ is a local operator, i.e., $\langle \vec{x}''|V(\vec{x})|\vec{x}\rangle\ =\ V(\vec{x})\ \delta^3\ (\vec{x}'' - \vec{x}')$, and $V(\vec{x})$ be a real function. The Schrödinger equation for the state (1.19) written in coordinate representation is

$$i\ \hbar\ \frac{\partial}{\partial t}\ \langle x'|\alpha, t_0; t\rangle\ = \langle x'|\ H\ |\alpha, t_0; t\rangle\ . \tag{1.38}$$

Inserting the Hamiltonian (1.37) leads to

$$i\ \hbar\ \frac{\partial}{\partial t}\ \langle \vec{x}'\ |\alpha, t_0; t\rangle\ =\ -\ \frac{\hbar^2}{2m}\ \nabla'^2\ \langle \vec{x}'\ |\alpha, t_0; t\rangle\ +\ V(\vec{x}')\ \langle \vec{x}'|\alpha, t_0; t\rangle\ . \tag{1.39}$$

This represents the time-dependent Schrödinger equation and is the starting point for the so-called *wave mechanics.*

Eigenfunctions of the Hamiltonian have the simple time dependence of (1.31),

$$\langle \vec{x}'\ |a', t_0; t\rangle\ =\ \langle \vec{x}'|a'\rangle \exp\left(-\ \frac{i}{\hbar}\ E_{a'}\ t\right), \tag{1.40}$$

where it is understood that initially the system is prepared in a simultaneous eigenstate of A and H with eigenvalues a' and $E_{a'}$. Substituting (1.40) into (1.39) leads to the time-independent Schrödinger equation

$$-\ \frac{\hbar^2}{2m}\ \nabla'^2\ \langle \vec{x}'\ |a'\rangle\ +\ V(\vec{x}')\ \langle \vec{x}'\ |a'\rangle\ =\ E_{a'}\ \langle \vec{x}'\ |a'\rangle\ . \tag{1.41}$$

This partial differential equation is satisfied by the energy eigenfunctions $\langle \vec{x}' | a' \rangle$ with energy eigenvalues $E_{a'}$.

Let us turn to the *interpretation of the wave function.* The expression $\langle \vec{x}' \, | \alpha, t_0; t \rangle$ is to be considered as the expansion coefficient of $|\alpha, t_0; t\rangle$ in terms of the position eigenstates $\{|\vec{x}'\rangle\}$. The quantity $\rho\ (\vec{x}', t)$ defined by

$$\rho\ (\vec{x}',\ t) \ \equiv\ |\psi\ (\vec{x}',\ t)|^2 \ =\ |\langle \vec{x}' \ |\alpha, t_0; t\rangle|^2 \tag{1.42}$$

is, therefore, regarded as the probability density in quantum mechanics, e.g., when using a detector to ascertain the presence of the particle within a volume element $d^3\ x'$ around $\vec{x}'$, the probability of recording a positive result at time t is given by $\rho\ (\vec{x}',\ t)\ d^3\ x'$.

Defining a *probability flux* $j(\vec{x}, t)$ by

$$\begin{aligned} \vec{j}\ (\vec{x},\ t) \ &=\ -\ \left(\frac{i\hbar}{2m}\right)\ [\psi^*\ \nabla\psi\ -\ (\nabla\psi^*)\ \psi] \\ &=\ \frac{\hbar}{m}\ \Im\ (\psi^*\ \nabla\psi) \end{aligned} \tag{1.43}$$

we can derive the continuity equation

$$\frac{\partial\rho}{\partial t}\ +\ \vec{\nabla}\ \cdot\ \vec{j}\ =\ 0\ . \tag{1.44}$$

In obtaining this result, the Hermiticity of V (a reality of V) plays a crucial role. A complex potential can phenomenologically account for the disappearance of particles.

Rewrite the wave function as

$$\psi(\vec{x},\ t)\ =\ \sqrt{\rho(\vec{x}, t)}\ exp\ \left(\ \frac{i}{\hbar}\ S(\vec{x}, t)\right) \tag{1.45}$$

with S real and $\rho > 0$. Any complex function of $\vec{x}$ and t can be represented this way. Consider

$$\psi^* \nabla\psi = \sqrt{\rho}\exp\left(-\frac{i}{\hbar}S\right)\left[\nabla\sqrt{\rho}\exp\left(\frac{i}{\hbar}S\right) + \sqrt{\rho}\frac{i}{\hbar}\nabla S\, exp\left(\frac{i}{\hbar}S\right)\right] \tag{1.46}$$
$$= \sqrt{\rho}\,\nabla\sqrt{\rho} + \frac{i}{\hbar}\rho\nabla S$$

and

$$\psi\nabla\psi^* = \sqrt{\rho}\,\nabla\sqrt{\rho} - \frac{i}{\hbar}\rho\nabla\rho \tag{1.47}$$

follows from (1.44)

$$\vec{j}(\vec{x},\ t) = \frac{\rho(\vec{x},t)}{m}\vec{\nabla}\, S(\vec{x},t)\ . \tag{1.48}$$

Thus, the gradient of the phase S, i.e., the *spatial variation of the phase* of the wave function characterizes the probability flux. The stronger the phase variations, the more intense the flux. The direction of $\vec{j}$ at some point $\vec{x}$ is seen to be normal to the surface of a constant phase that goes through that point.

Consider simple case of plane wave

$$\psi(\vec{x},t) \approx \exp\left(\frac{i}{\hbar}\vec{p}\cdot\vec{x} - \frac{i}{\hbar}E\,t\right)\ . \tag{1.49}$$

Then $\vec{\nabla}\, S = \vec{p}$. It is tempting to regard $\vec{\nabla}\, S/m$ as some kind of velocity, so that the continuity equation reads

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho''\,\vec{v}'') = 0 \tag{1.50}$$

as in fluid dynamics. But caution!

1.3 Propagators

In Section 1.1 we showed that the most general time-evolution with a time-independent Hamiltonian can be solved once the initial state is expanded in terms of eigenstates of an observable that commutes with H.

$$\begin{aligned} |\alpha, t_0; t\rangle &= \exp\left[-\frac{i}{\hbar}\, H\, (t-t_0)\right]\, |\alpha, t_0\rangle \\ &= \sum_{a'} |a'\rangle\langle a'\, |\alpha,\ t_0\rangle \exp\left[-\frac{i}{\hbar}\, E_{a'}\, (t-t_0)\right] \end{aligned} \tag{1.51}$$

Multiplying with $\langle \vec{x}\,'|$ gives

$$\langle \vec{x}'\, |\alpha, t_0; t\rangle = \sum_{a'} \langle \vec{x}'\, |a'\rangle\, \langle a'\, |\alpha, t_0\rangle \exp\left[-\frac{i}{\hbar}\, E_{a'}(t-t_0)\right] \tag{1.52}$$

which is of the form

$$\psi\, (\vec{x}, t) = \sum_{a'} c_{a'}\, (t_0)\, u_{a'}\, (\vec{x}')\, exp\, \left[-\frac{i}{\hbar}\, E_{a'}\, (t-t_0)\right]\ , \tag{1.53}$$

where $u_{a'}(\vec{x}') = \langle \vec{x}'\, |a'\rangle$ stands for the eigenfunctions of an operator A with eigenvalue a'. Note also that

$$\langle a'\, |\alpha, t_0\rangle = \int d^3\, x'\, \langle a'\, |\vec{x}'\rangle\, \langle \vec{x}'\, |\alpha, t_0\rangle \tag{1.54}$$

which corresponds to the usual rule for obtaining the expansion coefficients of an initial state:

$$c_{a'}\, (t_0) = \int d^3\, \vec{x}'\, u_a^*\, (\vec{x}')\, \psi\, (\vec{x}',\ t_0)\ . \tag{1.55}$$

Combining (1.52) and (1.54) can be written as some kind of integral operator, which acts on the initial wave function to yield a final wave function:

$$\psi\ (\vec{x}'',\ t)\ =\ \int\ d^3\ \vec{x}\ \mathbf{K}\ (\vec{x}'',\ t\ ;\ \vec{x}',\ t_0)\ \psi\ (\vec{x}',\ t_0) \tag{1.56}$$

where the kernel of the integral operator is given by

$$\mathbf{K}\ (\vec{x}'', t\ ;\ \vec{x}', t_0)\ =\ \sum_{a'}\ \langle \vec{x}''\ |a'\rangle\ \langle a'\ |\vec{x}'\rangle \exp\left[-\ \frac{i}{\hbar}\ E_{a'}\ (t - t_0)\right] \tag{1.57}$$

and is known as **propagator**.

In any given problem the propagator depends only on the potential (via H) and is independent of the initial wave function. The time-evolution of the wave function is completely determined once $\mathbf{K}\ (\vec{x}'', t; \vec{x}', t)$ is known and $\psi\ (\vec{x}', t_0)$ is given. In this sense, the Schrödinger theory is a *perfectly causal theory.* The time development of a wave function subjected to some potential is as deterministic as classical mechanics *provided the system is left undisturbed.* If a measurement is intervenes, the wave function changes abruptly.

Properties of the propagator:

1. For $t\ >\ t_0, \mathbf{K}\ (\vec{x}''; \vec{x}', t_0)$ satisfies Schrödinger's time-dependent wave equation in variables $\vec{x}'$ and t with $\vec{x}'$ and t_0 fixed (via construction).
2. For $t\ \rightarrow\ t_0$

$$\lim_{t \to t_0}\ \mathbf{K}\ (\vec{x}'', t;\ \vec{x}', t_0)\ =\ \delta^3\ (\vec{x}'\ -\ \vec{x}') \tag{1.58}$$

because of the completeness of $\{|a'\rangle\}$

In fact, (1.57) can be written as

$$\mathbf{K}\ (\vec{x}'', t; \vec{x}', t_0)\ =\ \langle \vec{x}''\ | \exp\left[-\ \frac{i}{\hbar}\ H\ (t - t_0)\right]\ |\vec{x}'\rangle\ . \tag{1.59}$$

This means the propagator acts on a state $|\vec{x}'\rangle$ at a given t_0 and propagates it to state $|\vec{x}''\rangle$ at time t.

To obtain information over the wave function $\psi\ (\vec{x}', t_0)$ distributed over a finite space, one multiples $\psi\ (\vec{x}', t_0)$ with $\mathbf{K}$ and integrates over the entire space.

The propagator is the Green's function for the time-dependent Schrödinger equation satisfying

$$\left[- \left(\frac{\hbar^2}{2m} \right) \nabla''^2 \; + \; V(\vec{x}'') \; - \; i\hbar \frac{\partial}{\partial t} \right] \; \mathbf{K} \; (\vec{x}'', t; \vec{x}', t_0) \; = -i\hbar \; d^3(\vec{x}'' - \vec{x}') \; \delta \; (t - t_0) \quad (1.60)$$

with the boundary condition

$$\mathbf{K} \; (\vec{x}'', t; \vec{x}', t_0) \; = \; 0 \quad \text{for} \;\; t \; < \; t_0 \; . \qquad (1.61)$$

The $\delta-$function $\delta \; (t - t_0)$ is needed in (1.60), since $\mathbf{K}$ varies discontinuous at $t \; = \; t_0$.

The particular form of $\mathbf{K}$ depends on the potential in the Hamiltonian. Consider the simplest case of free propagation, i.e.,

$$H_0 \; = \; \frac{p^2}{2m} \; . \qquad (1.62)$$

The obvious observable to commute with H are the momentum eigenstates with $p|p'\rangle \; = p'|p'\rangle$ Starting from (1.59) and setting $t_0 \; = \; 0$ for convenience, one obtains

$$\begin{aligned} \mathbf{K} \; (\vec{x}'', t; \vec{x}', 0) \; &= \; \langle \vec{x}''| \exp \left[- \; \frac{i}{\hbar} \; H_0 \; t \right] \; |\vec{x}'\rangle \qquad (1.63) \\ &= \; \int d^3 \; p' \; \langle \vec{x}''| \exp \left[- \; \frac{i}{\hbar} \; \frac{p^2}{2m} \; t \right] \; |\vec{p}'\rangle \; \langle \vec{p}' \; |\vec{x}'\rangle \\ &= \; \int d^3 \; p' \; \langle \vec{x}'' \; |\vec{p}''\rangle \; \langle \vec{p}' \; |\vec{x}'\rangle \exp \left[- \; \frac{i}{\hbar} \frac{p'^2}{2m} \; t \right] \\ &= \; \int d^3 \; p' \; \frac{1}{(2\pi \; \hbar)^3} \; e^{\frac{i}{\hbar} \; \vec{p}' \; \cdot \; (\vec{x}'' - \vec{x}')} \; e^{- \; \frac{i}{\hbar} \; \frac{p'^2}{2m} \; t} \\ &= \; \frac{1}{(2\pi \; \hbar)^3} \; \int d^3 \; p' \; e^{\frac{i}{\hbar} \; \left[\vec{p} \; \cdot \; (\vec{x}'' \; - \; \vec{x}') - \frac{p^2}{2m} \; t \right]} \; . \end{aligned}$$

Splitting up the integral into $\int \; d \; p_1' \; \cdots \; \int \; d \; p_2' \; \cdots \; \int \; d \; p_3' \; \cdots$, one has to solve

$$\int \; d \; p \; e^{\frac{i}{\hbar} \; \left(px \; - \; \frac{p^2}{2m} \; t \right)} \; . \qquad (1.64)$$

Consider exponent

$$\left(-\frac{p^2}{2m}t + px\right) = -\frac{t}{2m}\left(+p^2 - \frac{2m}{t}px + \frac{x^2m^2}{t^2}\right) + \frac{x^2}{2}\frac{m}{t} \tag{1.65}$$
$$= -\frac{t}{2m}\left(+p - \frac{mx}{t}\right)^2 + \frac{x^2m}{2t}$$
$$= -\left(\frac{x}{2}\sqrt{\frac{2m}{t}} - \sqrt{\frac{t}{2m}}p\right)^2 + \frac{x^2m}{2t} \ .$$

Thus

$$\int dp\, e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)} = \int d\left(p\cdot\sqrt{\frac{t}{2m\hbar}}\right) e^{\frac{i}{\hbar}\frac{mx^2}{2t}}\sqrt{\frac{2m\hbar}{t}} e^{-\frac{i}{\hbar}\left(\frac{x}{2}\sqrt{\frac{2m}{t}} - p\sqrt{\frac{t}{2m}}\right)^2} \tag{1.66}$$
$$= e^{\frac{i}{\hbar}\frac{mx^2}{2t}}\sqrt{\frac{2m\hbar}{t}}\int d\left(p\sqrt{\frac{t}{2m\hbar}} - \frac{x}{2}\sqrt{\frac{2m}{t\hbar}}\right) e^{-\frac{i}{\hbar}\left(\frac{x}{2}\sqrt{\frac{2m}{t}} - p\sqrt{\frac{t}{2m}}\right)^2}$$
$$= e^{\frac{i}{\hbar}\frac{mx^2}{2t}}\sqrt{\frac{2m\hbar}{t}}\int d\xi e^{-i\xi^2}$$
$$= e^{\frac{i}{\hbar}\frac{mx^2}{2t}}\sqrt{\frac{2m\hbar}{t}}\sqrt{\frac{\pi}{i}}$$
$$= e^{\frac{i}{\hbar}\frac{mx^2}{2t}}\sqrt{\frac{2m\pi\hbar}{it}}$$

.

Thus

$$\frac{1}{2\pi\hbar}\int dp e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)} = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} e^{\frac{i}{\hbar}\frac{mx^2}{2t}} \ . \tag{1.67}$$

Thus for (1.64) one obtains

$$\mathbf{K}\ (\vec{x}'', t, \vec{x}', 0) = \frac{1}{(2\pi\ \hbar)^3}\int d^3p'\ e^{\frac{i}{\hbar}\ \vec{p}'\cdot(\vec{x}'' - \vec{x}')}\ e^{-\frac{i}{\hbar}\frac{p'^2}{2m}\ t} \tag{1.68}$$
$$= \left(\frac{m}{2\pi i\hbar\ t}\right)^{3/2} e^{\frac{i}{\hbar}\ \frac{m}{2t}\ (\vec{x}'' - \vec{x}')^2}$$

.

This expression can be used to study the spreading of a Gaussian wave pocket over time.

Certain space and time integrals derivable from $\mathbf{K}\ (\vec{x}''t; \vec{x}', t_0)$ are of special interest. Set again $t_0 = 0$ and consider $\vec{x}'' = \vec{x}'$ and integrate over space, i.e.,

$$
\begin{aligned}
G\ (t) &= \int d^3\ x'\ \mathbf{K}\ (\vec{x}', t, \vec{x}', 0) && (1.69) \\
&= \int d^3\ x'\ \sum_{a'}\ |\langle \vec{x}' | a' \rangle|^2 \exp\left[-\frac{i}{\hbar}\ E_{a'}\ t\right] \\
&= \sum_{a'} \exp\left[-\frac{i}{\hbar}\ E_{a'}\ t\right] .
\end{aligned}
$$

Notice that setting $\vec{x}'' = \vec{x}'$ is equivalent to taking the trace of the time-evolution operator in the $\vec{x}$ representation. The trace is independent of the representation, thus one can use a basis in which H is diagonal. In a sense (1.70) is just a sum over states, reminiscent of a partition function in statistical mechanics. Continue the variable t analytically into the complex plane and define

$$
\beta = \frac{it}{\hbar} \tag{1.70}
$$

to be real and positive. Then (1.70) can be identified with the partition function

$$
Z = \sum_{a'} \exp(-\beta\ E_{a'}) \ . \tag{1.71}
$$

For this reason, some of the techniques used in statistical mechanics can provide useful in dealing with propagators in quantum mechanics.

Consider the Laplace-Fourier transform of $G(t)$

$$
\begin{aligned}
\tilde{G}(E) &= -\frac{-i}{\hbar} \int_0^\infty dt\ G(t) \exp\left(\frac{i}{\hbar}\ Et\right) && (1.72) \\
&= -\frac{i}{\hbar} \int_0^\infty dt\ \sum_{a'}\ exp\ \left(-\frac{i}{\hbar}\ E_{a'}\ t\right) \exp\left(\frac{i}{\hbar}\ Et\right) \ .
\end{aligned}
$$

The integrand is indefinitely oscillatory. Thus let E acquire a small imaginary part $E \rightarrow e + i\epsilon$. Then one obtains in the limit $\epsilon \rightarrow 0$

$$\tilde{G}\ (E)\ =\ \sum_{a'}\ \frac{1}{E - E_{a'}}\ . \tag{1.73}$$

Here the entire energy spectrum is exhibited as simple poles of $\tilde{G}(E)$ in the complex E-plane. Thus, to find the energy spectrum of a physical system, it is sufficient to study the analytic properties of $\tilde{G}\ (E)$.

To gain further insight into the physical meaning of the propagator, we wish to relate it to the concept of a transition amplitude. According to (1.59) $\mathbf{K}\ (\vec{x}'', t; \vec{x}', t_0)$ can be written as

$$\begin{aligned}
\mathbf{K}\ (\vec{x}'', t; \vec{x}', t_0) \ &= \ \langle \vec{x}' |\ \exp\left[-\ \frac{i}{\hbar}\ H\ (t\ -\ t_0) \right]\ \vec{x}' \rangle \\
&= \ \sum_{a'}\ \langle \vec{x}'' |\ \exp\left[-\ \frac{i}{\hbar}\ H\ t \right]\ |a'\rangle\ \langle a' |\ exp\ \left[\frac{i}{\hbar}\ H\ t_0 \right] |\vec{x}' \rangle \\
&= \ \langle \vec{x}''\ t\ |\ \vec{x}',\ t_0 \rangle\ .
\end{aligned} \tag{1.74}$$

where the states $|\vec{x}',\ t_0\rangle$ and $\langle \vec{x}'',\ t\rangle$ are to be understood as eigenket (bra) in the Heisenberg picture.

In this notation $\langle \vec{x}'', t\ |\vec{x}', t|$ can be identified as the probability amplitude for the particle prepared at time t_0 with position eigenvalue $\vec{x}'$ to be found at a later time t with position eigenvalue $\vec{x}''$.

Roughly speaking, $\langle \vec{x}'', t\ |\vec{x}', t_0\rangle$ is the **transition amplitude** for the particle to go from space time point $(\vec{x}', t_0)$ to another space time point $(\vec{x}'', t)$.

Yet another way to interpret $\langle \vec{x}''t|\ \vec{x}'t_0\rangle$ is to view $|\vec{x}'t_0\rangle$ as position Ket at t_0 with eigenvalue $\vec{x}''$. Thus $\langle \vec{x}''t\ |\vec{x}'t_0\rangle$ is a transformation function that connects two sets of base $Kets$ at different times, i.e., the time-evolution can be viewed as unitary transformation that connects one set of basis $Kets$ $\{|\vec{x}', t_0\rangle\}$ to one formed by $\{|\vec{x}'', t\rangle\}$.

To use a more systematic notation, we write $\langle \vec{x}'', t''\ |\vec{x}', t'\rangle$ Since at any given time those sets form a complete basis, the identity can be represented as

$$\int\ d^3\ x''\ |\ x'',\ t''\rangle\ \langle x'',\ t''|\ =\ \mathbf{1}\ . \tag{1.75}$$

Thus the time-evolution from t' to t''' can be divided into two intervals: $(t',\ t''')\ \rightarrow\ (t',\ t'')\ \cup\ (t'',\ t''')$ with $t'''\ >\ t''\ >\ t'$ as

$$\langle \vec{x}''', t''' | \vec{x}', t' \rangle \ = \ \int \ d^3\ x''\ \langle \vec{x}''', t''' | \vec{x}'', t'' \rangle\ \langle \vec{x}'',\ t'' | \vec{x}',\ t' \rangle \tag{1.76}$$

We call this the *composition property* of the transition amplitude. Clearly, the considered time interval can be divided into as many smaller subintervals as desired

$$\begin{aligned} \langle \vec{x}'''',\ t'''' |\ \vec{x}', t' \rangle \ &= \ \int \ d^3\ x''' \ \int \ d^3\ x'' \langle \vec{x}'''',\ t'''' | \vec{x}''', t''' \rangle \\ & \langle \vec{x}''', t''' \rangle \langle \vec{x}'', t'' \rangle\ \langle \vec{x}'', t'' | \vec{x}'', t'' | \vec{x}', t' \rangle \end{aligned} \tag{1.77}$$

1.4 Feynman Path Integral Formulation

Without loss of generality, consider here only one-dimensional problems. Consider the transition amplitude for a particle going from an initial space time point (x_1, t_1) to a final space time point (x_n, t_n). The entire time interval between t_1 and t_N is divided into $N-1$ equal parts

$$t_j - t_{j-1} \equiv \Delta t = \frac{(t_N - t_n)}{N-1} \tag{1.78}$$

Exploiting the composition property gives

$$\begin{aligned}\langle x_n, t_N | x_1, t_1 \rangle &= \int dx_{N-1} \int dx_{N-2} \cdots \int dx_2 \ \langle x_n, t_N | x_{N-1}, t_{N-1} \rangle \\ &\quad \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \cdots \langle x_2, t_2 | x_1, t_1 \rangle \ .\end{aligned} \tag{1.79}$$

To visualize, consider the $x - t$ plane in Fig. 1.1.

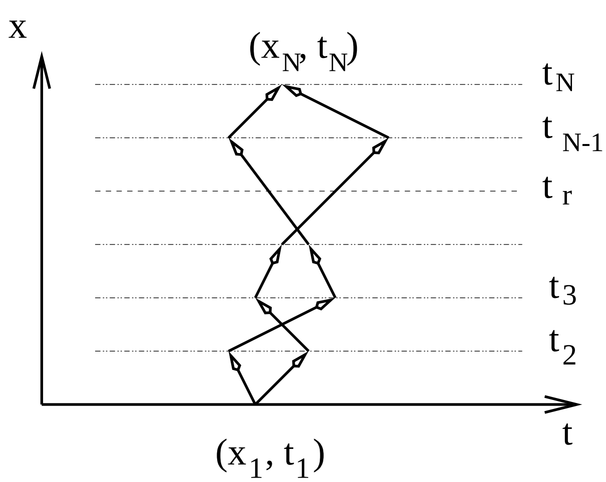

Fig. 1.1 Paths in the x-t-plane

The initial and final space-time points (x_1, t_1) and (x_N, t_N) are fixed. For each time segment, i.e., between t_{n-1} and t_n, we are instructed to consider the transition amplitude to go from (x_{n-1}, t_{n-1}) to (x_n, t_n) and then integrate over $x_2, x_3, \cdots x_{N-1}$. This means that we must *sum over all possible paths* in the space time plane with the end points fixed.

Short review of path integrals in classical mechanics:

The *classical Lagrangian* is written as

$$L_{classical}\,(x, \dot{x}) \;=\; \frac{m\dot{x}^2}{2} \;-\; V(x)\;. \tag{1.80}$$

Given this Lagrangian with fixed end points (x_1, t_x) and (x_N, t_N), we do **not** consider just any path joining the end points. There exists a *unique path* that corresponds to the actual path of a classical particle. E.g., for

$$\begin{aligned} V(x) &= mgx \\ (x_1, t_1) &= (h, 0) \\ (x_N, t_N) &= \left(0,\ \sqrt{\frac{2h}{g}}\right) \end{aligned} \tag{1.81}$$

the classical path in the $x-t$-plane can **only** be

$$x \;=\; h \;-\; \frac{gt^2}{2}\;. \tag{1.82}$$

According to Hamilton's principle, the unique path that minimizes the action is defined as

$$\delta \int_{t_1}^{t_2} dt\; L_{classical}\,(x, \dot{x}) \;=\; 0 \tag{1.83}$$

from which the equations of motion are obtained.

The basic difference between classical mechanics and quantum mechanics is that in classical mechanics a definite path is associated with the particle's motion. In contrast, in quantum mechanics all possible paths are included. Yet, classical mechanics must be reproduced in a smooth manner in the limit $\hbar \;\to\; 0$.

Introduce the notation

$$S(n, n-1) \;\equiv\; \int_{t_{n-1}}^{t_n} dt\; L_{classical}\;(x, \dot{x})\;. \tag{1.84}$$

Because $L_{classical}$ is a function of x and $\dot{x}$, $S(n, n-1)$ is defined only after a definite path is specified along which the integration is to be carried out.

Consider a small segment between (n_{n-1}, t_{n-1}). According to a suggestion by Dirac an "*evolution operator*" *exp* $\left(\frac{i}{\hbar}\; S(n, n-1)\right)$ should be associated with that segment. Going along a definite path, successively expressions of this type need to be multiplied:

$$\prod_{n=2}^{N} \exp\left[\frac{i}{\hbar}\; S(n, n-1)\right] \;= \exp\left[\frac{i}{\hbar}\; \sum_{n=2}^{N}\; S(n, n-1)\right] \;= \exp\left[\frac{i}{\hbar}\; S(N, 1)\right]\;. \tag{1.85}$$

This does not yet give $\langle x_n, t_n | x_1, t_1\rangle$ rather describes the contribution along a particular path. One still needs to integrate over $x_2,\; x_3,\; \cdots\; x_{N-1}$. At the same time, let time interval Δt be infinitesimally small. Thus, in a loose sense we may write

$$\langle x_N, t_N | x_1, t_1\rangle \;\approx\; \sum_{\text{all paths}} \exp\left[\frac{1}{\hbar}\; S(N, 1)\right]\;, \tag{1.86}$$

where the sum is to be taken over an infinite set of paths.

Qualitative remarks:

As $\hbar\;\; \rightarrow$ exponent in (1.82) oscillates strongly, thus destructive interference for most paths.

Consider a path that satisfies $\delta\; S(N, 1) \;=\; 0$, where the change in S is due to a slight deformation of the path with the end point fixed (i.e., the classical path of Hamilton's principle). As long as a deformation of the classical path is small, there will be constructive interference, even if $\hbar$ is small. For larger deformations, destructive interference. As a result, as long as we stay near the classical path constructive interference between neighboring paths is possible. In the limit $\hbar \;\rightarrow\; 0$ the major contributions must arise from a very narrow stripe containing the classical paths.

To formulate Feynman's conjecture more precisely, write

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \; = \; \frac{1}{\omega(\delta t)} \; exp \; \left[\frac{i}{\hbar} \; S(n, n-1) \right] \tag{1.87}$$

where the difference $\Delta t \; = \; t_n - t_{n-1}$ is assumed to be infinitesimally small, and $S(n, n-1)$ is evaluated in the limit $\delta t \; \rightarrow \; 0$. The weight factor $\frac{1}{\omega(\delta t)}$ is assumed to depend only on the time interval $t_n - t_{n-1}$, and is necessary since $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$ has the dimension of $\frac{1}{length}$. Consider the exponent $S(n, n-1)$ in the limit $\delta t \; \rightarrow \; 0$. Since δt small, a straight-line approximation between (x_{n-1}, t_{n-1}) and (x_n, t_n) is justified.

$$\begin{aligned} S(n, n-1) \; &= \; \int_{t_{n-1}}^{t_n} \; dt \; \left[\frac{m \dot{x}^2}{2} \; - \; V(x) \right] \\ &= \; \delta t \left\{ \left(\frac{m}{2} \right) \; \left[\frac{x_n - x_{n-1}}{\delta t} \right]^2 \; - \; V \; \left(\frac{x_n + x_{n-1}}{2} \right) \right\} \; . \end{aligned} \tag{1.88}$$

Consider the free motion case, $V = 0$, where (1.87) becomes

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \; = \; \frac{1}{\omega(\delta t)} \; exp \; \left[\frac{i}{\hbar} \; \frac{m}{2} \; \frac{(x_n - x_{n-1})^2}{\delta t} \right] \; . \tag{1.89}$$

This expression is equivalent to the one for free particle propagation given in (1.69).

The weight form $\frac{1}{\omega(\Delta t)}$ is assumed to be independent of $V(x)$, so it may well be evaluated for the free propagation. Because of the normalization $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \delta(x_n - x_{n-1})$, we obtain with

$$\int_{-\infty}^{\infty} \; d\xi \exp \left(\frac{im \; \xi^2}{2\hbar \; \Delta t} \right) \; = \; \sqrt{\frac{2\pi \; i \; \hbar \; \Delta t}{m}} \tag{1.90}$$

for $\frac{1}{\omega(\Delta t)}$

$$\frac{1}{\omega(\Delta t)} \; = \; \sqrt{\frac{m}{2\pi \; i \; \hbar \; \Delta t}} \tag{1.91}$$

and

$$\lim_{\Delta t\to 0} \sqrt{\frac{m}{2\pi\ i\ \hbar\ \Delta t}} \exp\left(\frac{im\ \xi^2}{2\hbar\ \Delta t}\right) \ = \ \delta(\xi)\ . \tag{1.92}$$

In summary, as $\Delta t \to 0$ one obtains

$$\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle \ = \ \sqrt{\frac{m}{2\pi\ i\ \hbar\ \Delta t}}\ exp\ \left[\frac{i}{\hbar}\ S(n, n-1)\right]\ . \tag{1.93}$$

Thus, the final expression for the transition amplitude, where $t_n - t_1$ is finite, is given as

$$\langle x_N, t_N | x_1, t_1\rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{(N-1)/2} \int dx_{N-1} \int dx_{N-2} \cdots \int dx_1 \prod_{n=2}^{N} \exp\left[\frac{i}{\hbar} S(n, n-1)\right] \tag{1.94}$$

where the limit $N \ \to \ \infty$ is taken with x_N and t_N fixed. It is customary to define the functional

$$\int_{x_n}^{x_N} \mathbf{D}[x(t)] \ \equiv \lim_{N\to\infty} \left(\frac{m}{2\pi\ i\hbar\ \Delta t}\right)^{(N-1)/2} \int\ dx_{N-1}\ \int\ dx_{N-2}\ \cdots\ \int\ dx_2 \tag{1.95}$$

and

$$\langle x_N, t_N | x_1, t_1\rangle \ = \ \int_{x_1}^{x_N} \mathbf{D}\ [x(t)] \exp\left[\frac{i}{\hbar}\ \int_{t_1}^{t_N}\ dt\ L_{classical}\ (x, \dot{x})\right]\ . \tag{1.96}$$

This expression is known as **Feyman's path integral**.

The above was not meant to be a shift derivation. The ideas borrowed from the conventional quantum mechanics are:

1. the superposition principle, used in summing up the conditions from various alternate paths
2. the composition property of the transition amplitude
3. the classical correspondence in the limit $\hbar \ \to \ 0$.

Now we need to show that Feynman's expression for $\langle x_N, t_N | x_1, t_1\rangle$ satisfies the time-dependent Schrödinger equation in the variables x_N, t_N. We show with

$$\begin{aligned}\langle x_N, t_N | x_1, t_1\rangle &= \int dx_{N-1} \langle x_N, t_N | x_{N-1}, t_{N-1}\rangle \langle x_{N-1}, t_{N-1} | x_1, t_1\rangle \\ &= \int_{-\infty}^{\infty} dx_{N-1} \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \exp\left[\frac{im}{2\hbar}\frac{(x_N - x_{N-1})^2}{\Delta t} - \frac{i}{\hbar} V \Delta t\right] \\ &\quad \langle x_{N-1}, t_{N-1} | x_1, t_1\rangle\end{aligned} \tag{1.97}$$

where $t_N - t_{N-1} = \Delta t$ is assumed to be infinitesimal. Further introduce $\xi \equiv x_N - x_{N-1}$ and let $x_N \rightarrow X$, and $t_N = t + \Delta t$. Then (1.97) becomes

$$\langle x, t + \Delta t | x_1, t_1\rangle = \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \int_{-\infty}^{\infty} d\xi \exp\left[\frac{im}{2\hbar}\frac{\xi^2}{\Delta t} - \frac{i}{\hbar} V \Delta t\right] \langle x - \xi, t | x_1, t_1\rangle \ . \tag{1.98}$$

From (1.92) it is obvious that in the lim $\delta t \rightarrow 0$ the major contributions to the integral come from the $\xi \approx 0$ region. Thus, we expand $\langle x - \xi, t | x_1, t_1\rangle$ in powers of ξ, and $\langle x, t + \Delta t | x_1, t_1\rangle$ and $exp\left[-\frac{i}{\hbar} V \Delta t\right]$ in powers of Δt.

$$\begin{aligned}&\langle x, t | x_1, t_1\rangle + \Delta t \frac{\partial}{\partial t} \langle x, t | x_1, t_1\rangle \\ &= \sqrt{\frac{m}{2\pi\ i\ \hbar\ \Delta t}} \int_{-\infty}^{\infty} d\xi \exp\left[\frac{im}{2\hbar}\ \frac{\xi^2}{\Delta t}\right] \left[\mathbf{1} - \frac{i}{\hbar} V \Delta t + \cdots\right] \\ &\times \left[\langle x, t | x_1, t_1\rangle + \left(\frac{\xi^2}{2}\right) \frac{\partial^2}{\partial \xi^2} \langle x, t | x_1, t_1\rangle + \cdots\right]\end{aligned} \tag{1.99}$$

where we dropped the term linear in ξ, since it vanishes when integrating with respect to ξ.

The first term on the right-hand side just integrates to $\langle x, t | x_1, t_1\rangle$ since the integral cancels the factor due to (1.91). In collecting the terms of first order in Δt gives

$$\Delta t \frac{\partial}{\partial t} \langle x, t | x_1, t_1\rangle = -\left(\frac{i}{\hbar}\right) \Delta t\ V \langle x, t | x_1, t_1\rangle \tag{1.100}$$

$$+ \quad \sqrt{\frac{m}{2\pi \ i \ \hbar \ \Delta t}} \ \sqrt{2\pi} \ \left(\frac{i \ \hbar \ \Delta t}{m} \right)^{3/2} \ \frac{1}{2} \ \frac{\partial^2}{\partial \xi^2} \ \langle xt | x_1, t_1 \rangle$$

where we have used

$$\int_{-\infty}^{\infty} d\xi \ \xi^2 \ \exp \ \left[\frac{im \ \xi^2}{2\hbar \ \Delta t} \right] \ = \ \sqrt{2\pi} \ \left(\frac{i\hbar \ \Delta t}{m} \right)^{3/2} \ . \tag{1.101}$$

Thus

$$i\hbar \ \frac{\partial}{\partial t} \ \langle x, t | x_1, t_1 \rangle \ = \ - \ \left(\frac{\hbar^2}{2m} \right) \ \frac{\partial^2}{\partial x^2} \ \langle x, t | x_1, t_1 \rangle \ + \ V \ \langle x, t | x_1, t_1 \rangle \tag{1.102}$$

which corresponds to the time-dependent Schrödinger equation for the quantity $\langle x, t | x_1, t_1 \rangle$

Chapter 2

Lorentz Transformations

2.1 Elementary Considerations

We assume we have two coordinate systems S and $S\prime$ with coordinates x, y, z, t and $x\prime, y\prime, z\prime, t\prime$, respectively. Physical events can be measured in either system, and the Lorentz transformation give the relation between the coordinates $x\prime, y\prime, z\prime, t\prime$ and x, y, z, t. The transformation has to fulfill the following requirements:

1. The transformation should be linear. Otherwise a specific system S, or a point in space or time would be distinct. (For a linear transformation, the inverse has the same form.)
2. Each point in $\mathbf{R}^3$ given by $x\prime, y\prime, z\prime$ in $S\prime$ moves with constant velocity $\vec{v}$ with respect to a point x, y, z in S.
3. A measurement of the speed of light should give C in both systems.

We assume the uniform motion is in $z-$direction. Then

$$x\prime = x \quad , \quad y\prime = y \; . \tag{2.1}$$

From (1) follows

$$\begin{aligned} z\prime &= a_1 z + a_2 t \\ t\prime &= b_1 t + b_2 z \end{aligned} \tag{2.2}$$