Chapter 2

Lorentz Transformations

2.1 Elementary Considerations

We assume we have two coordinate systems S and $S\prime$ with coordinates x, y, z, t and $x\prime, y\prime, z\prime, t\prime$, respectively. Physical events can be measured in either system, and the Lorentz transformation give the relation between the coordinates $x\prime, y\prime, z\prime, t\prime$ and x, y, z, t. The transformation has to fulfill the following requirements:

1. The transformation should be linear. Otherwise a specific system S, or a point in space or time would be distinct. (For a linear transformation, the inverse has the same form.)
2. Each point in $\mathbf{R}^3$ given by $x\prime, y\prime, z\prime$ in $S\prime$ moves with constant velocity $\vec{v}$ with respect to a point x, y, z in S.
3. A measurement of the speed of light should give C in both systems.

We assume the uniform motion is in $z-$direction. Then

$$x\prime \;=\; x \;\;,\quad y\prime \;=\; y \;. \tag{2.1}$$

From (1) follows

$$\begin{aligned} z\prime \;&=\; a_1 z \;+\; a_2 t \\ t\prime \;&=\; b_1 t \;+\; b_2 z \end{aligned} \tag{2.2}$$

From (2) follows, if $z\prime = 0$, then $z = vt$, thus

$$a_2 = -va_1 \tag{2.3}$$

Then 2.3 becomes

$$\begin{aligned} z\prime &= z_1 (z - vt) \\ t\prime &= b_1 t + b_2 z \end{aligned} \tag{2.4}$$

Consider the consequences of (3). A light pulse is sent at

$$x = y = z = t = 0 \quad \text{in systems } S \tag{2.5}$$

and

$$x\prime = y\prime = z\prime = t\prime = 0 \quad \text{in systems } S\prime\,, \tag{2.6}$$

where S is the rest frame and $S\prime$ some moving frame. In both systems, the speed of light is given by c, i.e.,

$$\text{rest frame } S: \quad \sqrt{x^2 + y^2 + z^2} = ct \tag{2.7}$$

$$\text{moving frame } S\prime: \quad \sqrt{x\prime^2 + y\prime^2 + z\prime^2} = ct\prime \tag{2.8}$$

From this follows

$$\begin{aligned} x\prime^2 + y\prime^2 + z\prime^2 &= x^2 + y^2 + a_1^2 (z - vt)^2 \\ &= x^2 + y^2 + a_1^2 \left(z - \frac{v}{c} \sqrt{x^2 + y^2 + z^2} \right)^2 \\ &= c^2 t\prime^2 \\ &= c^2 (b_1 t + b_2 z)^2 \\ &= c^2 \left(\frac{b_1}{c} \sqrt{x^2 + y^2 + z^2} + b_2 z \right)^2 \end{aligned} \tag{2.9}$$

This must be identical for x, y, z and $x^2 + y^2$, $z\sqrt{x^2 + y^2 + z^2}$, z^2.
Terms proportional to

$$\begin{aligned} x^2 + y^2 &: 1 - a_1^2 \frac{v^2}{c^2} = c^2 \frac{b_1^2}{c^2} \\ z\sqrt{x^2 + y^2 + z^2} &: -\frac{a_1^2}{c} = c^2 \frac{b_1 b_2}{c} \\ z^2 &: a_1^2 \left(1 + \frac{v^2}{c^2} \right) = c^2 \left(\frac{b_1^2}{c^2} + b_2^2 \right) \end{aligned} \tag{2.10}$$

These are three equations for three unknowns, which give for a_1, b_1, b_2:

$$\begin{aligned} a_1 \;=\; b_1 \;&=\; \frac{1}{\sqrt{1 \,-\, \frac{v^2}{c^2}}} \\ b_2 \;&=\; -\,\frac{v}{c^2}\, a_1 \end{aligned} \tag{2.11}$$

Thus the Lorentz transformations become

$$\begin{aligned} z\prime \;&=\; \frac{z \,-\, vt}{\sqrt{1 \,-\, \frac{v^2}{c^2}}} \\ t\prime \;&=\; \frac{t \,-\, \frac{v}{c^2 z}}{\sqrt{1 \,-\, \frac{v^2}{c^2}}} \end{aligned} \tag{2.12}$$

The inverse transformations are obtained by replacing v with $-v$:

$$\begin{aligned} z \;&=\; \frac{z\prime \,+\, vt\prime}{\sqrt{1 \,-\, \frac{v^2}{c^2}}} \\ t \;&=\; \frac{t\prime \,+\, \frac{v}{c^2}\, z\prime}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned} \tag{2.13}$$

2.2 Lorentz Transformations as Orthogonal Transformations in Four Dimensions

A Lorentz transformation leaves the expression

$$x_1^2 \,+\, x_2^2 \,+\, x_3^2 \,-\, c^2 t^2 \tag{2.14}$$

invariant. Proof by inserting

$$\begin{aligned} x\prime_1^2 \,+\, x\prime_2^2 \,+\, x\prime_3^2 \,-\, c^2 t\prime^2 \;&=\; x_1^2 \,+\, x_2^2 \,+\, \frac{(x_3 \,-\, vt)^2}{1 \,-\, \frac{v^2}{c^2}} \,-\, c^2\, \frac{\left(t \,-\, \frac{v}{c^2}\, x_2\right)^2}{1 \,-\, \frac{v^2}{c^2}} \\ &=\; x_1^2 \,+\, x_2^2 \,+\, x_3^2 \,-\, c^2 t^2 \end{aligned} \tag{2.15}$$

Defining $x_4 \ : \ ict$, the invariant quantity can be written as

$$\sum_{\alpha \ = \ 1}^{4} \ x_\alpha^2 \ = \ x_\alpha \ x^\alpha \ = \ x_\alpha\prime \ x\prime^\alpha \tag{2.16}$$

This means that the length of a vector x_α in $\mathbf{R}^4$ remains unchanged under Lorentz transformations. Thus, a Lorentz transformation can be interpreted as rotation in a four-dimensional space,

$$x_\mu\prime \ = \ a_{\mu\nu} \ x^\nu \tag{2.17}$$

with $a_{\mu\nu} \ a^{\mu\lambda} \ = \ \delta_\nu^\lambda$. For the special case that $S\prime$ moves with v in x_3-direction, one has

$$a_{\mu\nu}(\beta) \ = \ a_{\mu\nu} \ (-\beta) \ = \ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{pmatrix} \tag{2.18}$$

where $\beta \ \equiv \ \frac{v}{c}$ and $\gamma \ = \ \frac{1}{\sqrt{1 \ - \ \beta^2}}$.

Let us consider vectors and tensors in a four-dimensional space. A_μ is a four-dimensional vector, if its components transform as the coordinates of a four-dimensional vector space, i.e.,

$$A_\mu\prime \ = \ a_{\mu\nu} \ A^\nu \ . \tag{2.19}$$

A tensor of rank 2 transforms as

$$A_{\mu\nu}\prime \ = \ a_{\mu\sigma} \ a_{\nu\lambda} \ A^{\sigma\lambda} \ . \tag{2.20}$$

Those vectors have a special meaning. Since according to Einstein's principle of relativity all systems moving with a uniform velocity are equivalent, the laws of physics must obey the same equations in e.g., system S and system $S\prime$. Compare rotations in three dimensions. The laws of physics are independent from the position of one coordinate system with respect to the other. Physical quantities are described by scalars, vectors and tensors, and the physical laws are given by combination of those quantities. Since the physics is independent from the position of the coordinate system, the form of the equation is the same in each coordinate system.

Since the laws of physics can be described in each moving frame, and according to special relativity all moving frame are equivalent, the laws of physics must be described by

equations which do not change when going from one frame to another. Thus, they must be expressed as scalars, vectors and tensors in a four-dimensional space. (The Maxwell equations are Lorentz invariant. Newtonian mechanics not.)

The quantity $A_\mu\ B^\mu$ is a **four-scalar**. If $A_\mu\ A^\mu\ =\ 0$ in a Minkovski metric (imaginary components), then this does not imply that all components are zero. A vector is space-like if $A_\mu\ A^\mu\ >\ 0$, and time-like if $A_\mu\ A^\mu < 0$.

For the vector product in three dimension, the totally antisymmetric ε tensor

$$\varepsilon_{ijk}\ =\ \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \tag{2.21}$$

with

$$\varepsilon_{ijk}\ \varepsilon_{lmk}\ =\ \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix}\ =\ \delta_{il}\ \delta_{jm}\ -\ \delta_{jl}\ \delta_{im} \tag{2.22}$$

played a special role. Analogously we introduce a four-dimensional totally antisymmetric tensor

$$\varepsilon_{\alpha\beta\gamma\tau}\ =\ \begin{vmatrix} \delta_{\alpha1} & \delta_{\alpha2} & \delta_{\alpha3} & \delta_{\alpha4} \\ \delta_{\beta1} & \delta_{\beta2} & \delta_{\beta3} & \delta_{\beta4} \\ \delta_{\gamma1} & \delta_{\gamma2} & \delta_{\gamma3} & \delta_{\gamma4} \\ \delta_{\tau1} & \delta_{\tau2} & \delta_{\tau3} & \delta_{\tau4} \end{vmatrix} \tag{2.23}$$

with

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma\tau}\ \varepsilon_{\nu\lambda\sigma\tau}\ &=\ \begin{vmatrix} \delta_{\alpha\nu} & \delta_{\alpha\lambda} & \delta_{\alpha\sigma} \\ \delta_{\beta\nu} & \delta_{\beta\lambda} & \delta_{\beta\sigma} \\ \delta_{\gamma\nu} & \delta_{\gamma\lambda} & \delta_{\gamma\sigma} \end{vmatrix} \\ &=\ \delta_{\alpha\nu}\ \delta_{\beta\lambda}\ \delta_{\gamma\sigma}\ +\ \delta_{\alpha\lambda}\ \delta_{\beta\sigma}\ \delta_{\gamma\nu}\ +\ \delta_{\alpha\sigma}\ \delta_{\beta\nu}\ \delta_{\gamma\lambda} \\ &-\ \delta_{\alpha\nu}\ \delta_{\beta\sigma}\ \delta_{\gamma\lambda}\ -\ \delta_{\alpha\lambda}\ \delta_{\beta\nu}\ \delta_{\gamma\sigma}\ -\ \delta_{\alpha\sigma}\ \delta_{\beta\lambda}\ \delta_{\gamma\nu} \\ \varepsilon_{\alpha\beta\gamma\mu}\ \varepsilon_{\nu\lambda\gamma\mu}\ &=\ 2\ \delta_{\alpha\gamma}\ \delta_{\beta\lambda}\ -\ 2\ \delta_{\alpha\lambda}\ \delta_{\beta\gamma} \end{aligned} \tag{2.24}$$

Now we can define a '*vector product*'

$$C_{\alpha\beta}\ =\ \varepsilon_{\alpha\beta\mu\nu}\ A_\mu\ B_\nu \tag{2.25}$$

which is an antisymmetric tensor of rank 2 with six independent components. Every antisymmetric tensor of rank 2 can be associated with a dual tenor

$$\tilde{C}_{\alpha\beta} \;=\; \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma\tau}\,C_{\gamma\tau} \tag{2.26}$$

with $\tilde{\tilde{C}}_{\alpha\beta} \;=\; C_{\alpha\beta}$. If $C_{\alpha\beta}$ stands for a vector product, then

$$\tilde{C}_{\alpha\beta} \;=\; A_\alpha\,B_\beta \;-\; A_\beta\,B_\alpha\;. \tag{2.27}$$

Most operations can be carried through from three to four dimensions

$$\begin{aligned}
\frac{\partial\phi}{\partial x_\mu} \;&\equiv\; \text{gradient} \;\equiv\; \text{vector} \\
\frac{\partial A_\mu}{\partial x_\mu} \;&\equiv\; \text{gradient} \;=\; \text{scalar} \\
R_{\alpha\beta} \;&=\; \varepsilon_{\alpha\beta\gamma\tau}\,\frac{\partial A_\tau}{\partial x_\gamma} \;=\; \text{antisymmetric tensor of rank 2} \\
\tilde{R}_{\alpha\beta} \;&=\; \frac{\partial A_\beta}{\partial x_\alpha} \;-\; \frac{\partial A_\alpha}{\partial x_\beta}
\end{aligned} \tag{2.28}$$

2.3 Electrodynamics in Four Vectors

From the vector potential $\vec{A}$ and the scalar potential ϕ, one can construct a four vector $A_\mu = (A_1, A_2, A_3, i\phi)$ with

$$\frac{\partial A_\mu}{\partial x^\mu} = \vec{\nabla}\ \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0\ , \tag{2.29}$$

which states that the divergence of the four potential to zero (Lorentz condition). Similarly one introduces a four current $J_\mu = (J_1, J_2, J_3, ic\ \rho)$, and the continuity equation becomes

$$\frac{\partial J_\mu}{\partial x^\mu} = \vec{\nabla}\ \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0\ . \tag{2.30}$$

The wave equation can be written as

$$\Box A_\mu = -\frac{4\pi}{c} J_\mu\ , \tag{2.31}$$

where

$$\Box := \frac{\partial^2}{\partial x_\mu\ \partial x^\mu} = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\ . \tag{2.32}$$

This equation is identical with the two equations

$$\begin{aligned} \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\ \vec{A} &= -\frac{4\pi}{c}\ \vec{J} \\ \left(\Delta - \frac{1}{c^c}\frac{\partial^2}{\partial t^2}\right)\ \phi &= -4\pi\ \rho \end{aligned} \tag{2.33}$$

We now define a totally antisymmetric tensor of rank 2

$$F_{\mu\nu} := \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{2.34}$$

and the corresponding dual tensor

$$\tilde{F}_{\mu\nu} \;=\; \frac{1}{2}\,\varepsilon_{\mu\nu\lambda\sigma}\,F_{\lambda\sigma} \;=\; \varepsilon_{\mu\nu\lambda\sigma}\,\frac{\partial A_\sigma}{\partial x_\lambda} \tag{2.35}$$

Explicitly $F_{\mu\nu}$ is given by

$$\begin{aligned} F_{\mu\nu} \;&=\; \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \;=\; \begin{pmatrix} \varepsilon_{ijk}\,B_k & iE_i \\ iE_i & 0 \end{pmatrix} \\ \tilde{F}_{\mu\nu} \;&=\; \begin{pmatrix} 0 & -iE_3 & iE_2 & B_1 \\ iE_3 & 0 & -iE_1 & B_2 \\ -iE_2 & iE_1 & 0 & B_3 \\ -B_1 & -B_2 & -B^3 & 0 \end{pmatrix} \;=\; \begin{pmatrix} -i\varepsilon_{ijk}\,E_k & B_i \\ -B_i & 0 \end{pmatrix} \end{aligned} \tag{2.36}$$

These tensors allow to write the Maxwell equations in a very compact fashion

$$\begin{aligned} \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\nu} \;&=\; 0 \quad \Leftrightarrow \quad \nabla \,\times\, \vec{E} \,+\, \frac{1}{c}\,\frac{\partial B}{\partial t} \;=\; 0\;; \quad \nabla\,\vec{B} \;=\; 0 \\ \frac{\partial F_{\mu\nu}}{\partial x_\nu} \;&=\; 0 \quad \Leftrightarrow \quad \nabla \,\times\, \vec{B} \,-\, \frac{1}{c}\,\frac{\partial E}{\partial t} \;=\; 0\;; \quad \nabla\,\vec{E} \;=\; 4\pi\,\rho \end{aligned} \tag{2.37}$$

2.4 Lorentz Transformations in a Four Vector Notation

Aside of setting $x_4 = ict$, it is also customary to use $\vec{x} = (x_1, x_2, x_3)$ and $x^0 = x_1 = ct$. If one considers $x_i = -x^i,\ i = 1, 2, 3$, then the invariant line element $c^2t^2 - x_1^2 - x_2^2 - x_3^2$ can be written as $x^\nu x_\nu = c^2t^2 - \vec{x}^2$.

$x_\nu := (x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3)$ is the covariant four vector,
$x^\nu := (x^0, x^1, x^2, x^3)$ is the contravariant four vector.

The invariant line elements is written as

$$d\sigma^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{2.38}$$

where $g_{\mu\nu}$ is the covariant metric tensor with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.39}$$

In special relativity the metric tensor is constant. The contravariant metric tensor $g^{\mu\nu}$ is given by the relation

$$g^{\mu\nu}\, g_{\mu\alpha} = \delta^\nu_\alpha \tag{2.40}$$

which is equivalent to $g^{\mu\nu} = g_{\mu\nu}$. Thus one has

$$x_\nu = g_{\mu\nu}\, x^\mu \tag{2.41}$$
$$x^\nu = g^{\mu\nu}\, x_\mu \tag{2.42}$$
$$\tag{2.43}$$

Scalars, vectors and tensors are given by their transformation properties. An arbitrary tensor of higher rank may have covariant as well as contravariant components, e.g.,

$$A^{\sigma\lambda}_\mu = g_{\mu\nu}\, A^{\sigma\lambda\nu} . \tag{2.44}$$

2.5 Elements of Special Relativity

Explicitly a Lorentz transformation is given as

$$x'^{\mu} \;=\; \Lambda^{\mu}_{\nu}\; x^{\nu} \;, \tag{2.45}$$

which leaves the scalar product

$$x^2 \;=\; (x^0)^2 \;-\; \vec{x}^{\;2} \tag{2.46}$$

invariant. From the invariance of the scalar product, i.e., $x'^2 \;=\; x^2$ follows

$$\Lambda^T \; g \; \Lambda \;=\; g \;. \tag{2.47}$$

This condition is fulfilled by matrices of the form

$$\Lambda \;=\; \begin{pmatrix} \cosh\,\chi & -\sinh\,\chi & 0 & 0 \\ -\sinh\,\chi & \cosh\,\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.48}$$

The parameter χ is called rapidity and depends on the velocity $\vec{v}$. Explicitly

$$\tanh\,\chi \;=\; \frac{v}{c} \;. \tag{2.49}$$

and

$$\begin{aligned} \cosh\,\chi \;&=\; \frac{1}{\sqrt{1 \;-\; (\frac{v}{c})^2}} \;=\; \gamma \\ \sinh\,\chi \;&=\; \frac{v}{c}\,\frac{1}{\sqrt{1 \;-\; (\frac{v}{c})^2}} \;=\; \beta\gamma \end{aligned} \tag{2.50}$$

If one adds to the Lorentz transformations (2.45) translations in space and time

$$x'^{\mu} \;=\; x^{\mu} \;+\; a^{\mu} \tag{2.51}$$

where

$$a \;=\; \begin{pmatrix} a^0 \\ \vec{a} \end{pmatrix} \tag{2.52}$$

is an arbitrary four-vector, one obtains the symmetry group which characterizes the theory of special relativity:

$$x' \;=\; \Lambda x \;+\; a \quad \text{with} \quad \Lambda^T \; g\Lambda \;=\; g \;. \tag{2.53}$$

This group is called **Poincaré** or **inhomogeneous Lorentz group**. A general approach to relativistic quantum mechanics can be designed the following way: According to the general principles of quantum mechanics, the Poincaré group has to be represented in the Hilbert space of quantum states as unitary (or anti-unitary) operators:

$$(\Lambda, a) \longrightarrow U(\Lambda, a) \; . \tag{2.54}$$

The problem is then to determine the irreducible (unitary) representations of the Poincaré group. This problem was solved in 1939 by Wigner. He obtained the result:

For every real number $m \geq 0$ and each $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ exists an irreducible representation of the Poincaré group characterized by (m, j).

The "quantum numbers" m and j can here be associated with the mass and spin of a corresponding particle (particle physics).

A more elementary (and historical) approach is to look for a Schrödinger equation, which fulfills Lorentz covariance. Obviously, one cannot start from the well-known expressions

$$i\hbar \, \frac{d}{dt} \; | \, \psi_s(t) \rangle \; = \; H(\vec{P}, \vec{Q}) \; | \, \psi_s(t) \rangle \tag{2.55}$$

or

$$\frac{d}{dt} \, A_H(t) \; = \; \frac{i}{\hbar} \, [H(\vec{P}, \vec{Q}), \; A_h(t)] \; , \tag{2.56}$$

since both equations contain time t and space $\vec{Q}$ in an asymmetrical fashion, namely t as c-number and $\vec{Q}$ as hermitian operator. A symmetric form is only achieved if either t becomes an operator or Q a c-number vector. In the first case, one would have to introduce in analogy to

$$[P_j, Q_k] \; = \; \frac{\hbar}{i} \, \delta_{jk} \tag{2.57}$$

the commutator

$$[H, t] \; = \; \frac{\hbar}{i} \; . \tag{2.58}$$

However, the latter contradicts the physical requirement that the spectrum of H has to be bounded from below. Thus, we choose the second possibility and work exclusively in the coordinate representation, where the operator $\vec{Q}$ appears only as c-number vector $\vec{r}$, and the wave function is represented as

$$\psi(x) \; \equiv \; \psi(x^0, x^1, x^2, x^3) \; = \; \psi(ct, \vec{x}) \; . \tag{2.59}$$

Here space and time enter on the same footing.

The different equation for $\psi(x)$, which has to correspond to (2.55), has to contain the relativistic relation between energy and momentum, i.e., for a particle moving without the influence of a force, the relation

$$E \;=\; c\,\sqrt{\vec{p}^{\,2} \;+\; (mc)^2} \tag{2.60}$$

has to be valid. Energy and momentum are in quantum mechanics replaced as

$$\begin{aligned} E \;&\longrightarrow\; i\hbar\,\frac{\partial}{\partial t} \\ \vec{p} \;&\longrightarrow\; \frac{\hbar}{i}\,\vec{\nabla}_{\vec{r}}\;. \end{aligned} \tag{2.61}$$

The above relations correspond to

$$\begin{aligned} E \;&=\; \hbar\omega \\ \vec{p} \;&=\; \hbar\vec{k} \end{aligned} \tag{2.62}$$

which reflect the translational invariance with respect to space and time, and which shall stay valid in a relativistic quantum mechanics. In order to bring the relations (2.61) into a formal covariant form, dimensions and signs have to be considered:

$$\begin{aligned} \frac{1}{c}\,E \;&:=\; p_0 \;\longrightarrow\; i\hbar\,\frac{\partial}{\partial(ct)} \;=\; i\hbar\,\frac{\partial}{\partial x^0} \\ -(\vec{p})_k \;&:=\; p_k \;\longrightarrow\; -\,\frac{\hbar}{i}\,(\vec{\nabla}_{\vec{r}})_k \;=\; i\hbar\,\frac{\partial}{\partial x^k} \quad (k=1,2,3) \end{aligned} \tag{2.63}$$

As a Reminder: The derivative with respect to the contravariant components form the covariant components of a four-vector. Consider $\frac{\partial}{\partial x^\mu}$ applied to a scalar function of $x^2 \;=\; x^\nu\,x_\nu$

$$\frac{\partial}{\partial x^\mu}\,f(x^2) \;=\; f'(x^2)\,\frac{\partial}{\partial x^\mu}\,(x^\nu x_\nu) \;=\; f'(x^2)\,2x_\mu\;. \tag{2.64}$$

Thus, $\frac{\partial}{\partial x^\mu} \;:=\; \partial_\mu\;;\;\mu \;=\; 0,1,2,3$ and

$$p_\mu \;\longrightarrow\; i\hbar\,\partial_\mu\;, \tag{2.65}$$

which shows that p_μ transforms as a covariant vector. Inserting (2.63) into (2.60) leads to

$$\begin{aligned} \frac{1}{c}\,E \;-\; \sqrt{\vec{p}^{\,2} \;+\; (mc)^2} \;&\longrightarrow\; i\hbar\,\partial_0 \;-\; \sqrt{-\hbar^2\,\sum_k\,\partial_k^2 \;+\; (mc)^2} \\ &=\; i\hbar\,\partial_0 \;-\; \sqrt{-\hbar^2\,\Delta \;+\; (mc)^2}\;. \end{aligned} \tag{2.66}$$

The last expression is unfortunately again asymmetric with respect to the differentiation with respect to time and spatial coordinates. This asymmetry can be avoided when starting from the squared energy momentum relation

$$\left(\frac{1}{c}\,E\right)^2 \;=\; \vec{p}^{\,2} \;+\; (mc)^2 \tag{2.67}$$

in the form

$$(mc)^2 \;=\; \left(\frac{1}{c}\,E\right)^2 \;-\; \vec{p}^{\,2} \;=\; p_0^2 \;-\; \vec{p}^{\,2} \;=\; p_\mu p^\mu \;. \tag{2.68}$$

One obtains:

$$\begin{aligned} p_\mu p^\mu \;-\; (mc)^2 \;&\longrightarrow\; (i\hbar)^2\,\partial_\mu\partial^\mu \;-\; (mc)^2 \\ &=\; -\hbar^2\,\left[\partial_\mu\partial^\mu \;+\; \left(\frac{mc}{\hbar}\right)^2\right] \end{aligned} \tag{2.69}$$

Here the derivatives with respect to time and space coordinates are symmetric, and one defines

$$\Box \;:=\; \partial_\mu\partial^\mu \;=\; \frac{1}{c^2}\,\frac{\partial}{\partial t^2} \;-\; \Delta \;. \tag{2.70}$$

According to the previous considerations, the relativistic relation between energy and momentum is fulfilled, if the wave function $\psi(x)$ obeys the differential equation

$$\left(\Box \;+\; \left(\frac{mc}{\hbar}\right)^2\right)\;\psi(x) \;=\; 0 \;, \tag{2.71}$$

which is the so-called **Klein-Gordon equation**.

Remark on **Natural Units**:

In principle, one would like to have fundamental units for space, time and mass, which are derived from the fundamental laws of physics.

elementary length ℓ_0
elementary time t_0
elementary mass m_0

A modern candidate for ℓ_0 is the Planck length

$$\ell_{Planck} \;=\; \sqrt{\frac{g\hbar}{c^2}} \;=\; 1.6 \;\cdot\; 10^{-33}\;cm \tag{2.72}$$

(g: gravitational constant)

Even if ℓ_0 is not explicitly fixed, one can use c and $\hbar$ to fix units for t_0 and m_0:

$t_0 \;=\; \frac{\ell_0}{c} \;:=\;$ elementary time $\;=\;$ time, which light needs to pass ℓ_0.

$m_0 \;=\; \frac{\hbar}{\ell_0 c} \;:=\;$ elementary mass from the Compton relation.

If one chooses ℓ_0, t_0 and m_0 as elementary units, then:

The speed of light has the numerical value 1 : $c = 1$.

The Planck constant $\hbar$ has the numerical value 1 : $\hbar \;=\; 1$.

Starting from these natural units, all other units can be related to the unit of length, e.g., cm. Some important relations acquire the form

$$\begin{aligned} E &= \sqrt{p^2 \;+\; m^2} \\ E &= \omega \\ \vec{p} &= \vec{k} \\ \alpha &= e^2 \\ \vec{\mu} &= \frac{e}{2m}\; g_s\; \vec{S} \quad \text{(magnetic moment)} \\ (\Box \;+\; m^2)\; \psi(x) &= 0 \quad \text{(Klein} - \text{Gordon equation)} \end{aligned} \tag{2.73}$$