Chapter 4

The Dirac Equation

4.1 Towards the Dirac Equation

Dirac tried to find a partial differential equation with a positive probability density. He realized that the problem in (3.13) resulted from the fact that one started from the non-relativistic definition $\vec{j}_{NR}$ in (3.9). This equation cannot determine a definite sign. For solving the problem, one could start from ρ_{NR}, but then one would need a different equation of 1st order, since the possibility of defining $\rho = \psi^*\psi$ is related to the fact that the Schrödinger equation contains only a first derivative with respect to the time. Thus, one faces the following problem; that one needs a differential equation (1st order) for $\psi(x)$, which leads to the correct energy-momentum relation

$$p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2 \,. \tag{4.1}$$

This problem cannot be solved for a scalar function ψ, and the crucial point of Dirac's idea was to introduce a "vector" quantity

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \cdot \\ \cdot \\ \psi_n x \end{pmatrix} \tag{4.2}$$

and thus a positive definite density

$$\rho(x) \ := \ \sum_{\alpha=1}^{n} \ \psi_\alpha^*(x) \ \psi_\alpha(x) \ . \tag{4.3}$$

The most general form of a first-order differential equation for this $\psi(x)$ is given by

$$\sum_\beta \ [i\gamma^\mu_{\alpha\beta} \ \partial_\mu \ - \ m\Gamma_{\alpha\beta}] \ \psi_\beta(x) \ = \ 0 \tag{4.4}$$

for $\alpha \ = \ 1, 2, \cdots, n$ (notation consistent with the one of Bjorken/Drell).

Here $\gamma^\mu_{\alpha\beta}$ and $\Gamma_{\alpha\beta}(\mu \ = \ 0, 1, 2, 3 \ ; \ \alpha, \beta \ = \ 1, 2, \cdots, n)$ are coefficients, which we assume to be complex constants, since we do not consider forces. Note that $i\partial_\mu$ is the operator for the four-momentum. With the matrix notation $\gamma^\mu \ := \ (\gamma^\mu_{\alpha\beta})$ and $\Gamma \ := \ (\Gamma_{\alpha\beta})$, Eq. (4.4) can be written as

$$(i\gamma^\mu \ \partial_\mu \ - \ m\Gamma) \ \psi(x) \ = \ 0 \tag{4.5}$$

so that energy and momentum fulfill the relation (2.67). Dirac required that each component $\psi_\alpha(x)$ of the wave function separately fulfills the Klein-Gordon equation (2.71):

$$(\Box \ + \ m^2) \ \psi_\alpha(x) \ = \ 0 \tag{4.6}$$

for $\alpha \ = \ 1, 2, \cdots, n$. The requirement is compatible with (4.5) and leads to relations for the γ^μ and Γ.

To illustrate this further, we want to mention that a similar formal relation occurs when considering electromagnetic waves. There are free Maxwell equations

$$\begin{aligned} div \ \vec{E} \ &= \ 0 \qquad rot \ \vec{B} \ - \ \frac{1}{c} \ \dot{\vec{E}} \ = \ \vec{0} \\ div \vec{B} \ &= \ 0 \qquad rot \ \vec{E} \ + \ \frac{1}{c} \ \dot{\vec{B}} \ = \ 0 \end{aligned} \tag{4.7}$$

represent a first-order differential equation for $\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$. It can be recast into the form (4.4), where the $\gamma^\mu_{\alpha\beta}$ turn to six-dimensional matrices, and $\Gamma_{\alpha\beta}$ vanish. On the other hand follows from Maxwell's equations

$$\Box \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = 0 \,, \tag{4.8}$$

i.e., the same wave equation is valid for each component. We proceeded the other way. Starting from an equation

$$(\Box + m^2)\mathbf{1}\, \psi(x) = 0 \tag{4.9}$$

for the components, we want to determine the matrices γ^μ and γ. For this we apply onto (4.5) the 'conjugate' operator $(-i\gamma^\nu\, \partial_\gamma - m\Gamma)$:

$$\begin{aligned} (-i\gamma^\nu \partial_\nu - m\Gamma)(i\gamma^\mu\, \partial_\mu - m\Gamma)\, \psi &= 0 \\ \left[\gamma^\nu \gamma^\mu\, \partial^2_{\nu\mu} + i\, m(\gamma^\nu \Gamma - \Gamma\gamma^\nu)\, \partial_\nu + m^2\, \Gamma^{\,2}\right] \psi &= 0 \end{aligned} \tag{4.10}$$

where

$$\partial^2_{\nu\mu} := \partial_\nu \partial_\mu = \frac{\partial^2}{\partial x^\nu\, \partial x^\mu} \tag{4.11}$$

and ν being the common summation index. A comparison with (4.6) gives

$$\begin{aligned} \gamma^\nu \gamma^\nu\, \partial^2_{\nu\mu} &= \Box\mathbf{1} \\ \gamma^\nu\, \Gamma - \Gamma\gamma^\nu &= 0 \\ \Gamma^2 &= \mathbf{1} \end{aligned} \tag{4.12}$$

These equations determine five matrices, $\gamma^0, \gamma^1, \gamma^2, \gamma^3, \Gamma$ uniquely up to similarity transformations.

Together with γ^μ, Γ the quantities

$$\begin{aligned} \hat{\gamma}^\mu &= S\gamma^\mu\, S^{-1} \\ \hat{\Gamma} &= S\Gamma\, S^{-1} \end{aligned} \tag{4.13}$$

fulfill equations (4.12), where S is an arbitrary, non-singular matrix. Next, one has to explicitly construct the γ-matrices. First, from (4.12) follows that Γ is a non-singular matrix:

$$\Gamma^{-1} = \Gamma \,, \tag{4.14}$$

thus, multiplying (4.5) with Γ^{-1} gives

$$(i\tilde{\gamma}^\mu \, \partial_\mu \; - \; m\mathbf{1}) \; \psi(x) \; = \; 0 \; . \tag{4.15}$$

Without loosing generality, we can choose $\Gamma \; = \; \mathbf{1}$, and thus the second equation (4.12) is fulfilled.

Thus, the Dirac equation takes the form

$$(i\gamma^\mu \, \partial_\mu \; - \; m) \; \psi(x) \; = \; 0 \tag{4.16}$$

or, when introducing $\hbar$ or c explicitly

$$(i\hbar \; \gamma^\mu \, \partial_\mu \; - \; mc) \; \psi(x) \; = \; 0 \; . \tag{4.17}$$

The latter can be verified by dimensional considerations:

According to the first Eq. (4.12) the γ-matrices are dimensionless. $i\hbar \; \partial_\mu$ stands for the four-momentum, and thus the second term in (4.17) has to contain the momentum mc.

Considering the first Eq. (4.12) shows that due to the symmetry $\partial^2_{\mu\nu} \; = \; \partial^2_{\nu\mu}$

$$\gamma^\mu \gamma^\nu \; \partial^2_{\mu\nu} \; = \; \frac{1}{2} \, (\gamma^\mu \gamma^\nu \; + \; \gamma^\nu \gamma^\mu) \; \partial^2_{\mu\nu} \; . \tag{4.18}$$

On the other hand,

$$\Box \; = \; \partial^\nu \partial_\nu \; = \; g^{\mu\nu} \; \partial^2_{\mu\nu} \; , \tag{4.19}$$

from which follows

$$\{\gamma^\mu, \gamma^\nu\} \; := \; \gamma^\mu \gamma^\nu \; + \; \gamma^\nu \gamma^\mu \; = \; 2g^{\mu\nu} \; . \tag{4.20}$$

These equations define the **Dirac γ-matrices**. The Dirac problem is solved as soon as we find matrices fulfilling (4.20).

In Summary: With the help of the γ-matrices, it is possible to explicitly take the square root of the d'Alembert operator:

$$\sqrt{-\Box} \; = \; i\gamma^\mu \; \partial_\mu \; . \tag{4.21}$$

In fact:

$$\begin{aligned} (i\gamma^\mu \; \partial_\mu)^2 \; &= \; -\gamma^\mu\gamma^\nu \; \partial^2_{\mu\nu} \\ &= \; -\; \frac{1}{2} \; (\gamma^\mu\gamma^\nu \; + \; \gamma^\nu\gamma^\mu) \; \partial^2_{\mu\nu} \\ &= \; -g^{\mu\nu} \; \partial^2_{\mu\nu} \; = \; -\Box \; . \end{aligned} \tag{4.22}$$

This last calculation clearly stresses the importance of the anti-commutation relation (4.20).

4.2 Properties of the γ-Matrices

All physically relevant information about the γ-matrices are given in the anti-commutation relation (4.20). For $\mu \; \neq \; \nu$ follows

$$\gamma^\mu\gamma^\nu \; = \; -\gamma^\nu\gamma^\mu \tag{4.23}$$

$\mu \; = \; \nu \; = \; 0$:

$$(\gamma^0)^2 \; = \; \mathbf{1} \tag{4.24}$$

$\mu \; = \; \nu \; = \; k \; , \quad k \; = \; 1, 2, 3$

$$(\gamma^k)^2 \; = \; -\mathbf{1} \; . \tag{4.25}$$

From (4.23) follows that the dimension n of the matrix has to be larger than 1. If $n \; = \; 2$, then (4.20) reminds of the commutation relations of the Pauli matrix σ_k

$$\{\sigma_k, \sigma_\ell\} \; = \; 2\delta_{k\ell} \; \mathbf{1} \tag{4.26}$$

and one can show that

$$\gamma^k \; := \; i\sigma_k \tag{4.27}$$

fulfills (2.51) for $k \; = \; 1, 2, 3$. Since every 2×2 matrix can be written as linear combination of σ_k and the unit matrix, we start with

$$\gamma^0 \;=\; a\mathbf{1} \;+\; \sum_\ell \; b_\ell \sigma_\ell \;. \tag{4.28}$$

Using (4.27) and (4.26):

$$\begin{aligned} \{\gamma^0, \gamma^k\} \;&=\; ia\{\mathbf{1}, \sigma_k\} \;+\; i \sum_\ell \; b_\ell \; \{\sigma_\ell, \sigma_k\} \\ &=\; 2i(a\sigma_k \;+\; b_k \mathbf{1}) \\ &=\; 0 \end{aligned} \tag{4.29}$$

from which follows $a \;=\; b_\ell \;=\; 0$, thus $\gamma^0 \;=\; 0$, since σ_k and $\mathbf{1}$ are linearly independent. In this case (4.24) is not fulfilled, thus the γ-matrices cannot be two dimensional.

Instead of discussing $n \;=\; 3$ we want to show that n has to be an even number. We define

$$\gamma^5 \;:=\; i\; \gamma^0\gamma^1\gamma^2\gamma^3 \;. \tag{4.30}$$

γ^5 anti-commutes with all other γ-matrices:

$$\{\gamma^5, \gamma^\mu\} \;=\; 0 \;. \tag{4.31}$$

For the proof, (4.23) is applied several times, i.e.,

$$\begin{aligned} \gamma^5\gamma^0 \;&=\; i\; \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 \\ &=\; -i\; \gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 \\ &=\; +i\; \gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 \\ &=\; -i\; \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 \\ &=\; -\gamma^0\gamma^5 \end{aligned} \tag{4.32}$$

In addition, one can show that

$$(\gamma^5)^2 \;=\; \mathbf{1} \;. \tag{4.33}$$

Finally, one obtained from (4.31) and (4.33)

$$\gamma^\mu \;=\; -\gamma^5\gamma^\mu\gamma^5 \tag{4.34}$$

and

$$\begin{aligned} tr(\gamma^\mu) \;&=\; -tr\; (\gamma^5\gamma^\mu\gamma^5) \\ &=\; -tr\; ((\gamma^5)^2\gamma^\mu) \\ &=\; -tr\; (\gamma^\mu) \end{aligned} \tag{4.35}$$

from which follows

$$tr(\gamma^\mu) \;=\; 0 \; . \tag{4.36}$$

From (4.36) follows: n has to be even.

γ^0 can be chosen as hermitian matrix, thus has only ± 1 as eigenvalues (see (4.24)). After diagonalization, one obtains the following structure

$$\gamma^0 \;=\; \left(\begin{array}{ccccc} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & -1 & \\ 0 & & & & -1 \\ & n_+ & & & n_- \end{array} \right) \tag{4.37}$$

with $n \;=\; n_+ \,+\, n_-$. Because of (4.36) one has in addition the condition $n_+ \,-\, n_- \;=\; 0$, thus $n \;=\; 2n_+ \;=\; 2n_-$.

Now we consider $n \;=\; 4$ for the γ-matrices, i.e., we consider matrices

$$\left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \; , \tag{4.38}$$

where A, B, C, D are 2×2 matrices. We will show that the standard representation of the γ-matrices fulfills (4.20):

$$\begin{aligned} \gamma^0 \;&=\; \left(\begin{array}{cc} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{array} \right) \\ \gamma^k \;&=\; \left(\begin{array}{cc} 0 & \sigma_k \\ -\sigma_k & 0 \end{array} \right) \end{aligned} \tag{4.39}$$

where σ_k are the usual Pauli matrices

$$\sigma_1 \;=\; \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \;\; ; \;\; \sigma_2 \;=\; \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) \;\; ; \;\; \sigma_3 \;=\; \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \; . \tag{4.40}$$

One has, e.g., then

$$\gamma^k \gamma^\ell \;=\; - \left(\begin{array}{cc} \sigma_k \sigma_\ell & 0 \\ 0 & \sigma_k \sigma_\ell \end{array} \right) \tag{4.41}$$

and thus

$$\{\gamma^k, \gamma^\ell\} \;=\; -\begin{pmatrix} 2\delta_{k\ell}\mathbf{1} & 0 \\ 0 & 2\delta_{k\ell}\mathbf{1} \end{pmatrix} \;=\; -2\delta_{k\ell}\mathbf{1} \tag{4.42}$$

which corresponds to (4.20) for $\mu \;=\; k$ and $\nu \;=\; \ell$. But now γ^0 also fulfills the relations:

$$\begin{aligned} \gamma^0\gamma^k \;&=\; \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \;=\; -\gamma^k\gamma^0 \\ (\gamma^0)^2 \;&=\; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & (-\mathbf{1})^2 \end{pmatrix} \;=\; \mathbf{1} \end{aligned} \tag{4.43}$$

In addition, one has because of $\sigma_k^\dagger \;=\; \sigma_k$

$$\begin{aligned} (\gamma^0)^\dagger \;&=\; \gamma^0 \\ (\gamma^k)^\dagger \;&=\; -\gamma^k \,. \end{aligned} \tag{4.44}$$

Thus, the γ-matrices are anti-hermitian. For γ^5 one obtains

$$\begin{aligned} \gamma^5 \;&=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \\ &=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{aligned} \tag{4.45}$$

As mentioned earlier, the representation of the γ-matrices is not unique. A different set of important γ-matrices, which is used in the so-called spinor analysis is given by

$$\gamma^5 \;=\; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{4.46}$$

$$\gamma^0 \;=\; \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \tag{4.47}$$

$$\gamma^k \;=\; \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \,. \tag{4.48}$$

The two representations are connected through the map

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \tag{4.49}$$

with $S^2 = \mathbf{1}$; $S^\dagger = S = S^{-1}$.

There are also γ-matrices of higher dimensions. They only exist in a trivial generalization of the case $n = 4$, where the sub-matrices are the four-dimensional γ-matrices:

$$\begin{pmatrix} \gamma^\mu & & & 0 \\ & \gamma^\mu & & \\ & & \gamma^\mu & \\ 0 & & & \ddots \end{pmatrix} . \tag{4.50}$$

The proof follows from the fact that the relations (4.20) build a Clifford algebra with 16 independent elements together with the Theorem of Burnside:

$$N = \sum_i n_i^2 , \tag{4.51}$$

where n_i is the dimension of i-th inequivalent representation: With $n_i = 4$ the sum is saturated so that there are no more inequivalent representations.

In the following we want to analyze the physical consequences of the Dirac equation. The four-component quantity

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \tag{4.52}$$

is called **Dirac spinor** since it transforms under rotations analogously to spin-$\frac{1}{2}$ spinors.

The Dirac equation (4.17) was introduced for particles without a force. However, it is easy to consider the effect of an electromagnetic field. In a non-relativistic theory the electromagnetic field can be introduced through the following substitution:

$$\begin{aligned} \frac{\hbar}{i} \vec{\nabla} &\longrightarrow \frac{\hbar}{i} \vec{D} = \frac{\hbar}{i} \vec{\nabla} - \frac{e}{c} \vec{A} \\ i\hbar \frac{\partial}{\partial t} &\longrightarrow i\hbar\, D_t = i\hbar \frac{\partial}{\partial t} - e\phi . \end{aligned} \tag{4.53}$$

This can be rewritten in a four-dimensional form

$$i\hbar\, \partial_\mu \;\longrightarrow\; i\hbar\, D_\mu \;:=\; i\hbar\, \partial_\mu \;-\; \frac{e}{c}\, A_\mu\ . \tag{4.54}$$

According to Gell-Mann, this rule is called the **principle of minimal electromagnetic interaction**. It is a covariant rule and thus can be applied to the Dirac equation. One obtains

$$\left[\gamma^\mu (i\hbar\, \partial_\mu \;-\; \frac{e}{c}\, A_\mu) \;-\; mc\right]\, \psi(x) \;=\; 0 \tag{4.55}$$

and with $\hbar\ = c\ =\ 1$

$$[\gamma^\mu (i\partial_\mu \;-\; e\, A_\mu) \;-\; m]\ \psi(x) \;=\; 0\ . \tag{4.56}$$

Eqs. (4.55) are invariant if the following gauge transformations are carried out

$$\begin{aligned} A_\mu \;&\longrightarrow\; A'_\mu \;=\; A_\mu \;+\; \partial_\mu \chi \\ \psi \;&\longrightarrow\; \psi' \;=\; e^{ie\chi}\ \psi \end{aligned} \tag{4.57}$$

4.3 Covariant Bilinear Forms

Through products of γ-matrices it is possible to form 16 linearly independent 4×4 matrices $\Gamma^u_{\alpha\beta}$, which often occur in Dirac theory. These matrices are

$$\begin{aligned} \Gamma^s &= \mathbf{1} \\ \Gamma^V_\mu &= \gamma_\mu \\ \Gamma^T_{\mu\nu} &= \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \\ \Gamma^A_\mu &= \gamma_5 \gamma_\mu \\ \Gamma^p &= i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; \gamma_5 \;\equiv\; \gamma^5 \end{aligned} \tag{4.58}$$

Using the commutation relation $\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu}\mathbf{1}$, one can show that the Γ^n are linearly independent. Brief arguments are

1) For each Γ^n holds $(\Gamma^n)^2 = \pm 1$.

2) For each Γ^n (except Γ^s), there is a Γ^m such that

$$\Gamma^n \Gamma^m = -\Gamma^m \Gamma^n . \tag{4.59}$$

From this follows that $tr\ \Gamma^n = 0$:

$$\pm tr\ \Gamma^n = tr\ \Gamma^n (\Gamma^m)^2 = -tr\ \Gamma^m \Gamma^n \Gamma^m = -tr\ \Gamma^n (\Gamma^m)^2 = 0 . \tag{4.60}$$

3) For fixed Γ^a and $\Gamma^b (a \neq b)$, there exists a $\Gamma^n \neq \Gamma^s$ such that

$$\Gamma^a \Gamma^b = \Gamma^n . \tag{4.61}$$

4) Suppose there exist numbers a_n such that

$$\sum_n a_n\ \Gamma^n = 0 . \tag{4.62}$$

Then, via multiplication with $\Gamma^n \neq \Gamma^s$ and taking the trace, one finds because of (4.60) that $a_m = 0$. In the case of $\Gamma^n = \Gamma^s$ one finds $a_s = 0$, and all coefficients vanish. This proves the linear independence of the Γ^n. From this follows that every 4×4 matrix can be expressed as a linear combination of the Γ^n.

4.4 The Dirac Current

As the next step, we need to show that the current density defined in (4.3) corresponds to the Dirac equation. Thus we have to show that $\rho(x)$ constitutes the time-like component of the four vector $j^\mu(x)$, whose divergence vanishes according to (3.7). We start from

$$j^0(x) \;=\; \rho(x) \;=\; \sum_{\alpha=1}^{4} \psi_\alpha^*(x)\;\psi_\alpha(x) \;=\; \psi^\dagger(x)\;\psi(x) \tag{4.63}$$

with

$$\psi^\dagger \;=\; (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\;. \tag{4.64}$$

To express clearly that $\psi^\dagger\psi$ is a time-like component, we write

$$j^0 \;=\; \psi^\dagger\psi \;=\; \psi^\dagger\;\gamma^0\gamma^0\;\psi \;=\; \bar{\psi}\;\gamma^0\;\psi \tag{4.65}$$

with

$$\bar{\psi} \;:=\; \psi^\dagger\;\gamma^0\;, \tag{4.66}$$

the **adjoint Dirac Spinor**. Formally (4.65) can be generalized to a four-component quantity

$$j^\mu \;:=\; \bar{\psi}\;\gamma^\mu\;\psi\;. \tag{4.67}$$

In the following we want to show that the above defined **Dirac current** is indeed divergence free, as well as that it transforms as four-vector under Lorentz transformations.

To calculate the divergence of (4.67), we need to consider

$$\partial_\mu j^\mu \;=\; \bar{\psi}\;\gamma^\mu(\partial_\mu\psi) \;+\; (\partial_\mu\bar{\psi})\;\gamma^\mu\;\psi\;. \tag{4.68}$$

For the first term on the right-hand side, we use the Dirac equation directly:

$$(\gamma^\mu\;(i\partial_\mu -\;eA_\mu)\;-\;m)\;\psi \;=\; 0 \tag{4.69}$$

gives

$$i\gamma^\mu \, \partial_\mu \, \psi \; = \; (\gamma^\mu \, eA_\mu \; + \; m) \, \psi \; . \tag{4.70}$$

For the second term, we start from the adjoint equation

$$\psi^\dagger(\gamma^{\mu\dagger} \, (-i\overleftarrow{\partial}_\mu \; - \; eA_\mu) \; - \; m) \; = \; 0 \; . \tag{4.71}$$

Multiplication from the right with γ^0 and considering

$$\bar{\gamma}^\mu \; := \; \gamma^0 \gamma^{\mu\dagger} \gamma^0 \tag{4.72}$$

gives

$$\bar{\psi}(\bar{\gamma}^\mu \, (-i\overleftarrow{\partial}_\mu \; - \; eA_\mu) \; - \; m) \; = \; 0 \; . \tag{4.73}$$

Using the properties of the γ-matrices (4.44), (4.23, 4.24, 4.25) gives $\bar{\gamma}^\mu \; = \; \gamma^\mu$ and thus

$$\begin{aligned} \bar{\psi}(\gamma^\mu \, (i\overleftarrow{\partial}_\mu \; + \; eA_\mu) \; + \; m) \;\; &= \;\; 0 \\ i\partial_\mu \, \bar{\psi} \, \gamma^\mu \;\; &= \;\; -\bar{\psi}(\gamma^\mu \, eA_\mu \; + \; m) \; . \end{aligned} \tag{4.74}$$

With (4.70) and (4.74) follows

$$\begin{aligned} i\partial_\mu \, j^\mu \;\; &= \;\; \bar{\psi} \, (\gamma^\mu \, eA_\mu \; + \; m) \, \psi \; - \; \bar{\psi} \, (\gamma^\mu \, eA_\mu \; + \; m) \, \psi \\ &= \;\; 0 \end{aligned} \tag{4.75}$$

Thus

$$\partial_\mu j^\mu \; = \; \partial_\mu \, (\bar{\psi} \, \gamma^\mu \, \psi) \; = \; 0 \tag{4.76}$$

and the Dirac spinor can be used via the Dirac current as quantum mechanical probability amplitude.

We need to consider the space-like part of the probability current j^μ:

$$\vec{j} \; = \; (j^k) \; = \; \bar{\psi} \, \vec{\gamma} \, \psi \; = \; \psi^\dagger \, \gamma^0 \, \vec{\gamma} \, \psi \; = \; \psi^\dagger \, \vec{\alpha} \, \psi \tag{4.77}$$

with

$$\vec{\gamma} \; := \; \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix} \tag{4.78}$$

$$\vec{\alpha} \; := \; \gamma^0 \, \vec{\gamma} \; . \tag{4.79}$$

From the explicit forms of γ^0 and $\vec{\gamma}$ follows

$$\vec{\alpha} \; = \; \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \; , \tag{4.80}$$

thus $\vec{\alpha}$ consists of hermitian matrices. In addition, the trace vanishes, so that (4.77) only contains mixed products $\psi^*_\alpha \psi_\beta$ with $\alpha \; \neq \; \beta$.

Gordon-Decomposition of the Dirac Current

For $A_\mu \; = \; 0$ (4.70) and (4.74) gives

$$\begin{aligned} \psi \; &= \; \frac{1}{m} \, i \gamma^\nu \partial_\nu \, \psi \\ \bar{\psi} \; &= \; - \, \frac{1}{m} \, i \partial_\nu \, \bar{\psi} \, \gamma^\nu \end{aligned} \tag{4.81}$$

Inserting this into

$$j^\mu \; = \; \frac{1}{2} \, (\bar{\psi} \, \gamma^\mu \, \psi \; + \; \bar{\psi} \, \gamma^\mu \, \psi) \tag{4.82}$$

gives

$$j^\mu \; = \; \frac{1}{2m} \, (\bar{\psi} \, \gamma^\mu \gamma^\nu) \, (\partial_\nu \, \psi) \; - \; (\partial_\nu \, \bar{\psi}) \, \gamma^\nu \gamma^\mu \, \psi) \; . \tag{4.83}$$

Rewriting the products of γ-matrices as

$$\begin{aligned} \gamma^\mu \gamma^\nu \; &= \; \frac{1}{2} \, \{ \gamma^\mu , \gamma^\nu \} \; + \; \frac{1}{2} \, [\gamma^\mu , \gamma^\nu] \\ \gamma^\nu \gamma^\mu \; &= \; \frac{1}{2} \, \{ \gamma^\mu , \gamma^\nu \} \; - \; \frac{1}{2} \, [\gamma^\mu , \gamma^\nu] \end{aligned} \tag{4.84}$$

With the anti-commutation relations (4.20) and the definition of the **spin tensor**

$$\sigma^{\mu\nu} \ := \ \frac{i}{2} \ [\gamma^\mu, \gamma^\nu] \tag{4.85}$$

follows

$$\begin{aligned} \gamma^\mu \gamma^\nu \ &= \ g^{\mu\nu} \ + \ \frac{1}{i} \ \sigma^{\mu\nu} \\ \gamma^\nu \gamma^\mu \ &= \ g^{\mu\nu} \ - \ \frac{1}{i} \ \sigma^{\mu\nu} \ . \end{aligned} \tag{4.86}$$

Inserting into (4.83) leads to

$$j^\mu \ = \ \frac{i}{2m} \ \bar{\psi} \ \overset{\leftrightarrow}{\partial^\mu} \ \psi \ + \ \frac{1}{2m} \ \partial_\nu \ (\bar{\psi} \ \sigma^{\mu\nu} \ \psi) \ . \tag{4.87}$$

The first term is called 'orbital current'

$$j^\mu_{orbit} \ := \ \frac{i}{2m} \ \bar{\psi} \ \overset{\leftrightarrow}{\partial^\mu} \ \psi \tag{4.88}$$

the second, the spin part of the current,

$$j^\mu_{spin} \ := \ \frac{1}{2m} \ \partial_\nu \ (\bar{\psi} \ \sigma^{\mu\nu} \ \psi) \ . \tag{4.89}$$

Let us consider the spatial part j^k. With $\partial^k \ = \ -\partial_k \ = \ -(\vec{\nabla})_k$, the orbital part is given by

$$\begin{aligned} j^k_{orbit} \ &= \ \frac{1}{2mi} \ \bar{\psi} \ (\overset{\leftrightarrow}{\nabla})_k \ \psi \\ &= \ \frac{1}{2mi} \ \psi^\dagger \ \gamma^0 \ (\overset{\leftrightarrow}{\nabla})_k \ \psi \end{aligned} \tag{4.90}$$

This is obviously a generalization of the non-relativistic current (3.9), which we encountered with the Klein-Gordon equation, and will describe the probability current due to the spatial movement of an electron. For the second term, (4.89), we write

$$j^k_{spin} \ = \ \frac{1}{2m} \ \partial_\ell \ (\bar{\psi} \ \sigma^{k\ell} \ \psi) \ + \ \frac{1}{2m} \ \partial_0 \ (\bar{\psi} \ \sigma^{k0} \ \psi) \ . \tag{4.91}$$

The matrices $\sigma^{k\ell}$ and σ^{k0} fulfill

$$\begin{aligned} \sigma^{k\ell} &= \epsilon_{k\ell m} \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix} := \epsilon_{k\ell m} \, \Sigma^m \\ \sigma^{k0} &= (-i) \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = (-i) \, \alpha^k \end{aligned} \tag{4.92}$$

We introduced here the spin matrices $\sum^m$, which describe the spin in a Dirac theory. Thus the first term on the right-hand side of (4.91) contains directly the spin. The second term is of relativistic nature, since here spatial and time components are mixed.

4.5 Solutions of the Free Dirac Equation and Interpretation of the Dirac Spinors

Solutions of $(i\gamma^\mu \, \partial_\mu \; - \; m) \, \psi(x) \; = \; 0$ will be studied next. The combination $\gamma^\mu a_\mu$ of the γ-matrices with an arbitrary four-vector is abbreviated as

$$\gamma^\mu a_\mu \; := \; \not{a} \, , \tag{4.93}$$

and with this the Dirac equation takes the following form:

$$(i\not{\partial} \; - \; m) \, \psi(x) \; = \; 0 \; . \tag{4.94}$$

The coefficients of the differential equation are constant, thus the ansatz

$$\psi(x) \; = \; e^{-ip_\mu x^\mu} \; u(p) \; = \; e^{-ipx} \; u(p) \tag{4.95}$$

will be a solution. Here $u(p)$ is a spinor which does not depend on x^μ. Inserting of (4.95) into (4.94) gives

$$(\not{p} \; - \; m) \, u(p)) \; = \; 0 \; , \tag{4.96}$$

or

$$\sum_{\beta=1}^{4} \, (\gamma^\mu_{\alpha\beta} \, p_\mu \; - \; m \; \delta_{\alpha\beta}) \; u_\beta(p) \; = \; 0 \; , \tag{4.97}$$

i.e., a homogeneous system of four equations for the four components of u. The p_μ stands at present for four arbitrary parameters. The system of equations (4.97) has a non-trivial solution if

$$det(\not{p} \; - \; m) \; = \; 0 \; . \tag{4.98}$$

This condition leads to a fourth-order equation for p

$$(p^2 \; - \; m^2)^2 \; = \; 0 \; , \tag{4.99}$$

which can be written with $p^2 \; = \; (p^0)^2 \; - \; \vec{p}^{\;2}$ as

$$\left[(p^0 \; - \; \sqrt{\vec{p}^{\;2} \; + \; m^2}) \; (p^0 \; + \; \sqrt{\vec{p}^{\;2} \; + \; m^2}) \right]^2 \; = \; 0 \; . \tag{4.100}$$

Considering p_μ as four-vector, the above equation is fulfilled for momenta $\vec{p}$ obeying $p^0 = \pm \sqrt{\vec{p}^{\,2} + m^2}$ as energy eigenvalues. Eq. (4.96) has for each of the two values a two-dimensional solution space, with a basis, which can depend on $\vec{p}$:

$$
\begin{aligned}
\mu_1(\vec{p}) \quad , \quad \mu_2(\vec{p}) \quad & \text{for} \;\; p^0 = + \sqrt{\vec{p}^{\,2} + m^2} \\
\mu_3(\vec{p}) \quad , \quad \mu_4(\vec{p}) \quad & \text{for} \;\; p^0 = - \sqrt{\vec{p}^{\,2} + m^2}
\end{aligned}
\tag{4.101}
$$

For the explicit calculation of the spinors, we choose

$$
u = \begin{pmatrix} u^L \\ u^S \end{pmatrix} , \tag{4.102}
$$

where u^L and u^S are two-component spinors. Inserting the explicit representation of the γ-matrices leads to

$$
\left[\begin{pmatrix} p^0 - m & 0 \\ 0 & -p^0 - m \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ -\vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} \right] \begin{pmatrix} u^L \\ u^S \end{pmatrix} = 0 , \tag{4.103}
$$

i.e., to

$$
\begin{aligned}
(p^0 - m)\, u^L &= \vec{\sigma} \cdot \vec{p}\, u^S \\
(p^0 + m)\, u^S &= \vec{\sigma} \cdot \vec{p}\, u^L
\end{aligned}
\tag{4.104}
$$

Expressing u^S through u^L gives

$$
u^S = \frac{\vec{\sigma} \cdot \vec{p}}{p^0 + m}\, u^L \tag{4.105}
$$

and inserting this into the first equation (4.104) leads to (after multiplication with $(p^0 + m)$)

$$
\left[(p^0)^2 - m^2 \right] u^L = (\vec{\sigma} \cdot \vec{p})^2\, u^L = \vec{p}^{\,2}\, u^L . \tag{4.106}
$$

This equation has non-trivial solutions for u^L if

$$(p^0)^2 \;=\; \vec{p}^{\,2} \;+\; m^2 \;. \tag{4.107}$$

Here u^L can be chosen arbitrarily and u^S can be calculated using (4.105). However, (4.107) allows $p^0 \;=\; +\sqrt{\vec{p}^{\,2} \;+\; m^2}$ and $p^0 \;=\; -\sqrt{\vec{p}^{\,2} \;+\; m^2}$. For the different signs, one obtains different solutions u^S.

In case of very small momenta and negative energies, numerator and denominator of (4.105) can become very small, in the limit $\vec{p} \;\to\; \vec{0}$ one obtains $\frac{0}{0}$. Thus for $p^0 \;<\; 0$ it is better to use the first equation of (4.104) to determine u^L

$$u^L \;=\; \frac{\vec{\sigma} \;\cdot\; \vec{p}}{p^0 \;-\; m}\; u^S \tag{4.108}$$

and first determine u^S.

In Summary:

For arbitrary momenta $\vec{p}$ there exists solutions of the Dirac equation with positive and negative energies. For each $\vec{p}$ and for each sign of the energy exists a two-dimensional solution space.

It is tempting to connect the dual degeneracy to the sign and the conclusion is:

The Dirac equation describes spin $\frac{1}{2}$ particles.

This is a surprising and non-trivial result, since during the entire derivation there was no assumption about the spin of the particle described by ψ.

We study now the spin-properties for the physically easier case of positive energies. According to (4.105), u^S can be always expressed via u^L for the case $p^0 > 0$.

For $|\,\vec{p}\,| \leq\; m$, one has

$$u^S \;\approx\; \frac{\vec{\sigma}\cdot\vec{p}}{2m}\, u^L \;=\; \frac{1}{2}\,\vec{\sigma}\cdot\vec{v}\; u^L \;. \tag{4.109}$$

Since $|\;\vec{v}\;| = \frac{|\vec{p}|}{m}$ is small compared to 1 (= speed of light), one can neglect the **small component** u^S with respect to the **large component** u^L:

$$\psi(x) \;=\; e^{-ipx} \begin{pmatrix} u^L \\ 0 \end{pmatrix} \qquad \text{for } \frac{v}{c} \;<<\; 1 \;. \tag{4.110}$$

The wave function is solely determined through the four-momentum p and the two arbitrary two-spinors u^L. The latter can be interpreted as Pauli spinors, and especially as eigenvectors of

$$S_3 \;=\; \frac{1}{2}\,\sigma_3 \;. \tag{4.111}$$

This leads to $u^L_\pm$ with

$$\sigma_3 \; u^L_\pm \;=\; \pm \; u^L_\pm \;. \tag{4.112}$$

If one cannot neglect u^S (which is true for $\vec{p} \neq 0$), one has to determine the spin operator. The choice

$$\vec{S} \;:=\; \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \;=\; \frac{1}{2}\,\vec{\Sigma} \tag{4.113}$$

is obvious.

First we need to show that the two-spinors u^L really are equivalent to Pauli spinors, and then we need to show that

$$\vec{J} \;=\; \vec{L} \;+\; \vec{S} \;=\; \vec{r} \;\times\; \frac{1}{i}\,\vec{\nabla} \;+\; \frac{1}{2}\,\vec{\Sigma} \tag{4.114}$$

describes the total angular momentum for the Dirac spinor, since

$$\frac{d}{dt}\,\vec{J} \;=\; \vec{0} \tag{4.115}$$

and $\vec{J}$ is the generator of rotations in the space of Dirac spinors.

First we want to study the spin properties of ψ or u for arbitrary velocities. Choosing u^L as eigenvector of σ_3 leads with (4.105) to the conclusion that u^S is only then eigenvector of σ_3, if the momentum $\vec{p}$ points into the three-direction. Thus, it is obvious to work with states, for which the component of the spin in the direction of the momentum is measured. We introduce the operator

$$\Lambda \;:=\; \frac{\vec{\sigma} \,\cdot\, \vec{p}}{|\,\vec{p}\,|} \tag{4.116}$$

and choose u^L to be its eigenvector u_λ with

$$\Lambda\, u_\lambda \;=\; \lambda\, u_\lambda\,; \quad \lambda \;=\; \pm\, 1\,. \tag{4.117}$$

In the following, we consider u^L_λ. From (4.105) and

$$\left[\Lambda,\, \frac{\vec{\sigma} \,\cdot\, \vec{p}}{p^0 \,+\, m}\right] \;=\; 0 \tag{4.118}$$

follows that also the small components u^S are eigenvectors of Λ:

$$u^S_\lambda \;:=\; \frac{\vec{\sigma} \,\cdot\, \vec{p}}{p^0 \,+\, m}\; u^L_\lambda \tag{4.119}$$

with

$$\Lambda\; u^S_\lambda \;=\; \lambda\; u^S_\lambda\;. \tag{4.120}$$

Thus the spinor

$$u_\lambda \;:=\; \begin{pmatrix} u^L_\lambda \\ u^S_\lambda \end{pmatrix} \tag{4.121}$$

is according to (4.113) eigenvector of the operator $\frac{\vec{S}\,\cdot\,\vec{p}}{|\vec{p}|}$ with the eigenvalue $\frac{1}{2}\,\lambda$.

The operator Λ is called **helicity** operator, its eigenstates u_k are called **helicity states**. Pictorially they define a screw motion, for $\lambda \;=\; +1$ a right-handed, and for $\lambda \;=\; -1$ a left-handed. Explicitly, one obtains from (4.116) and (4.119) for u_λ:

$$u_\lambda \;=\; \begin{pmatrix} u^L_\lambda \\ \frac{|\vec{p}|}{p^0 \,+\, m}\,\lambda\; u^L_\lambda \end{pmatrix}\;. \tag{4.122}$$

For explicit calculations, one needs the normalization of (4.122). If η_λ is a two-dimensional unit vector **and** eigenvector of Λ, i.e.,

$$\eta_\lambda^\dagger \, \eta_\lambda \;=\; \mathbf{1} \, , \qquad \Lambda \, \eta_\lambda \;=\; \lambda \, \eta_\lambda \tag{4.123}$$

then

$$u_\lambda \;=\; N \begin{pmatrix} \eta_\lambda \\ \frac{|\vec{p}|}{p^0 \, + \, m} \, \lambda \, \eta_\lambda \end{pmatrix} . \tag{4.124}$$

The factor N only depends on the normalization condition for the Dirac spinor u_λ, namely if the probability density was normalized to one via

$$u_\lambda^\dagger \; u_\lambda \;=\; 1 \tag{4.125}$$

or if via

$$\bar{u}_\lambda \; u_\lambda \;=\; 1 \tag{4.126}$$

one has chosen a Lorentz invariant normalization.

In the first case (4.125), one obtains

$$N \;=\; \sqrt{\frac{p^0 \, + \, m}{2p^0}} \tag{4.127}$$

and in the second (4.126)

$$N \;=\; \sqrt{\frac{p^0 \, + \, m}{2m}} \, . \tag{4.128}$$

With the above developed expression we want to prove:
Fast moving electrons cannot have a transverse polarization.

For the proof we use the Dirac spinor

$$u(\vec{p}) \;:=\; N \begin{pmatrix} \eta \\ \frac{\vec{\sigma} \, \cdot \, \vec{p}}{p^0 \, + \, m} \, \eta \end{pmatrix} \tag{4.129}$$

with $\eta^\dagger \eta = 1$; $N = \sqrt{\frac{(p^0 \ + \ m)}{(2p^0)}}$. The momentum $\vec{p}$ points in 1-direction: $\vec{p} = \mid \vec{p} \mid e_1$. For the expectation value Σ_3 from (4.113) follows:

$$
\begin{aligned}
u^\dagger(\vec{p}) \ \Sigma_3 \ u(\vec{p}) &= N^2 \left[\eta^\dagger \ \sigma_3 \eta \ + \ \left(\frac{\mid \vec{p} \mid}{p^0 \ + \ m} \right)^2 \ \eta^\dagger \ \sigma_1 \sigma_3 \sigma_1 \ \eta \right] \\
&= N^2 \left[\eta^\dagger \ \sigma_3 \eta \ - \ \left(\frac{\mid \vec{p} \mid}{p^0 \ + \ m} \right)^2 \ \eta^\dagger \ \sigma_3 \ \eta \right] \\
&= N^2 \ \eta^\dagger \ \sigma_3 \eta \ \left[1 \ - \ \left(\frac{\mid \vec{p} \mid}{p^0 \ + \ m} \right)^2 \right] \\
&= N^2 \ \frac{2m}{p^0 \ + \ m} \ \eta^\dagger \ \sigma_3 \eta
\end{aligned}
\tag{4.130}
$$

and using the explicit expression for N gives

$$
u^\dagger(\vec{p}) \ \Sigma_3 \ u(\vec{p}) \ = \ \frac{m}{p^0} \ \eta^\dagger \ \sigma_3 \eta \ . \tag{4.131}
$$

For $p^0 \ \rightarrow \ \infty$ this expression vanishes independent of $\eta^\dagger \ \sigma_3 \eta$, the result which was to be shown.

One can understand this result also pictorially with the help of Lorentz contractions.

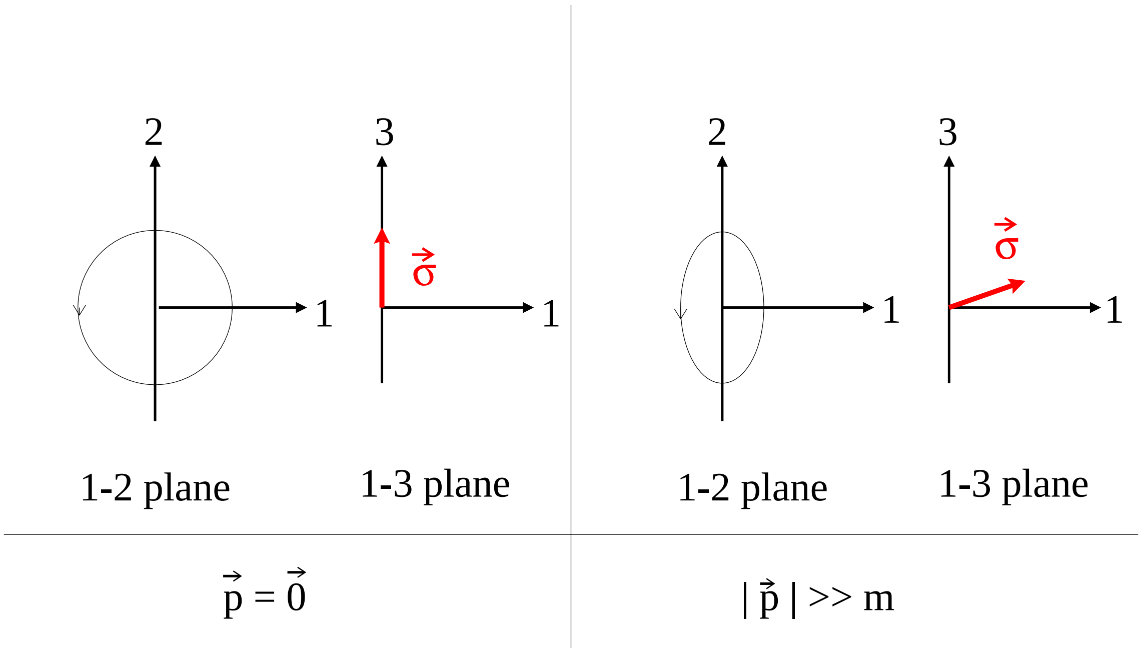

Figure 9.1 Transversal depolarization due to high momenta.

Fig. 9.1 shows an electron at rest with $\sigma_3 = +1$ moving in the 1-2 plane. If the electron moves with momentum $\vec{p}$, the circular movement deforms to an ellipse, which can be interpreted as tilting of the spin into the direction of $\vec{p}$.

For the Dirac current (4.87)

$$j^\mu = \frac{1}{2m} \bar{\psi} \overset{\leftrightarrow}{\partial}^\mu \psi + \frac{1}{2m} \partial_\nu \left(\bar{\psi} \sigma^{\mu\nu} \psi \right) \tag{4.132}$$

the solution is

$$\psi = e^{-ipx} u . \tag{4.133}$$

Here

$$\bar{\psi} \sigma^{\mu\nu} \psi = \bar{u} \sigma^{\mu\nu} u \tag{4.134}$$

independent of x, which means the second term does not give any contribution. The first term leads to (after differentiation)

$$j^\mu_{orbit} = \frac{p^\mu}{m} \bar{u}u . \tag{4.135}$$

With (4.129) and the explicit form of γ^0, one obtains

$$\begin{aligned} \bar{u}u &= u^\dagger\, \gamma^0 u \\ &= N^2 \left[\eta^\dagger \eta - \frac{1}{(P^0 + m)^2}\, \eta^\dagger\, (\vec{\sigma} \cdot \vec{p})^2\, \eta \right] \\ &= N^2 \left[1 - \frac{|\, \vec{p}\,|^2}{(P^0 + m)^2} \right] \\ &= \frac{m}{p^0} \end{aligned} \tag{4.136}$$

Thus for the free solution, one has

$$j^\mu = \frac{p^\mu}{p^0} \tag{4.137}$$

and especially

$$\begin{aligned} \rho &= j^0 = 1 \\ \bar{j} &= \frac{\vec{p}}{p^0} = \vec{v}\, . \end{aligned} \tag{4.138}$$

Thus the plan wave (4.95) describes a probability density which is normalized to **1**. The probability current is given by the relativistically defined velocity $\bar{v} = \frac{\vec{p}}{p^0}$.

4.6 Successes of the Dirac Theory: G-Factor of the Electron and Spin Orbit Coupling

First, we need to write the Dirac equation in the presence of an electromagnetic field A_ν, where we use minimal coupling.

$$\left(\gamma^\nu(p_\nu - \frac{e}{c}A_\nu) - mc\right)\psi = 0, \tag{4.139}$$

or with $p_\nu = -\hbar/i\partial/\partial x^\nu$

$$\left(\gamma^\nu(\frac{\partial}{\partial x^\nu} + \frac{ie}{\hbar c}A_\nu) + i\frac{mc}{\hbar}\right)\psi = 0. \tag{4.140}$$

In the following we use the abbreviation $k = mc/\hbar$. The adjoint Dirac equation gives

$$\left(\frac{\partial}{\partial x^\nu} - \frac{ie}{\hbar c}A_\nu\right)\psi^\dagger(\gamma^\nu)^\dagger - ik\psi^\dagger = 0, \tag{4.141}$$

with $(\gamma^\nu)^\dagger\gamma^0 = \gamma^0\gamma^\nu$ and $\bar{\psi} = \psi^\dagger\gamma^0$ we get

$$\left(\frac{\partial}{\partial x^\nu} - \frac{ie}{\hbar c}A_\nu\right)\bar{\psi}\gamma^\nu - ik\bar{\psi} = 0 \tag{4.142}$$

or

$$\left(p_\nu + \frac{e}{c}A_\nu\right)\bar{\psi}\gamma^\nu - mc\bar{\psi} = 0 \tag{4.143}$$

Starting from (4.140), we can write the Dirac equation in noncovariant form as differential equation of first order in time

$$\left(\gamma^0\left(\frac{\partial}{\partial x^0} + \frac{ie}{\hbar c}A_0\right) + \sum_{i=1,3}\gamma^i\left(\frac{\partial}{\partial x^i} + \frac{ie}{\hbar c}A_i\right) + ik\right)\psi = 0. \tag{4.144}$$

Multiplication with $\gamma^0 i\hbar c$ from the left and considering that $\partial/\partial x^0 \equiv \partial/\partial ct$ gives

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-i\hbar c\gamma^0\left(\sum_{i=1,3}\gamma^i\frac{\partial}{\partial x^i}\right) + e\varphi + e\gamma^0\sum_{i=1,3}\gamma^i A_i + \gamma^0 mc^2\right)\psi \tag{4.145}$$

This equation (4.145) has the form

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi, \tag{4.146}$$

where H is Hermitian, since $\gamma^0\gamma^i = (\gamma^0\gamma^i)^\dagger$. With $\mathbf{p} = (p_1, p_2, p_3)$, $\gamma^0\gamma^i = -\gamma^i\gamma^0 \equiv \alpha^i$, and $\gamma^0 \equiv \beta$ we can write

$$H = c\alpha(\mathbf{p} - \frac{e}{c}\mathbf{A}) + mc^2\beta e\varphi. \tag{4.147}$$

The following relations hold:

$$\begin{aligned} \beta^\dagger &= \beta \\ (\alpha^k)^\dagger &= \alpha^k \\ \alpha\beta + \beta\alpha &= 0 \\ \alpha^k\alpha^l + \alpha^l\alpha^k &= 2\delta^{kl} \end{aligned} \tag{4.148}$$

The importance of H is the following. If one takes expectation values of operators, i.e.

$$\langle A \rangle = \int d^3x \psi^\dagger A \psi, \tag{4.149}$$

then

$$\begin{aligned} \frac{\partial}{\partial t}\langle A \rangle &= \int d^3x \left[\frac{\partial}{\partial t}\psi^\dagger A\psi + \psi^\dagger A \frac{\partial}{\partial t}\psi\right] \\ &= \int d^3x \psi^\dagger \frac{i}{\hbar}[H, A]\psi \\ &= \langle \frac{\partial A}{\partial t} \rangle \end{aligned} \tag{4.150}$$

so that

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar}[H, A] \tag{4.151}$$

Let us consider the particle velocity as example, which should give

$$\mathbf{v} = \frac{\partial}{\partial t}\mathbf{x} = \frac{i}{\hbar}[H, \mathbf{x}] \tag{4.152}$$

For the components we have

$$\frac{\partial}{\partial t}x^k = \frac{i}{\hbar}[H, x^k] = \frac{i}{\hbar}c\alpha^l[p^l, x^k] = \frac{i}{\hbar}c\alpha^l\frac{\hbar}{i}\delta^{kl} = c\alpha^k. \tag{4.153}$$

Because of $(\alpha^k)^2 = (\gamma^0\gamma^k)^2 = \gamma^0\gamma^k\gamma^0\gamma^k = -(\gamma^0)^2(\gamma^k)^2 = (-1)(-1) = 1$ follows that the eigenvalues of $\alpha^k = \pm 1$, and thus the eigenvalues of $v^k = \pm c$. Thus in general the expectation values for the velocity can take any value between $+c$ and $-c$.

Using $\hbar$ and c equal 1, (4.147) reads

$$\begin{aligned} i\hbar\,\frac{\partial\psi}{\partial t} &= H\psi \\ H &= c\vec{\alpha}\,(\vec{p} - \frac{e}{c}\,\vec{A}) + \beta mc^2 + e\varphi\,\mathbf{1}\,. \end{aligned} \tag{4.154}$$

Considering the dimensions, we have

$$[c\vec{p}] \;=\; \left[mc^2\right] \;=\; [e\varphi] \;=\; [\text{energy}] \;. \tag{4.155}$$

For small velocities (4.147) should become an equation of non-relativistic quantum mechanics, i.e., the Pauli equation. In the following this aspect should be considered.

According to the experiences of the previous chapter, we decompose the Dirac spinor into two parts. Further, we split off the time dependence of ψ, which results from the rest energy m. Thus, we make the ansatz

$$\psi(x) \;=\; e^{-imx^0} \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \;. \tag{4.156}$$

From (4.147) follows

$$\begin{aligned} i\partial_0 \; \psi(x) \;&=\; i\partial_0 \left(e^{-imx^0} \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \right) \\ &=\; -\, e^{-imx^0} \; m \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \;+\; e^{-imx^0} \; i\partial_0 \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \\ &=\; e^{-imx^0} \; H \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \;, \end{aligned} \tag{4.157}$$

i.e.,

$$i\partial_0 \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \;=\; (H \;-\; m\,\mathbf{1}) \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \;. \tag{4.158}$$

If one rewrites the operator on the right-hand side by using the abbreviation

$$\vec{\pi} \;:=\; \vec{p} \;-\; e\vec{A} \;, \tag{4.159}$$

one finds

$$H \;-\; m\mathbf{1} \;=\; \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\pi} \\ \vec{\sigma}\cdot\vec{\pi} & 0 \end{pmatrix} \;+\; \begin{pmatrix} 0 & 0 \\ 0 & -2m \end{pmatrix} \;+\; \begin{pmatrix} e\varphi & 0 \\ 0 & e\varphi \end{pmatrix} \;. \tag{4.160}$$

Thus one has

$$
\begin{aligned}
i\partial_0 \, \phi \;&=\; \vec{\sigma} \cdot \vec{\pi} \, \chi \;+\; e\varphi\phi \\
i\partial_0 \, \chi \;&=\; \vec{\sigma} \cdot \vec{\pi} \, \phi \;-\; (2m \;-\; e\varphi) \, \chi \; .
\end{aligned} \tag{4.161}
$$

It is important to note that the mass m only occurs in the second equation. If the kinetic energies and the field-interaction energies are small compared to the rest mass m, one can write (since then also $| \, e\varphi \, | \leq \; 2m$)

$$
| \, i\partial_0 \, \chi \, | \approx \; (E \;-\; m) \;\; | \, \chi \, | \leq | \, \chi \, | \leq (2m \;-\; e\varphi) \;\; | \, \chi \, | \;\; . \tag{4.162}
$$

From the second equation of (4.161) follows

$$
\begin{aligned}
(2m \;-\; e\varphi) \, \chi \;\; &\approx \;\; \vec{\sigma} \cdot \vec{\pi} \, \phi \\
\chi \;\; &\approx \;\; \frac{\vec{\sigma} \cdot \vec{\pi}}{2m \;-\; e\varphi} \, \phi \; \approx \; \frac{1}{2m} \, \left(1 \;+\; \frac{e}{2m} \, \varphi \right) \, \vec{\sigma} \cdot \vec{\pi} \, \phi \; .
\end{aligned} \tag{4.163}
$$

As a remark: for $\varphi \;=\; 0$, $\vec{A} \;=\; 0$, Eq. (4.163) corresponds to the relation (4.109) of the previous chapter, what should be expected. Inserting this into the upper equation of (4.161) leads to

$$
i\partial_0 \, \phi \;=\; \frac{1}{2m} \, \vec{\sigma}\vec{\pi} \, \left(1 \;+\; \frac{e}{2m} \, \varphi \right) \, \vec{\sigma}\vec{\pi}\phi \;+\; e\varphi \, \phi \; . \tag{4.164}
$$

This equation allows to establish a relation to non-relativistic quantum mechanics. This shall be done in two steps:

1) **Consider the operator** $\frac{1}{2m} \, (\vec{\sigma}\vec{\pi}) \, (\vec{\sigma}\vec{\pi})$

Standard σ-matrix algebra gives

$$
(\vec{\sigma}\vec{\pi}) \, (\vec{\sigma}\vec{\pi}) \;=\; \pi^2 \;+\; i\vec{\sigma} \, (\vec{\pi} \;\times\; \vec{\pi}) \; . \tag{4.165}
$$

The first term is according to its definition (4.159)

$$
\vec{\pi}^2 \;=\; (\vec{p} \;-\; e\vec{A})^2 \; . \tag{4.166}
$$

The second term does **not** vanish! One obtains

$$
\begin{aligned}
(\vec{\pi} \times \vec{\pi})\, \phi &= -\frac{e}{i} \left[\vec{\nabla} \times \vec{A} + \vec{A} \times \vec{\nabla}\right] \phi \\
&= -\frac{e}{i} \left[(\vec{\nabla} \times \vec{A})\, \phi + (\vec{\nabla}\, \phi) \times \vec{A} + \vec{A} \times (\vec{\nabla} \phi)\right] \\
&= -\frac{e}{i} \text{ rot } \vec{A}\, \phi \\
&= -\frac{e}{i} \vec{B}\, \phi \qquad (4.167)
\end{aligned}
$$

Thus one has

$$\frac{1}{2m} (\vec{\sigma}\vec{\pi})(\vec{\sigma}\vec{\pi}) = \frac{1}{2m} (\vec{p} - e\vec{A})^2 - \frac{e}{2m} \vec{\sigma} B\,. \qquad (4.168)$$

The second term on the right-hand side can be written as

$$-2\, \frac{e}{2m}\, \vec{S} \cdot \vec{B} = -g\, \mu_B\, \vec{S} \cdot \vec{B} \qquad (4.169)$$

with $g = 2$ as a g-factor of the electron. Thus, the interaction of the magnetic moment with the $\vec{B}$-field and $g = 2$ are a consequence of the Dirac equation.

2) Consider the Operator $\frac{e}{4m^2}\, (\vec{\sigma}\vec{\pi})\, \varphi\, (\vec{\sigma}\vec{\pi})$

This operator contains terms of the second- and third-order in the potentials, which shall not be considered here. Here we want to consider the linear part corresponding to the approximations in non-relativistic physics. In this approximation, one obtains

$$\frac{e}{4m^2}\, (\vec{\sigma}\vec{\pi})\, \varphi\, (\vec{\sigma}\vec{\pi}) \approx \frac{e}{4m^2}\, (\vec{\sigma}\vec{p})\, \varphi\, (\vec{\sigma}\vec{p})\,. \qquad (4.170)$$

First consider

$$(\vec{\sigma}\vec{p})\, \varphi\, (\vec{\sigma}\vec{p}) = \sigma_k \sigma_\ell\, p_k\, \varphi\, p_\ell\,. \qquad (4.171)$$

According to

$$
\begin{aligned}
\sigma_k \sigma_\ell &= \frac{1}{2} \{\sigma_k, \sigma_\ell\} + \frac{1}{2} [\sigma_k, \sigma_\ell] \\
&= \delta_{k\ell} + i\epsilon_{k\ell m}\, \sigma_m \qquad (4.172)
\end{aligned}
$$

we obtain

$$(\vec{\sigma}\vec{p})\ \varphi\ (\vec{\sigma}\vec{p})\ =\ p_k\ \varphi\ p_k\ +\ i\epsilon_{k\ell m}\ \sigma_m\ p_k\ \varphi\ p_\ell\ . \tag{4.173}$$

For the second term, we obtain

$$\begin{aligned} i\epsilon_{k\ell m}\ \sigma_m\ p_k\ \varphi\ p_\ell\ &=\ i\epsilon_{k\ell m}\sigma_m\ [p_k,\varphi]\ p_\ell\ +\ i\epsilon_{k\ell m}\ \sigma_m\ \varphi\ p_k p_\ell \\ &=\ i\epsilon_{k\ell m}\ \sigma_m\ [p_k,\varphi]\ p_\ell \\ &=\ \epsilon_{k\ell m}\ \sigma_m\ \{(\vec{\nabla}_k\varphi)\ p_\ell\ -\ \varphi(\vec{\nabla}_k p_\ell)\} \\ &=\ \epsilon_{k\ell m}\ \sigma_m\ (\vec{\nabla}_k\varphi)\ p_\ell \\ &=\ \vec{\sigma}\ (\vec{\nabla}\varphi\ \times\ \vec{p})\ , \end{aligned} \tag{4.174}$$

where the second term vanished due to the symmetry to $p_k p_\ell$ together with the anti-symmetric tensor $\epsilon_{k\ell m}$.

Thus, one has

$$\begin{aligned} \frac{e}{4m^2}\ (\vec{\sigma}\vec{p})\ \varphi\ (\vec{\sigma}\vec{p})\ &=\ \frac{e}{4m^2}\ \vec{p}\ \varphi\ \vec{p}\ +\ \frac{e}{4m^2}\ \vec{\sigma}\ (\vec{\nabla}\varphi\ \times\ \vec{p}) \\ &:=\ V_1\ +\ V_2\ . \end{aligned} \tag{4.175}$$

Here V_2 describes a spin-orbit coupling. One can see this when considering a static, rotational invariant potential with

$$\vec{E}\ =\ -\vec{\nabla}\varphi\ =\ -\varphi'\ \frac{\vec{r}}{r}\ . \tag{4.176}$$

In this case, one has

$$\begin{aligned} V_2\ &=\ \frac{e}{4m^2}\ \frac{1}{r}\ \varphi'\ \vec{\sigma}\ (\vec{r}\ \times \vec{p}) \\ &=\ \frac{e}{4m^2}\ \frac{1}{r}\ \varphi'\ \vec{\sigma}\ \cdot\ \vec{L} \\ &=\ \frac{1}{2}\ \frac{e}{m^2}\ \frac{1}{r}\ \varphi'\ \vec{L}\ \cdot\ \vec{S}\ =:\ V_{LS}\ . \end{aligned} \tag{4.177}$$

This is the correct expression for the spin-orbit coupling with the Thomas factor $\frac{1}{2}$:

$$V_{LS}\ =\ \frac{1}{2}\ \left(\frac{1}{m}\right)^2\ \frac{1}{r}\ V'\ \vec{L}\ \cdot\ \vec{S} \tag{4.178}$$

where $V \; := \; e\varphi$.

The term V_1 in Eq (4.175) is new, i.e., does not have a non-relativistic correspondence. Since V_1 contains the square of the velocity $\frac{|p|}{m}$, it corresponds to a correction of the order $\left(\frac{v}{c}\right)^2$. However, such corrections are due to the approximation (4.163) not completely considered. One has to expect further contributions in this order, and for an exact calculation one needs systematic approximations. These are provided by the Foldy-Wouthuysen transformation.

Considering the approximation (4.163), one can say that the physical success of the Dirac theory manifests itself in the non-relativistic approximation for the total Hamiltonian

$$H \;=\; \frac{1}{2m}\,(\vec{p} \;-\; e\vec{A})^2 \;+\; V \;-\; 2\mu_B\,\vec{S}\;\cdot\;\vec{B} \;+\; \frac{1}{2}\left(\frac{1}{m}\right)^2\,\frac{1}{r}\,V'\,\vec{L}\;\cdot\;\vec{S}\;, \qquad (4.179)$$

where $V \; = \; e\varphi$.

4.7 Non-Relativistic Limit - Foldy-Wouthuysen Transformation

To investigate the non-relativistic limit of the Dirac equation, we work out the expansion to order $\left(\frac{v}{c}\right)^2 \approx \left(\frac{P}{m}\right)^2 \times$ *leading terms.* In making estimates, we assume all potentials V^0 and $\vec{V}$ to be of the same order as the kinematic energy term. Since all leading order terms are of order $\frac{P^2}{m}$, we need all terms up to order $\frac{P^4}{m^3}$.

We assume positive energy solutions

$$\psi(\vec{x},t) \; = \; e^{-iEt} \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \tag{4.180}$$

where the total energy is $E = m + T$. Inserting (4.180) into the Dirac equation (4.147) and assuming $\vec{p} \longrightarrow \vec{p} - \vec{V}$ $(V \equiv eA)$ gives

$$\left[\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} (\vec{p}-\vec{V}) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} m + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (V^0 - E)\right] \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} = 0. \tag{4.181}$$

This leads to

$$\begin{aligned} \vec{\sigma}\cdot(\vec{p}-\vec{V})\;\chi + (m + V^0 - E)\;\phi \;&=\; 0 \\ \vec{\sigma}\cdot(\vec{p}-\vec{V})\;\phi \;+\; (-m + V^0 - E)\;\chi \;&=\; 0 \end{aligned} \tag{4.182}$$

and with $E = m + T$ to

$$\begin{aligned} T\phi \;&=\; V^0\phi \;+\; \vec{\sigma}\cdot(\vec{p}-\vec{V})\chi \\ (2m+T)\chi \;&=\; V^0\chi \;+\; \vec{\sigma}\cdot(\vec{p}-\vec{V})\;\phi\;. \end{aligned} \tag{4.183}$$

In the non-relativistic limit $T, \mid \vec{p} \mid$ and all components of $\mid V^\mu \mid$ are assumed to be much smaller than m. Thus, the second equation of (4.184) shows that the lower component of the Dirac spinor is much smaller than the upper, and, therefore, the equations are solved approximately by eliminating the lower component. Proceeding directly by solving the equation for χ and substituting the solution into the equation for the upper component leads to the formal solution

$$T\phi \;=\; \left[V^0 \;+\; \vec{\sigma}\cdot(\vec{p}-\vec{V})\;\left(\frac{1}{2m+T-V^0}\right)\;\vec{\sigma}\cdot(\vec{p}-\vec{V})\right]\;\phi. \tag{4.184}$$

Since T is of the same order as $V^0(\cong \frac{p^2}{m})$, it is necessary to expand the denominator of the second term if we want to collect all terms of order p^4/m^3. This expansion gives

$$\begin{aligned} T\phi \;&=\; \Big[V^0 + \frac{1}{2m}\,\vec{\sigma}\cdot(\vec{p}-\vec{V})\,\vec{\sigma}\cdot(\vec{p}-\vec{V}) \\ &-\; \frac{1}{4m^2}\vec{\sigma}\cdot(\vec{p}-\vec{V})(T-V^0)\vec{\sigma}\cdot(\vec{p}-\vec{V})\Big]\;\phi \end{aligned} \tag{4.185}$$

The presence of T in (4.185) means that the effective Hamiltonian defined by (4.185) depends on the energy. Generally energy-dependent Hamiltonians lead to complications, so it is desirable to eliminate the energy dependence. Since the T dependence occurs only in the highest order term, it might seem that it could be removed by replacing it by an estimate obtained from the solution of the lower order equation, i.e.,

$$T \;\cong\; V^0 \;+\; \frac{1}{2m}\,\vec{\sigma}\cdot(\vec{p}-\vec{V})\,\vec{\sigma}\cdot(\vec{p}-\vec{V})\;. \tag{4.186}$$

Note: $\vec{V} \approx p^2/m$ is assumed.

However, this method will not give a unique answer because T is a number and commutes with $\vec{\sigma}\cdot(\vec{p}-\vec{V})$, while V^0 does not. A better method for obtaining a non-relativistic "limit" is known as the Foldy-Wouthuysen (FW) transformation (L.L. Foldy and S.A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950)). Here the idea is to transform the equations (4.183), (4.181) to a new form in which the off-diagonal elements of the Hamiltonian are so small that the leading order estimate of the lower components (which does not depend on T) is sufficient to get the effective Hamiltonian up to order $\frac{p^4}{m^3}$. To do this it is be sufficient to reduce the off-diagonal elements to order $\frac{p^2}{m}$. If this is the case, the leading combination from the lower component would be of order $\frac{p^2}{m}$, and their contribution to the equation for ϕ would thus be of order $\frac{p^4}{m^3}$.

To prepare for the application of the FW transformation, rewrite (4.181) in terms of Dirac matrices

$$T\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \big(-m + V^0 + \vec{\alpha}\cdot(\vec{p}-\vec{V}) + m\beta\big)\begin{pmatrix}\phi\\ \chi\end{pmatrix} = H\begin{pmatrix}\phi\\ \chi\end{pmatrix}. \tag{4.187}$$

The off-diagonal terms are those involving $\vec{\alpha}$, and they are of order m. The goal is to transform the equation in such a way that they are of order m^{-1}. Then, when the equation is solved, T will not enter into the m^{-3} terms.

Eq. (4.187) will be transformed using unitary transformations constructed from Dirac matrices. Since the large off-diagonal terms depend on $\vec{\alpha}\cdot\vec{p}$, it is sufficient to use a transformation of the form

$$U \; = \; U^\dagger \; = \; A\beta \; + \; \frac{\lambda}{m} \; \vec{\alpha} \; \cdot \; \vec{p} \tag{4.188}$$

with

$$A \; = \; \sqrt{1 \; - \; \frac{\lambda^2 p^2}{m^2}} \; , \tag{4.189}$$

where λ is a parameter to be determined. Recall, that $\alpha_i, \alpha_j = 2\delta_{ij}, \alpha_i, \beta = 0, \beta^2 = 1$. Using the anti-commutation rules of the Dirac matrices, it is easy to show that $UU^\dagger = U^\dagger U = \mathbf{1}$ for any λ. The fact that U is unitary means that the transformed wave function

$$\begin{pmatrix} \phi' \\ \chi' \end{pmatrix} \; := \; U \; \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{4.190}$$

has the same norm. Transforming (4.187) gives

$$T \; \begin{pmatrix} \phi' \\ \chi' \end{pmatrix} \; = \; U \; H \; U^{-1} \; \begin{pmatrix} \phi' \\ \chi' \end{pmatrix} \; = \; H' \; \begin{pmatrix} \phi' \\ \chi' \end{pmatrix} \; . \tag{4.191}$$

The individual contributions to H become

$$\begin{aligned} U(-m)U^{-1} \; &= \; -m \\ U \; V^0 \; U^{-1} \; &= \; AV^0 A + \beta\frac{\lambda}{m}(AV^0\vec{\alpha}\cdot\vec{p} - \vec{\alpha}\cdot\vec{p}V^0 A) + \left(\frac{\lambda}{m}\right)^2 \vec{\alpha}\cdot\vec{p}V^0\vec{\alpha}\cdot\vec{p} \\ U\vec{\alpha}\cdot(\vec{p}-\vec{V})U^{-1} \; &= \; -A\vec{\alpha}\cdot(\vec{p}-\vec{V})A \\ &+ \; \beta \; \frac{\lambda}{m} \; \left[A\vec{\alpha}\cdot(\vec{p}-\vec{V}) \; \vec{\alpha}\cdot\vec{p} + \vec{\alpha}\cdot\vec{p}(\vec{p}-\vec{V}) \; A\right] \\ &+ \; \left(\frac{\lambda}{m}\right)^2 \; \vec{\alpha}\cdot\vec{p} \; \vec{\alpha}\cdot(\vec{p}-\vec{V}) \; \vec{\alpha}\cdot\vec{p} \\ U \; m\beta \; U^{-1} \; &= \; m\beta \; A^2 \; + \; 2\lambda \; A\vec{\alpha}\cdot\vec{p} \; - \; \beta\lambda^2 \; \frac{p^2}{m} \; . \end{aligned} \tag{4.192}$$

The off-diagonal terms are those proportional to an **odd power** of $\vec{\alpha}$, and they only need to be calculated to order m^{-1}. Expanding A from (4.189)

$$A \;\cong\; 1 \;-\; \frac{\lambda^2 p^2}{2m^2} \tag{4.193}$$

one finds that the only off-diagonal terms which survive came from the first term on the RHS of the third equation of (4.192) and the second term on the RHS of the fourth equation, and that $A \;\approx\; 1$ is sufficient to get all terms of $\mathcal{O}(m^{-1})$. Collecting all relevant off-diagonal terms gives

$$H^1_{off-diagonal} \;=\; -\vec{\alpha}\cdot(\vec{p}-\vec{V}) + 2\lambda\;\vec{\alpha}\cdot\vec{p} + \mathcal{O}\left(\frac{1}{m^2}\right)\;. \tag{4.194}$$

The choice $\lambda \;=\; \frac{1}{2}$ gives

$$H^1_{off-diagonal} \;\cong\; \vec{\alpha}\cdot\vec{V} \tag{4.195}$$

which is $\mathcal{O}(m^{-1})$ by assumption. With these assumptions the coupled equations (4.184) become

$$\begin{aligned} T\phi' \;&=\; H'_{11}\phi' \;+\; \vec{\sigma}\;\cdot\;\vec{V}\;\chi' \\ T\chi' \;&=\; \vec{\sigma}\;\cdot\;\vec{V}\;\phi' \;-\; 2m\;\chi'\;. \end{aligned} \tag{4.196}$$

With exception of H'_n only the largest terms have been retained in (4.197). Neglecting $T\chi'$ in the second equation and inserting into the first equation gives

$$T\phi' \;=\; \left(H'_{11} \;+\; \frac{\vec{\sigma}\;\cdot\;\vec{V}\;\vec{\sigma}\cdot\;\vec{V}}{2m}\right)\;\phi' \;=\; \left(H'_{11} \;+\; \frac{V^2}{2m}\right)\;\phi'\;. \tag{4.197}$$

It remains to determine H'_{11} using $\lambda \;=\; \frac{1}{2}$. Up to $\mathcal{O}(m^{-3})$ one obtains from (4.192)

$$\begin{aligned} H'_{11} \;&\cong\; V^0 - \frac{p^2}{8m^2}V^0 - V^0\frac{p^2}{8m^2} \;+\; \frac{1}{4m^2}\;\vec{\sigma}\cdot\vec{p}\;V^0\;\vec{\sigma}\cdot\vec{p} \\ &+\; \frac{1}{2m}\;\left[\vec{\sigma}\cdot(\vec{p}-\vec{V})\;\vec{\sigma}\cdot\vec{p} + \vec{\sigma}\cdot\vec{p}\;\vec{\sigma}\cdot(\vec{p}-\vec{V})\right] - \frac{p^4}{8m^3} - \frac{p^2}{2m}\;. \end{aligned} \tag{4.198}$$

Here the first three terms on the RHS come from $A\ V^0\ A$, the first two in the second line on the expansion of the contributions from $U\ \vec{\alpha}\cdot(\vec{p}-\vec{V})\ U^{-1}$, and the last is the combined contribution from $U\ m\beta\ U^{-1}$. Using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ gives for

$$\begin{aligned}
\vec{\sigma}\cdot(\vec{p}-\vec{V})\ \vec{\sigma}\cdot\vec{p}\ \ &+\ \ \vec{\sigma}\cdot\vec{p}\ \vec{\sigma}\cdot(\vec{p}-\vec{V}) \\
&=\ 2p^2\ -\ \vec{\sigma}\cdot\vec{p}\ \vec{\sigma}\cdot\vec{V}\ -\ \vec{\sigma}\cdot\vec{V}\ \vec{\sigma}\cdot\vec{p} \\
&=\ 2p^2\ -\ \vec{p}\cdot\vec{V}\ -\ \vec{V}\cdot\vec{p}\ -\ i\vec{\sigma}\cdot(\vec{p}\times\vec{V})\ -\ i\vec{\sigma}\cdot(\vec{V}\times\vec{p}) \\
&=\ (\vec{p}-\vec{V})^2\ +\ p^2\ -\ V^2\ -\ \vec{\sigma}\cdot(\vec{\nabla}\times\vec{V})\ . \qquad (4.199)
\end{aligned}$$

For $V^\mu \equiv eA^\mu$ the term $\vec{\sigma}\cdot(\vec{\nabla}\times\vec{V}) = e\vec{\sigma}\cdot\vec{B}$, describing a magnetic moment interaction. The second through fourth term in (4.199) reduce to

$$\begin{aligned}
p^2V^0\ +\ V^0p^2\ &=\ \left[p^2V^0\right]\ +\ 2\left[\vec{p}\ V^0\right]\cdot\vec{p}\ +\ 2V^0\ p^2 \\
\vec{\sigma}\ \cdot\ \vec{p}\ V^0\ \vec{\sigma}\ \cdot\ \vec{p}\ &=\ \vec{\sigma}\ \cdot\ \left[\vec{p}\ V^0\right]\ \vec{\sigma}\cdot\vec{p}\ +\ V^0p^2 \\
&=\ \left[\vec{p}\ V^0\right]\cdot\vec{p}\ +\ i\vec{\sigma}\ (\left[\vec{p}\ V^0\right]\times\vec{p})\ +\ V^0p^2\ . \qquad (4.200)
\end{aligned}$$

Thus

$$-\ \frac{1}{8m^2}\ (p^2V^0\ +\ V^0p^2)\ +\ \frac{1}{4m^2}\ \vec{\sigma}\cdot\vec{p}\ V^0\ \vec{\sigma}\cdot\vec{p}\ =\ -\ \frac{[p^2V^0]}{8m^2}\ +\ \frac{i\vec{\sigma}\cdot([\vec{p}\ V^0]\times\vec{p})}{4m^2}\ . \qquad (4.201)$$

Here the use of the square brackets means that $\vec{p}$ (or $\vec{\nabla}$) operate only within the brackets. Replacing $\vec{p} = -i\vec{\nabla},\ \vec{V} = e\vec{A}\ V^0 = e\varphi(r)$ gives

$$H'_{11} + \frac{V^2}{2m} = \frac{(\vec{p}-e\vec{A})^2}{2m} + e\varphi - \frac{p^4}{8m^2} - \frac{e}{2m}\ \vec{\sigma}\cdot\vec{B}\ +\ \frac{e\,[\nabla^2\varphi]}{8m^2}\ +\ \frac{e}{4m^2}\ \vec{\sigma}\cdot([\nabla\phi]\times\vec{p}). \qquad (4.202)$$

Here the term $\frac{(\vec{p}-e\vec{A})^2}{2m}$ contains the usual Zeeman effect for spinless particles. Assuming that the potential is spherically symmetric, i.e., $\varphi \equiv \varphi(r)$, leads to $\vec{\nabla}\varphi = \frac{\vec{r}}{r}\ \frac{\partial\varphi}{\partial r}$ and gives the final effective Hamiltonian of (4.179).

The additional terms in (4.179) are

- $p^4/8m^3\ \ \equiv$ a relativistic mass increase

- $\frac{e}{8m^2}\nabla^2\varphi = -\frac{e}{8m^2}\vec{\nabla}\cdot E = \frac{Ze^2}{8m^2}\delta^3(r) \quad \equiv$ Darwin term for an electron in an atom with a point nucleus.
  Because of $\delta^3(r)$ this term is non-zero only for S-states. Physically it comes from quantum fluctuations in the position of the electron (*Zitterbewegung*), which make the electron sensitive to the average charge density in the vicinity of its average position.

- $\frac{e}{4m^2}\frac{1}{r}\frac{d\varphi}{dr}\vec{\sigma}\cdot\vec{L} = \frac{1}{2m^2}(\frac{1}{r}\frac{d\varphi}{dr})\vec{S}\cdot\vec{L} \quad \equiv$ spin-orbit term.

The last problem to address is whether there are general guidelines to construct the unitary operator U (4.188), which removes from (4.187) all operators of type $\vec{\alpha}$, which mix the small component with the large. Transforming the Hamiltonian for a Dirac particle so that the separation into positive and negative states is diagonal is called the **Foldy-Wouthuysen** transformation. It is possible to give the exact form of the transformation only in a few cases: those in which the Dirac equation can be solved exactly. In general, it is given iteratively, diagonalizing H to any desired order in $\frac{1}{c}$. This means that the method provides a systematic way to evaluate the relativistic corrections to the non-relativistic Schrödinger equation.

Consider the Dirac equation for a particle in a potential

$$H \;=\; m\beta \;+\; V \;+\; V_s\beta \;+\; \vec{\alpha}\;\cdot\;(\vec{p}\;-\;e\vec{A}) \tag{4.203}$$

where V_s is a scalar potentials, and $V \;\equiv\; V^0 \;+\; A^0$.

The Hamiltonian is of the form

$$H \;=\; mc^2\left(\beta \;+\; \frac{\mathbf{D}}{c^2} \;+\; \frac{\Omega}{c}\right)\;, \tag{4.204}$$

where the operator $\mathbf{D}$ is "even" or diagonal, i.e.,

$$\mathbf{D} \;=\; \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \tag{4.205}$$

and Ω is "odd" or anti-diagonal

$$\Omega \;=\; \begin{pmatrix} 0 & \omega_1 \\ \omega_2 & 0 \end{pmatrix} \;, \tag{4.206}$$

and the d_i, ω_j are 2×2 matrices. Given an operator like (4.204), i.e., the sum of a diagonal and anti-diagonal one, one can find a unitary transformation, e^T (with $T^\dagger \;=\; -T$), such that

$$H' \;=\; e^T \; H \; e^{-T} \tag{4.207}$$

is also of the form (4.204), but with the odd term of higher order in $\frac{1}{c}\left(\frac{1}{m}\right)$. In the specific case (4.204), one will get

$$H' \;=\; mc^2 \; \left(\beta \;+\; \frac{1}{c^2}\,\mathbf{D}' \;+\; \frac{1}{c^3}\,\Omega'\right) \;. \tag{4.208}$$

Iterating these transformations, one can get the odd term to be of order $\frac{1}{c^n}$ with n as large as one wishes. Therefore, to a given order, the resulting Hamiltonian will be equivalent to one where positive and negative energies are separated. This is the Foldy-Wouthuysen method. To implement this procedure, note that for any operators, F, T one has the relation

$$\begin{aligned} e^T \; F \; e^{-T} \;&=\; F \;+\; [T,\; F] \;+\; \frac{1}{2!}\;[T,\;[T,\;F]] \qquad (4.209)\\ &+\; \frac{1}{3!}\;[T,\;[T,\;[T,\;F]]] \;+\; \cdots \\ &=\; \sum_{n=0}^{\infty} n![T,[T,[\cdots[T,F]]\cdots]] \qquad (4.210) \end{aligned}$$

This can be shown by considering

$$F(\lambda) \;=\; e^{\lambda T} \; F \; e^{-\lambda T} \;=\; \sum_{n=0}^{\infty} \frac{\lambda^u}{n!}\;\left(\frac{\partial^n F}{\partial \lambda^n}\right)_{\lambda=0} \tag{4.211}$$

which gives, e.g.,

$$\frac{\partial F}{\partial \lambda} \;=\; e^{\lambda T}\;[T,\;F]\;e^{-\lambda T} \;. \tag{4.212}$$

On the other hand, for any odd operator O,

$$[\beta O,\; \beta] \;=\; -2\;O \tag{4.213}$$

one can apply a transformation like (4.210) with H replacing F, and profit from (4.213) to choose T proportional to $\beta\mathbf{O}$, so that the dominating odd term of H gets canceled by a piece of $[T, H]$. Let $\mathbf{h} = (\frac{1}{mc^2})H$, where H is the Dirac Hamiltonian. Then

$$\begin{aligned} \mathbf{h} &= \beta + \frac{1}{c^2}\,\mathbf{D} + \frac{1}{c}\,\Omega \\ \mathbf{D} &= \frac{1}{m}\,(V + V_s\,\beta) \\ \Omega &= \frac{1}{m}\,\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) . \end{aligned} \tag{4.214}$$

We then apply the transformation

$$T = \frac{1}{2c}\,\beta\Omega . \tag{4.215}$$

As a reminder, $(\beta\mathbf{O})^\dagger = -(\beta\mathbf{O})$ for any Hermitean, odd operator $\mathbf{O}$. Thus $\exp\left(\frac{\beta\Omega}{2c}\right)$ is unitary.

All the transformations will have the structure (4.215). Using (4.210) the transformed operator is given as

$$\begin{aligned} \mathbf{h}' &\equiv e^T\,\mathbf{h}\,e^{-T} = \beta + \frac{1}{c^2}\,\mathbf{D} + \frac{1}{c}\,\Omega + \frac{1}{2c}\,[\beta\Omega,\ \mathbf{h}] \\ &+ \frac{1}{2!}\,\frac{1}{(2c)^2}\,[\beta\Omega,\ [\beta\Omega,\ \mathbf{h}]] + \frac{1}{3!}\,\frac{1}{2c^3}\,[\beta\Omega,\ [\beta\Omega,\ [\beta\Omega,\ \mathbf{h}]]] \\ &+ \frac{1}{4!}\,\frac{1}{(2c)^4}\,[\beta\Omega,\ [\beta\Omega,[\beta\Omega,[\beta\Omega,\mathbf{h}]]]] + \mathcal{O}\left(c^{-5}\right) . \end{aligned} \tag{4.216}$$

In order to evaluate the first relativistic corrections, one has to go to order $\frac{1}{c^2}$ in H, i.e., one has to be exact to order $\frac{1}{c^4}$ in $\mathbf{h}, \mathbf{h}', \mathbf{h}''$ . . . Because of (4.210) and (4.213) the commutator $[\beta\Omega,\ \mathbf{h}] \approx [\beta\Omega,\ \beta]$ up to higher orders in $\frac{1}{c}$, and this term

$$\frac{1}{2c}\,[\beta\Omega,\ \beta] = -\frac{1}{c}\,\Omega \tag{4.217}$$

will cancel the term $(\frac{1}{c})\ \Omega$ in $\mathbf{h}$.

This (4.216) gives

$$\mathbf{h}' = \beta + \frac{1}{c^2}\,\mathbf{D} + \frac{1}{2c^2}\,[\beta\Omega,\ \Omega] + \frac{1}{2c^3}\,[\beta\Omega,\ \mathbf{D}] \tag{4.218}$$

$$+ \quad \frac{1}{2!} \, \frac{1}{(2c)^2} \; [\beta\Omega, \; [\beta\Omega, \; \mathbf{h}]] \; + \; \frac{1}{3!} \, \frac{1}{(2c)^3} \; [\beta\Omega, \; [\beta\Omega, \; [\beta\Omega, \; \mathbf{h}] \,] \,]$$

$$+ \quad \frac{1}{4!} \, \frac{1}{(2c)^4} \; [\beta\Omega, \; [\beta\Omega, \; [\beta\Omega, [\beta\Omega, \; \mathbf{h}] \,] \,] \,] \; + \; \mathcal{O}(c^-5)$$

We have even $\times$ even = even = odd $\times$ odd, odd $\times$ even = odd, so, the odd term in (4.219) is $\mathcal{O}(c^{-3})$.

For the explicit calculation, one can use the fact that in our case

$$\{\Omega, \beta\} \; = \; 0 \quad ; \quad [\mathbf{D}, \; \beta] \; = \; 0 \tag{4.219}$$

to obtain the relations

$$\begin{aligned}
[\beta\Omega, \; \beta] \; &= \; -2\Omega \\
[\beta\Omega, \; [\beta\Omega, \; \beta] \,] \; &= \; 4\beta\Omega^2 \\
[\beta\Omega, \; [\beta\Omega, \; [\beta\Omega, \; \beta] \,] \,] \; &= \; 8\Omega^3 \\
[\beta\Omega, \; [\beta\Omega, \; [\beta\Omega, \; [\beta\Omega, \; \beta] \,] \,] \,] \; &= \; 16\beta\Omega^4 \\
[\beta\Omega, \; \Omega] \; &= \; 2\beta\Omega^2 \\
[\beta\Omega, [\beta\Omega, \; \Omega] \,] \; &= \; 4\Omega^3 \\
[\beta\Omega, \; [\beta\Omega, \; [\beta\Omega, \; \Omega] \,] \,] \; &= \; 8\beta\Omega^4 \\
[\beta\Omega, \; [\beta\Omega, \; \mathbf{D}] \,] \; &= \; - \; [\Omega, \; [\Omega, \; \mathbf{D}] \,]
\end{aligned} \tag{4.220}$$

With these we get

$$\begin{aligned}
\mathbf{h}' \; = \; & \beta \; + \; \frac{1}{2c^2} \, \mathbf{D} \; + \; \frac{1}{2c^2} \, \beta\Omega^2 \; - \; \frac{1}{8c^4} \, \beta\Omega^4 \; - \; \frac{1}{4c^2} \; [\Omega, \; [\Omega, \; \mathbf{D}]] \\
- & \; \frac{1}{3c^3} \, \Omega^3 \; + \; \frac{1}{2c^3} \; \beta \; [\Omega, \; \mathbf{D}] \; + \; \mathcal{O}(c^{-5})
\end{aligned} \tag{4.221}$$

which we rewrite as

$$\begin{aligned}
\mathbf{h}' \; &= \; \beta \; + \; \frac{1}{c^2} \, \mathbf{D}' \; + \; \frac{1}{c^3} \, \Omega' \; + \; \mathcal{O}(c^{-5}) \\
\mathbf{D}' \; &= \; \mathbf{D} \; + \; \frac{1}{2} \, \beta\Omega^2 \; - \; \frac{1}{8c^2} \, \beta\Omega^4 \; - \; \frac{1}{8c^4} \; [\Omega, \; [\Omega, \; \mathbf{D}]] \\
\Omega' \; &= \; - \, \frac{1}{3} \, \Omega^3 \; + \; \frac{1}{2} \, \beta \; [\Omega, \; \mathbf{D}] \; .
\end{aligned} \tag{4.222}$$

Performing a new transformation

$$\begin{aligned}
\mathbf{h}'' \; &= \; e^{T\prime} \, \mathbf{h}' \, e^{-T'} \\
T' \; &= \; \frac{1}{2c^3} \, \beta\Omega'
\end{aligned} \tag{4.223}$$

we eliminate the term $\left(\frac{1}{c^3}\right)\ \Omega'$ in $\mathbf{h}'$ and obtain

$$\begin{aligned} \mathbf{h}'' &= \mathbf{h}' + \frac{1}{2c^3}\,[\beta\Omega,\ \mathbf{h}'] + \frac{1}{(2c^3)^2}\,[\beta\Omega',\ [\beta\Omega',\ \mathbf{h}']\,] + \ \ldots \\ &= \beta + \frac{1}{c^2}\,\mathbf{D}' + \mathcal{O}(c^{-5})\ . \end{aligned} \tag{4.224}$$

We have found that up to terms of order c^{-3} the Hamiltonian H of (4.203) is unitarily related to

$$\begin{aligned} \mathbf{H}'' &= mc^2\,\mathbf{h}'' = mc^2\,\beta + m\mathbf{D}' + \mathbf{O}(c^{-2}) \\ \mathbf{D}' &= \mathbf{D} + \frac{1}{2}\,\beta\Omega^2 - \frac{1}{8c^2}\,\beta\Omega^4 - \frac{1}{8c^2}\,[\Omega,\ [\Omega,\ \mathbf{D}]\,]\ . \end{aligned} \tag{4.225}$$

With the explicit expressions for $\mathbf{D},\ \Omega$, one obtains

$$\begin{aligned} \Omega^4 &= \frac{1}{m^4}\,(\vec{p}^{\,2})^2 + \mathcal{O}\left(\frac{1}{c}\right) \\ \Omega^2 &= \frac{1}{m^2}\,\left[\vec{\alpha}\cdot\left(\vec{p}-\frac{e}{c}\,\vec{A}\right)\right]^2 = \frac{1}{m^2}\,\left(\vec{p}-\frac{e}{c}\,\vec{A}\right)^2 - \frac{e}{m^2c^2}\,\vec{\sigma}\cdot\vec{B} \end{aligned} \tag{4.226}$$

and thus we see that $H'' = H_{FW} + \mathcal{O}(c^{-3})$, where the Foldy-Wouthuysen Hamiltonian H_{FW} is defined as

$$\begin{aligned} H_{FW} &= mc^2\beta + V_s\beta + \frac{1}{2m}\beta\left(\vec{p}-\frac{1}{c}\vec{A}\right)^2 - \frac{e}{2mc^2}\beta\vec{\sigma}\cdot\vec{B} \\ &- \frac{1}{8m^3c^2}\beta\vec{p}^4 - \frac{1}{8m^2c^2}\,[\vec{\alpha}\cdot\vec{p},\ [\vec{\alpha}\cdot\vec{p},\ V]\,] \\ &- \frac{1}{8m^2c^2}\,\beta\,\{\vec{\alpha}\cdot\vec{p},\ \{\vec{\alpha}\cdot\vec{p},\ V_s\}\}\,. \end{aligned} \tag{4.227}$$

The positive energy solutions of the Dirac equation will be represented by wave functions

$$\psi_{FW} = e^{T'}\,e^{T}\,\psi + \mathcal{O}(c^{-3}) \tag{4.228}$$

with $\beta\ \psi_{FW}\ =\ \psi_{FW}$. In this physical subspace β can be replaced by unity and one can use two component wave functions

$$\psi_{FW} \ = \ \begin{pmatrix} \psi_{s.r.} \\ 0 \end{pmatrix} \tag{4.229}$$

where $\psi_{s.r.}$ is the semi-relativistic wave function correct to order c^{-2}.

4.8 Physical Problems of the Dirac Theory: Spin Conservation and Zitterbewegung

After having seen the successes of the Dirac theory, we study some "strange" effects. For doing this, consider the time dependence of operators in the Heisenberg representation.

For an operator A, which does not explicitly depend on the time t, one has ($\hbar \;=\; 1$)

$$\frac{d}{dt}\, A \;=\; i\,[H, A] \;. \tag{4.230}$$

Let us consider the time dependence of the operators

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ \vec{S} &= \frac{1}{2}\,\vec{\Sigma} \\ \vec{r} & \end{aligned} \tag{4.231}$$

for the free Dirac equation, where the Hamiltonian is of the form

$$\vec{\alpha}\vec{p} \;+\; \beta m \;. \tag{4.232}$$

For the orbital angular momentum $\vec{L}$, one obtains

$$\frac{d}{dt}\,\vec{L} \;=\; i\left[H, \vec{L}\right] \;=\; i(\vec{\alpha})_\ell \;\left[p_\ell, \vec{L}\right] \;. \tag{4.233}$$

Since $\vec{p}$ is a vector:

$$[L_k, p_\ell] \;=\; i\;\epsilon_{k\ell m}\;p_m \tag{4.234}$$

and thus

$$\frac{d}{dt}\,\vec{L} \;=\; \vec{\alpha} \;\times\; \vec{p}\;. \tag{4.235}$$

The right-hand side of this equation is in general a non-vanishing quantity, and thus the orbital angular momentum changes over time without any external force. For Dirac this was an argument that his equation contains an additional angular momentum, namely the spin. Indeed, one has

$$\begin{aligned} \frac{d}{dt}\,\vec{S} &= \frac{i}{2}\;\left[(\vec{\alpha}\vec{p} \;+\; \beta m),\; \vec{\Sigma}\right] \\ &= \frac{i}{2}\;\left[(\alpha)_k,\; \vec{\Sigma}\right]\;p_k \;+\; \frac{i}{2}\;\left[\beta m,\; \vec{\Sigma}\right] \;. \end{aligned} \tag{4.236}$$

The standard representation of the matrices leads to the following relations

$$\begin{aligned} \left[(\vec{\alpha})_k,\ (\vec{\Sigma})_\ell\right] &= 2i\ \epsilon_{k\ell m}\ (\vec{\alpha})_m \\ \left[\beta,\ (\vec{\Sigma})_\ell\right] &= 0\ . \end{aligned} \tag{4.237}$$

Thus, one obtains

$$\frac{d}{dt}\,\vec{S} \;=\; -\vec{\alpha}\ \times\ \vec{p}\ , \tag{4.238}$$

and the operator

$$\vec{J}\ :=\ \vec{L}\ +\ \vec{S} \tag{4.239}$$

is time independent, i.e., $\frac{d}{dt}\,\vec{J}\ =\ 0$. $\vec{J}$ is interpreted as total angular momentum, the sum of orbital angular momentum $\vec{L}$ and spin $\vec{S}$. At first glance, the above result is convincing. However, it leads to the question why the spin should change in a free motion. Before trying to answer this question, consider the velocity of a Dirac particle:

$$\begin{aligned} \vec{v} &= \frac{d}{dt}\,\vec{r}\ =\ i\ [H,\vec{r}] \\ &= i(\vec{\alpha})_k\ [p_k,\vec{r}]\ . \end{aligned} \tag{4.240}$$

The canonical commutation relations lead to

$$\vec{v}\ =\ \vec{\alpha}\ . \tag{4.241}$$

The velocity is obviously described through the hermitian matrix $\vec{\alpha}$, i.e., using c one had $\vec{v}\ =\ c\vec{\alpha}$.
This equation has two problems:

1. For the components of $\vec{\alpha}$, one has due to

$$(\vec{\alpha})_k^2\ =\ \mathbf{1} \tag{4.242}$$

only the eigenvalues $+\,1$ and $-\,1$. Thus, one can expect for the values of the velocity components only $+\,c$ and $-\,c$, i.e.,

$$v_k\ =\ \pm\ c\ . \tag{4.243}$$

2. Due to

$$[(\vec{\alpha})_k,\ (\vec{\alpha})_\ell]\ \neq\ 0 \qquad \text{for}\ \ k\ \neq\ \ell \tag{4.244}$$

different components of $\vec{v}$ cannot be measured simultaneously.

Schrödinger interpreted the result of (4.243) through the so-called "Zitterbewegung." In order to reach the point $(ct_0;\ a, 0, 0)$ starting from $(0; 0, 0, 0)$ an electron has to move alternately with $+\ c$ and $-\ c$, as indicated in Fig. 11.1.

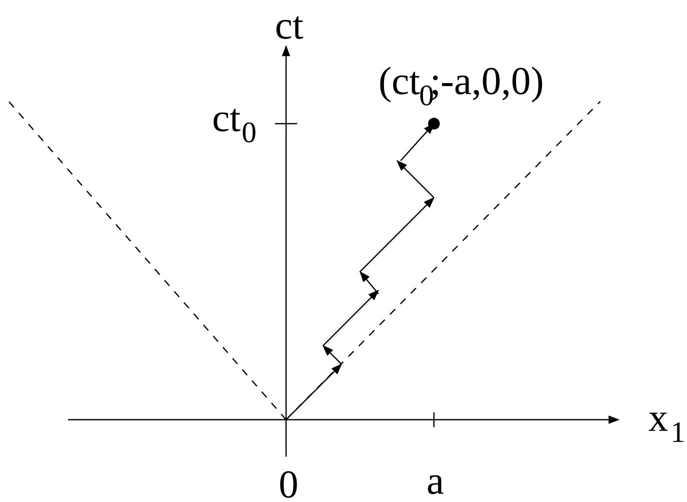

Figure 11.1 Illustration of the Zitterbewegung in a space-time diagram.

The following considerations show that with this concept, one can overcome the above discussed conceptual difficulties.

An electron with a fixed energy $p^0\ >\ 0$ and momentum $\vec{p}$ is described as an eigenstate of the Hamiltonian. Its velocity has because of

$$[H,\ \vec{\alpha}]\ \neq\ 0 \tag{4.245}$$

no sharp eigenvalue. Using the explicit representation of the spinor (4.129) we can calculate the expectation value of $\vec{\alpha}$.

$$\langle\vec{v}\rangle\ =\ \langle\vec{\alpha}\rangle$$

$$
\begin{aligned}
&= u^\dagger\,(\vec{p})\vec{\alpha}\;u(\vec{p}) \\
&= N^2\;\eta^\dagger\;\frac{\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\;+\;(\vec{\sigma}\cdot\vec{p})\;\vec{\sigma}}{p^0\;+\;m}\;\eta \\
&= N^2\;\frac{2\vec{p}}{p^0\;+\;m}\;\eta^\dagger\;\eta\;. \qquad (4.246)
\end{aligned}
$$

Here $\sigma_k\;(\vec{\sigma}\cdot\vec{p})\;+\;(\vec{\sigma}\cdot\vec{p})\;\sigma_k\;=\;(\sigma_k\sigma_\ell\;+\;\sigma_\ell\sigma_k)\;p_\ell\;=\;2p_k$ was used. With $\eta^\dagger\eta\;=\;1$ and (4.128) follows

$$
\langle\vec{v}\rangle\;=\;\frac{\vec{p}}{p_0}\;=\;\vec{v}_{classical}\;. \qquad (4.247)
$$

As far as the expectation value is concerned, we obtain the usual physical result. Similarly, this holds for the somewhat unexpected results (4.235) and (4.238). We obtain for the expectation values

$$
\begin{aligned}
\langle\frac{d}{dt}\;\vec{L}\rangle\;&=\;u^\dagger\;(\vec{p})\;(\vec{\alpha}\;\times\;\vec{p})\;u(\vec{p}) \\
&=\;u^\dagger\;\vec{\alpha}u\;\times\;\vec{p} \\
&=\;\frac{\vec{p}}{p_0}\;\times\;\vec{p} \\
&=\;\vec{0} \qquad (4.248)
\end{aligned}
$$

thus

$$
\langle\frac{d\vec{L}}{dt}\rangle\;=\;\vec{0}\;=\;\langle\frac{d\vec{S}}{dt}\rangle\;. \qquad (4.249)
$$

The change of $\vec{L}$ and $\vec{S}$ is thus caused by the Zitterbewegung. In general, the above calculations can be carried out for arbitrary wave packages with positive energies:

$$
\psi(\vec{x})\;=\;\frac{1}{(2\pi)^{\frac{3}{2}}}\;\int\;d^3p\;\left[e^{-i\vec{p}\vec{x}}\;\sum_{\lambda=\pm1}\;a_\lambda(\vec{p})\;u_\lambda(\vec{p})\right]\;. \qquad (4.250)
$$

One obtains:

$$
\begin{aligned}
\langle\vec{\alpha}\rangle\;&:=\;\int\;d^3x\;\psi^\dagger\;(\vec{x})\;\vec{\alpha}\;\psi(x) \\
&=\;\sum_\lambda\;\int\;d^3p\;\mid a_\lambda(\vec{p})\mid^2\;\frac{\vec{p}}{p_0} \qquad (4.251)
\end{aligned}
$$

and correspondingly

$$\langle\frac{d\vec{L}}{dt}\rangle \; := \; \int \; d^3x \; \psi^\dagger \; (\vec{x}) \; \frac{d\vec{L}}{dt} \; \phi(\vec{x}) \; = \; \vec{0} \tag{4.252}$$

and

$$\langle\frac{d\vec{S}}{dt}\rangle \; = \; \vec{0} \; . \tag{4.253}$$

It is important that in (4.250) only positive energies occur. If one allows negative energies, one encounters again the problems described in (4.65), (4.66) and (4.70).

4.9 Complications with Negative Energies, the Klein Paradox

Since the negative energy solutions proved to be the formal reason for the Zitterbewegung, we have to study them in more detail.

First we rewrite the two spinors u^L and u^S of a solution of the system of equations (4.104) for

$$p^0 \;=\; -\sqrt{\vec{p}^{\,2} \;+\; m^2} \tag{4.254}$$

in the form

$$\begin{aligned} u^S &= \eta \\ u^L &= \frac{\sigma \;\cdot\; \vec{p}}{p^0 \;-\; m}\,\eta \end{aligned} \tag{4.255}$$

with $\eta^\dagger \eta \;=\; 1$. Thus, we have

$$u \;=\; N \begin{pmatrix} \frac{\vec{\sigma} \,\cdot\, \vec{p}}{p^0 \,-\, m}\,\eta \\ \eta \end{pmatrix} \tag{4.256}$$

with $p^0 \;<\; 0$. From $u^\dagger u \;=\; 1$ follows

$$N \;=\; \sqrt{\frac{\mid p^0 \mid \;+ m}{2 \mid p^0 \mid}}\;. \tag{4.257}$$

From this one finds for the expectation value

$$\langle \vec{v} \rangle \;=\; \langle \vec{\alpha} \rangle \;=\; \frac{\vec{p}}{p^0} \;=\; -\;\frac{\vec{p}}{\mid p^0 \mid}\;. \tag{4.258}$$

Due to $p^0 \;<\; 0$, the expectation values of the velocity and the momentum have opposite directions, which contains further problems. The real danger of the negative energies is illustrated in Fig. 12.1: The allowed energy eigenvalues, $E \;>\; mc^2$ and $E \;<\; -mc^2$ are separated by an energy gap of width $2mc^2$.

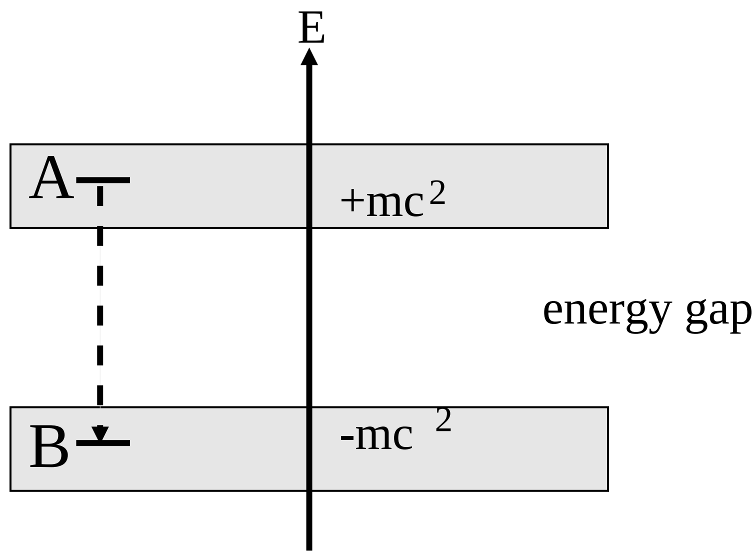

Figure 12.1 The energy spectrum of the free Dirac equation.

Under suitable conditions an electron could jump from level A with $p^0 > 0$ to level B with $p^0 < 0$, e.g., by emitting a photon with energy $E_\gamma \geq 2mc^2$. For satisfying momentum conservation, a nucleon or nucleus could take the recoil momentum. Since the spectrum is not bounded from below, $E_{electron} \rightarrow -\infty$ and matter in our usual understanding could not exit.

The problem with negative energies shall be demonstrated at a specific example, the so-called **Klein paradox**. Here the problems with the negative energies occur with respect to the potential V_0, which increases by an amount larger than $2mc^2$ within a finite space region. The situation is illustrated in Fig. 12.2.

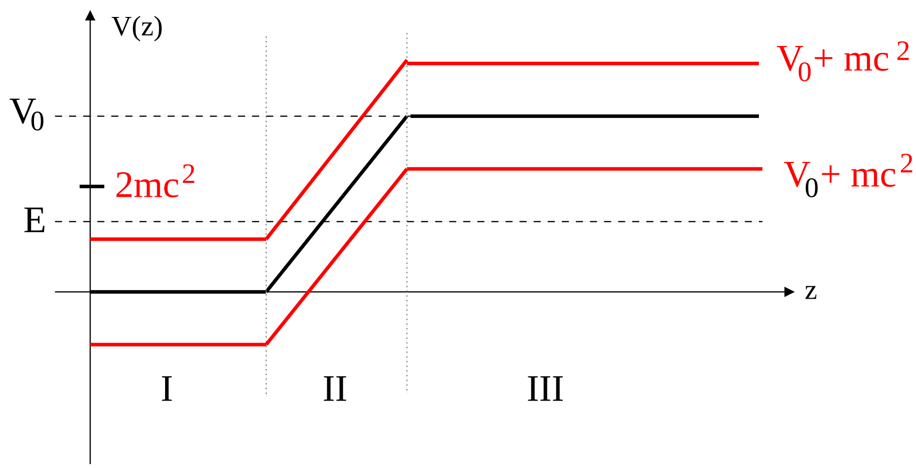

Figure 12.2 Potential difference and tunneling through the energy gap.

For the potential, one has

$$\begin{aligned} V &= 0 \qquad \text{in region I} \\ V &= V_0 > 2mc^2 \qquad \text{in region III} \end{aligned} \tag{4.259}$$

In the intermediate region II, V is supposed to increase linearly. If there is a particle in region I with energy $E = cp^0$ and

$$V_0 + mc^2 > E > V_0 - mc^2 \ , \tag{4.260}$$

it can not enter the region III, since its energy would be in the forbidden region. Its spinor decays exponentially. For smaller particle energies,

$$V_0 - mc^2 > E \geq mc^2 \tag{4.261}$$

region II is allowed, i.e., oscillating solutions are expected. For the explicit calculation, we consider a potential with a jump of the site V_0 at $z = 0$ (Fig. 12.3).

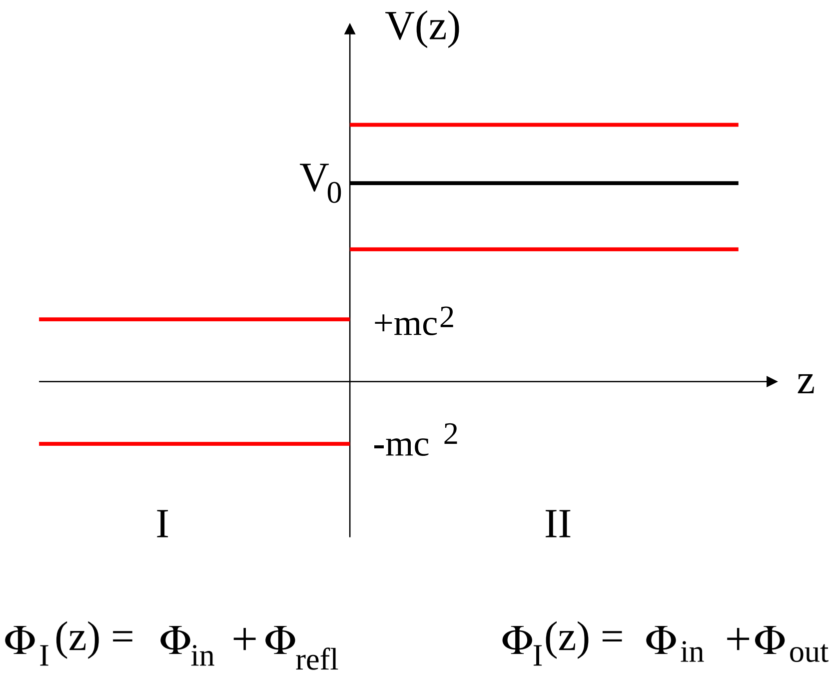

Figure 12.3 Potential jump for the explicit illustration of the Klein paradox.

In region I we have the free Dirac equation, while in region II a potential $V_0 = e\phi_0$ acts on the particle. We consider only one dimension and a fixed energy E, thus we have ($\hbar = c = 1$):

$$\begin{aligned} \text{Region I :} \quad & \left[\gamma^0 E \;+\; i\gamma^3 \frac{\partial}{\partial z} \;-\; m\right] \psi(x) \;=\; 0 \\ \text{Region II :} \quad & \left[\gamma^0 (E \;-\; V_0) \;+\; i\gamma^3 \frac{\partial}{\partial z} \;-\; m\right] \psi(x) \;=\; 0 \; . \end{aligned} \tag{4.262}$$

The solution of (4.262) in region I contains an incoming plane wave (moving to the right) and the term containing a reflected wave going to the left.

$$\begin{aligned} \psi_I(z) &= \psi_{in} \;+\; \psi_{refl} \\ &= a\, e^{ipz}\, u_1 \;+\; b\, e^{-ipz}\, u_2 \end{aligned} \tag{4.263}$$

with $p = \mid \vec{p} \mid = +\sqrt{E^2 - m^2}$.

According to (4.124), we can write for a spin $+\frac{1}{2}$ in z-direction

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} \quad ; \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{-p}{E+m} \\ 0 \end{pmatrix} . \tag{4.264}$$

In region II, one has an outgoing wave to the right, which has to fulfill (4.262). We make the ansatz

$$\begin{aligned} \psi_{II}(z) &= \psi_{out} \\ &= d\, e^{ip'z}\, u_3 \end{aligned} \tag{4.265}$$

with $p' = |\vec{p}\,'| = +\sqrt{(E-V_0)^2 - m^2}$ and

$$u_3 = \begin{pmatrix} 1 \\ 0 \\ \frac{p'}{E_0 - V_0 + m} \\ 0 \end{pmatrix} . \tag{4.266}$$

Since the Dirac equation is of first order with respect to z, $\psi(x)$ itself, i.e., each of the four components, has to be continuous at $z = 0$

$$\psi_I(0) = \psi_{II}(0) \tag{4.267}$$

or

$$\begin{aligned} a + b &= d \quad \text{(1. component)} \\ \frac{p}{E + m}(a-b) &= \frac{p'}{E - V_0 + m}\, d \quad \text{(3. component)} \\ (a-b) &= \frac{p'}{p} \frac{E + m}{E - V_0 + m}\, d \\ &= rd \end{aligned} \tag{4.268}$$

with

$$r := \frac{p'}{p} \frac{E + m}{E + m - V_0} . \tag{4.269}$$

Without loosing generality, the coefficients a, b, d can be chosen real. Multiplying the first two equations yields

$$a^2 - b^2 = rd^2 . \tag{4.270}$$

Since in the energy region under consideration $r < 0$, one has

$$a^2 < b^2 . \tag{4.271}$$

Thus the incoming wave is weaker than the reflected one. This is the Klein paradox: Upon reflection on the potential well $V_0 > E + mc^2$ probability is produced.

The formal reason for this physically unreasonable result becomes clear when considering the probability currents of the three waves. In all cases the spin contribution of

$$j^u = \frac{i}{2m} \bar{\psi} \overset{\leftrightarrow}{\partial^\mu} \psi + \frac{1}{2m} \partial_v (\bar{\psi} \sigma^{\mu\nu} \psi) \tag{4.272}$$

vanishes, and one obtains

$$\begin{aligned} j_{in} &= \frac{1}{2mi} \psi_{in} \frac{\overset{\leftrightarrow}{d}}{dz} \psi_{in} \\ &= a^2 \frac{p}{m} \bar{u}_1 u_1 \\ &= a^2 \frac{p}{m} \left(1 - \left(\frac{p}{E + m} \right)^2 \right) \\ &= a^2 \frac{2p}{E + m} \end{aligned} \tag{4.273}$$

$$\begin{aligned} j_{refl} &= \frac{1}{2mi} \bar{\psi}_{refl} \frac{\overset{\leftrightarrow}{d}}{dz} \psi_{refl} \\ &= -b^2 \frac{p}{m} \bar{u}_2 u_2 \\ &= -b^2 \frac{2p}{E + m} . \end{aligned} \tag{4.274}$$

The last part follows because $\bar{u}_2 u_2 = \bar{u}_1 u_1$.

$$j_{out} = \frac{1}{2mi} \psi_{out} \frac{\overset{\leftrightarrow}{d}}{dz} \psi_{out}$$

$$
\begin{aligned}
&= d^2 \, \frac{p'}{m} \, \bar{u}_3 \, u_3 \\
&= d^2 \, \frac{p'}{m} \left(1 \; - \; \left(\frac{p'}{E \; - \; V_0 \; + \; m} \right)^2 \right) \\
&= d^2 \, \frac{2p'}{E \; + \; m \; - \; V_0} \; . \qquad (4.275)
\end{aligned}
$$

In (4.275) the current is oriented opposite to the momentum due to the negative numerator (cp. 4.258). From (4.273) and (4.274) follows

$$
\begin{aligned}
j_{in} \; + \; j_{refl} \;\; &= \;\; (a^2 \; - \; b^2) \, \frac{2p}{E \; + \; m} \\
&= \;\; d^2 \, \frac{p'}{p} \, \frac{E \; + \; m}{E \; + \; m \; - \; V_0} \, \frac{2p}{E \; + \; m} \\
&= \;\; d^2 \, \frac{2p'}{E + \; m \; - \; V_0} \; , \qquad (4.276)
\end{aligned}
$$

i.e., one has

$$
j_{in} \; + \; j_{refl} \; = \; j_{out} \; . \qquad (4.277)
$$

Furthermore, one has with $\frac{d}{a} \; = \; \frac{2}{1 \; + \; r}$ and $\frac{b}{a} \; = \; \frac{1 \; - \; r}{1 \; + \; r}$ from (4.268) and (4.269) that

$$
\begin{aligned}
\frac{j_{in}}{j_{out}} \;\; &= \;\; \frac{d^2}{a^2} \, \frac{p'}{p} \, \frac{E \; + \; m}{E \; + \; m \; - \; V_0} \\
&= \;\; \frac{d^2}{a^2} \, r \\
&= \;\; \left(\frac{2}{1 \; + \; r} \right)^2 \, r \\
&= \;\; \frac{4r}{(1 \; + \; r)^2} \; . \qquad (4.278)
\end{aligned}
$$

In addition, one obtains

$$
\frac{|\, j_{refl} \,|}{j_{in}} \; = \; \frac{b^2}{a^2} \; = \; \left(\frac{1 \; - \; r}{1 \; + \; r} \right)^2 \; . \qquad (4.279)
$$

Due to $r \; < \; 0$ this means that incoming and outgoing current have opposite direction, and the reflected current is larger than the incoming.

In summary:
When tunneling through the energy gap in the region of a potential step $V_0 \; > \; 2mc^2$, one has anomalies in the probability currents.

4.10 Interpretation of the Negative Energies

The difficulties shown in the last chapter are removed if one considers the **Dirac sea**.

Here Dirac applies the Pauli-principle, where every quantum state can only be occupied with one electron. For each momentum $p = \mid \vec{p} \mid$, there exist only four electrons, namely two with the same sign for the energy, one with spin $+\frac{1}{2}$ and one with spin $-\frac{1}{2}$. According to Dirac the states with the negative energies is completely filled. Thus, one has an "underworld," the **Dirac sea** with the following properties:

$$\begin{aligned} \text{total energy} &: -\infty \\ \text{total charge} &: -\infty \\ \text{total momentum} &: 0 \\ \text{total angular momentum} &: 0\ . \end{aligned} \tag{4.280}$$

Total momentum and angular momentum vanish, since for each electron with momentum $+\vec{p}$ and spin $\pm\frac{1}{2}$ within the sea there exists one with $-\vec{p}$ and spin $\mp\frac{1}{2}$. As far as the first two singular properties are concerned, they are not observable: One does not measure absolute energies, but rather energy differences. The infinite charge is uniformly distributed in space, and thus leads to fields which are experimentally not measurable.

On the one hand, the interpretation of the Dirac sea can be combined with physical (observable) reality. On the other hand, it has the advantage that it forbids the transition to negative energies, since all states with $E < 0$ are already occupied. Taken alone, the theory of Dirac sea would not have had a breakthrough. However, one can explain new phenomena.

It should be possible to lift an electron out of the Dirac sea via a high energy γ-particle. Then there would be a hole in the Dirac sea. If the original electron in the sea had the properties

$$\begin{aligned} \text{momentum} &: \vec{p} \\ \text{energy} &: E = -\sqrt{\vec{p}^{\,2} + m^2} \\ \text{charge} &: e = -\mid e \mid \end{aligned} \tag{4.281}$$

then one obtains the hole (the infinite portions are not observables):

$$
\begin{aligned}
\text{momentum} &: \; -\vec{p} \\
\text{energy} &: \; \sqrt{\vec{p}^{\,2} \; + \; m} \\
\text{charge} &: \; \mid e \mid \; .
\end{aligned}
\tag{4.282}
$$

The energy is due to $-\infty \; - \; E \; = \; -\infty \; + \; \sqrt{\vec{p}^{\,2} \; + \; m}$ and the charge due to $-\infty \; - \; e \; = \; -\infty \; + \; \mid e \mid$. One obtains a state with positive energy and positive charge. Since the direction of momentum changed, the negative sign of (4.258) is compensated and velocity and momentum have again the same orientation. Thus the "hole" represents the anti-particle to the electron e^- with the mass m_e and a positive charge $+ \mid e \mid$, namely the **positron** e^+. According to the preceding consideration, it should be created with a high energy γ-particle. To fulfill momentum conservation one needs, e.g., the nucleus of an atom, the reaction

$$
\gamma \; + \; \text{nucleus} \; \longrightarrow \; e^- \; + \; e^+ \; + \; \text{nucleus}^* \tag{4.283}
$$

should produce an e^+e^- pair, and the recoil momentum is absorbed by the nucleus. Anderson and Blackett proved in 1932 the existence of the positron experimentally. The high energy γ-rays were those from cosmic radiation.

In addition, Dirac's theory allows that an existing hole can be filled with an electron with positive energy. To fulfill momentum conservation, at least two γ have to produced in the reaction

$$
e^+ \; + \; e^- \; \longrightarrow \; \gamma \; + \; \gamma \; . \tag{4.284}
$$

This process is experimentally well established.

With Dirac's theory of the negative energy sea, the original Dirac equation describing a one-electron system turned into a system of **infinitely many particles**. This transition had to be made to remove physical contradictions from the relativistic wave equation. Thus, one can expect a relativistic quantum theory will always have to deal with many-body systems, where creation and annihilation processes will play an important role. In these processes, the equivalence of energy and mass, $E \; = \; mc^2$, manifests itself in a very concrete fashion.

A consequent mathematical treatment of the Dirac sea is only possible with tools that allow to describe infinitely many particles and the Pauli-principle in an orderly fashion.

This is the case for the so-called "Second Quantization," in which the Dirac spinors became operators, which fulfill commutation relations. Within such a formulation, the infinities can be treated in a theoretically satisfactory way.

Here only the spinors for the hole states shall be given. A hole state with the positive (!) energy p^0 and momentum $\vec{p}$ is created by removal of an electron with negative (!) energy $-p^0$ and momentum $-\vec{p}$ from the sea, i.e.,

$$\text{a hole state with } +p \text{ is created from an electron with } -p \,. \tag{4.285}$$

Thus it appears natural to describe a hole with

$$v(p) \;:=\; u(-p) \,. \tag{4.286}$$

Instead of (4.96), $(\not{p} \,-\, m)\; u(p) \;=\; 0$ we obtain

$$(\not{p} \,+\, m)\; v(p) \;=\; 0 \,. \tag{4.287}$$

The solution can be read off (4.256) as

$$v(\vec{p}) \;=\; N \begin{pmatrix} \frac{\vec{\sigma}\,\cdot\,\vec{p}}{p^0\,+\,m}\,\eta \\ \eta \end{pmatrix} \,. \tag{4.288}$$

If we want to consider the space-time dependence of the spinor, we have to set because of (4.285)

$$\psi_{hole}(x) \;:=\; e^{ipx}\; v(\vec{p}) \,. \tag{4.289}$$

Only with this exponential factor the Dirac equation

$$(i\not{\partial} \,-\, m)\; \psi_{hole}(x) \;=\; 0 \tag{4.290}$$

is fulfilled together with (4.287). However, the time factor in (4.289),

$$e^{ip^0 x_0} \;=\; e^{iEt} \;=\; e^{-i(-E)t} \tag{4.291}$$

shows that the negative energy states are not yet completely eliminated.

This is only possible if one interprets the Dirac spinor as field operator. Here only the basic idea shall be presented. The expansion (4.250) is extended to incorporate the solutions with negative energies and becomes

$$\psi(x) \;=\; \frac{1}{(2\pi)^{3/2}} \sum_{\lambda} \int d^3p \, \left[e^{-ipx} \; a_\lambda(\vec{p}) \; u_\lambda(\vec{p}) \;+\; e^{ipx} \; b_\lambda^*(\vec{p}) \; v_\lambda(\vec{p}) \right] \;, \tag{4.292}$$

where $a_\lambda(\vec{p})$ is considered as annihilation operator for an electron with $\vec{p}, \lambda$ and $b_\lambda^*(\vec{p}) \;\rightarrow\; b_\lambda^\dagger(\vec{p})$ as creation operator for a positron with $\vec{p}, \lambda$. Thus, charges and signs of energies are properly taken care of.

4.11 Particle Anti-Particle Conjugation

In the Hole theory every electron, represented by $\psi(x)$, is associated with a positron state, which shall be described here by a spinor $\psi^c(x)$. The transition $\psi(x) \rightarrow \psi^c(x)$ is called **Particle Anti-Particle Conjugation** and is associated with a transformation in the Hilbert space of the particles.

We want to construct this transformation C, especially construct $\psi^c(x)$ from $\psi(x)$. $\psi^c(x)$ describes a particle with charge $+ \mid e \mid$. $\psi(x)$ fulfills

$$[\gamma^\mu (i\partial^\mu - eA_\mu) - m] \;\; \psi(x) \;\; = \;\; 0 \; . \tag{4.293}$$

Thus $\psi^c(x)$ should obey

$$[\gamma^\mu (i\partial_\mu + eA_\mu) - m] \;\; \psi^c(x) \;\; = \;\; 0 \; . \tag{4.294}$$

Starting from (4.293) the change of sign can be achieved in the following way:

According to (4.74) we have for the conjugate spinor

$$\bar{\psi} \left[\gamma^\mu (i \overleftarrow{\partial}_\mu + eA_\mu) + m \right] \; = 0 \; , \tag{4.295}$$

i.e., when introducing transposed spinors and matrices

$$\left[\gamma^{\mu^T} (i\partial_\mu + e \; A_\mu) \; + \; m \right] \; \bar{\psi}^T \;\; = \;\; 0 \; . \tag{4.296}$$

Comparing with (4.294), we see that the mass term has to change sign. To achieve this we need to find a matrix C with the property

$$C \gamma^{\mu^T} \; C^{-1} \;\; = \;\; -\gamma^\mu \; . \tag{4.297}$$

Such a matrix exists. From the commutation relations (4.20) follows

$$\{ \gamma^{\mu^T}, \gamma^{\nu^T} \} \;\; = \;\; 2 g^{\mu\nu} \; \mathbf{1} \; . \tag{4.298}$$

One can easily show that

$$C \; := \; (-i)\;\gamma^0\gamma^2 \; = \; (-i)\; \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{4.299}$$

has the desired properties. In addition, one has

$$C \; = \; -C^{-1} \; = \; -C^\dagger \; = \; -C^T \; . \tag{4.300}$$

Thus (4.296) can be rewritten as

$$C^{-1}\;\left[C\;\gamma^{\mu^T}\;C^{-1}\;(i\partial_\mu + e\;A_\mu) + m\right]\;C\bar{\psi}^T \; . \tag{4.301}$$

Together with (4.297) and after multiplication with $-C$ from the left, one obtains

$$[\gamma^\mu\;(i\partial_\mu\;+\;e\;A_\mu)\;-\;m]\;C\bar{\psi}^T\;=\;0\;. \tag{4.302}$$

Thus, one has

$$\psi^c(x)\;=\;C\bar{\psi}^T(x)\;. \tag{4.303}$$

As example, let us take $\psi(x)$ from (4.302),

$$\psi_{hole}(x)\;=\;e^{ipx}\;v(\vec{p})\;, \tag{4.304}$$

i.e., we consider the case of the absence of an electron with negative energy or the presence of a hole state with positive energy. One has

$$\begin{aligned} \psi_{positron}(x) \;&=\; \psi^c_{hole}(x) \\ &=\; C\bar{\psi}^T_{hole}(x) \\ &=\; C\gamma^0\;\psi^*_{hole}(x) \\ &=\; e^{-ipx}\;C\gamma^0\;v^*(\vec{p}) \\ &=\; e^{-ipx}\;v^c(\vec{p}) \end{aligned} \tag{4.305}$$

The time exponent, $e^{-ip^0x_0} = e^{iEt}$ indeed corresponds to a positive energy E. Furthermore, one can show that the spinor $v^c(\vec{p})$ fulfills the equation

$$(\not{p} - m)\, v^c(\vec{p}) = 0 , \tag{4.306}$$

where $p^0 > 0$. Thus, $v^c(\vec{p})$ has to coincide with $u(\vec{p})$ from (4.129)

$$v^c(\vec{p}) = u(\vec{p}) . \tag{4.307}$$

Thus, the wave function of a free positron is, according to (4.305), identical with the one of a free electron. In a field-free environment, there is no distinction between electron and positron.

4.12 Relativistic Covariance of the Dirac Equation

pretty lengthy – to be added at some later time.

4.13 Observables in the Dirac Theory

The observables in the Dirac theory must be hermitian operators. In addition to operators similar to the nonrelativistic theory, these must be objects, which are constructed from γ-matrices. This means, quantities of the form $\bar{\psi}O\psi$, where the operator O is a combination of γ-matrices or their products and is a fundamental observable for all field theories. Generically they are called *currents* (in analogy to the Dirac current $\bar{\psi}\gamma^\mu\psi$).

Important is that there exists only a finite, well defined number, namely 16, which are linearly independent.

Theorem from γ-matrix algebra:
The products of n γ-matrices $O_n = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}$ are only linearly independent for $n = 4$. The reason is that for $n > 4$ at least 2 matrices in the product are equal, which reduces the dimension by $n - 2$ due to $(\gamma^{\mu_i})^2 = \pm 1$.

Consider the cases $n = 1..4$:

n=0; $O_0 = \mathbf{1}$

which gives as current $\bar{\psi}\psi$.
This is the only Lorentz invariant quantity, i.e. it does neither change under proper Lorentz transformations nor under reflections. It was used for the Lorentz invariant normalization for the spinors. In field theory this quantity plays a fundamental role in the form of the mass term $m\bar{\psi}\psi$.

The number of independent matrices is 1.

n=1; $O_1 = \gamma^\mu$

This case contains the four γ-matrices and gives the four-vector $j^\mu = \bar{\psi}\gamma^\mu\psi$, which behaves under reflection as a polar vector.

The number of independent matrices is 4.

n=2; $O_2 = \gamma^\mu\gamma^\nu$

Only the antisymmetric piece is unequal to the unit matrix. This leads to $\bar{\psi}\sigma^{\mu\nu}\psi$, which is a tensor of rank 2.

The number of independent matrices is 6.

n=3; $O_3 = \gamma^\mu\gamma^\nu\gamma^\rho$

Here one has to choose three different indices from 0,1,2,3. For this $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4$ options exist. The corresponding γ-matrices can be represented by the product $\gamma^5\gamma^\mu$.
For a Lorentz transformation one has

$$S^{-1}\gamma_5\gamma^\mu S = \det(\Lambda)\Lambda^\mu_\nu\gamma_5\gamma^\mu. \tag{4.308}$$

The current $j^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi$ is an axial current and plays together with j^μ an important role in the theory of weak interactions.

The number of independent matrices is 4

n=4; $O_4 = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$

Here all 4 γ-matrices must be different. Thus, only the matrix γ^5 is a candidate. To determine its transformation properties write

$$\gamma^5 = i\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{4.309}$$

For the transformed matrix $S^{-1}\gamma^5 S$ one has to apply $S^{-1}\gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu$ on each matrix. Due to the totally antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ this gives the determinant of Λ:

$$S^{-1}\gamma^5 S = \det(\Lambda)\gamma^5. \tag{4.310}$$

Thus, under proper Lorentz transformations γ^5 is invariant. However under reflections it changes sign (due to $\det(\Lambda)$). This means that $j^5 = \bar{\psi}\gamma^5\psi$ transforms as pseudoscalar.

The number of independent matrices is 1.

In Summary: in total one has 1 + 4 + 6 + 4 + 1 = 16 independent matrices. Each element of the Dirac algebra cane by represented by $\sum_{A=1}^{16} c_A \gamma_A$, where c_A are constants.

4.14 Chirality

The invariance of γ^5 under proper Lorentz transformation s has important mathematical and physical consequences. Since γ^5 is a hermitian 4x4 matrix, it has 4 orthogonal eigenvectors, and because of $(\gamma^5)^2 = 1$, they belong to the eigenvalues ± 1.

Introduce the following spinors:

$$\psi_R(x) =: \frac{1}{2}(1+\gamma^5)\psi(x) \tag{4.311}$$
$$\psi_L(x) =: \frac{1}{2}(1-\gamma^5)\psi(x). \tag{4.312}$$

Because of $(\gamma^5)^2 = 1$ one has

$$\gamma^5\psi_R(x) = \psi_R(x) \tag{4.313}$$
$$\gamma^5\psi_L(x) = -\psi_L(x) \tag{4.314}$$

and

$$\psi_R(x) + \psi_L(x) = \psi(x). \tag{4.315}$$

The spinors $\psi_R(x)$ and $\psi_L(x)$ can be linked with a 'screw-direction', and thus are called *chiral spinors*. The eigenvalues ± 1 are called *chiralities* of the spinors. Under reflection, the chiralities change

$$\hat{P}\frac{1}{2}(1+\gamma^5)P = \frac{1}{2}(1-\gamma^5), \tag{4.316}$$

under proper Lorentz transformations, the chiralities remain. For massless particles this concept is especially important:

Consider the helicity $\vec{\Sigma}\cdot\vec{p}/|\vec{p}|$ is equal to the chirality γ^5, and

$$\vec{\Sigma} = \gamma^5\gamma^0\vec{\gamma} = \gamma^5\vec{\alpha}. \tag{4.317}$$

Consider the free Dirac equation

$$\gamma^\mu p_\mu u_\lambda(p) = 0 \tag{4.318}$$

For $m = 0$ this leads to

$$\vec{\gamma}\cdot\vec{p}u_\lambda(p) = \gamma^0 E u_\lambda(p) \tag{4.319}$$

and thus to

$$u_\lambda(p) = \gamma^0\vec{\gamma}\cdot\frac{\vec{p}}{|\vec{p}|}u_\lambda(p). \tag{4.320}$$

Applying γ^5 leads to

$$\gamma^5 u_\lambda(p) = \gamma^5\gamma^0\vec{\gamma}\cdot\frac{\vec{p}}{|\vec{p}|}u_\lambda(p) = \vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|}u_\lambda(p) = \lambda u_\lambda(p), \tag{4.321}$$

which shows explicitly the equality of chirality and helicity for massless particles.

For particles with $m \neq 0$ one has

$$\begin{aligned} i\gamma^\mu\partial_m u\psi_R &= m\psi_L \\ i\gamma^\mu\partial_m u\psi_L &= m\psi_R, \end{aligned} \tag{4.322}$$

i.e. the mass term couples the two chiralities.

4.15 The Weyl Equation – The Neutrino

In 1930 Pauli postulated the existence of the neutrino in order to Φ guarantee the energy and momentum conservation for the weak interaction, which at the time seemed to be violated in β-decay experiments. Since the energy of the neutrino could not be determined the interaction of this postulated particle with mater must be extremely small, e.g. it must not have an electric charge, mass and magnetic moment must be vanishingly small or zero.

Neutrinos are seen in the reactions

$$\begin{aligned} n &\longrightarrow p + e^- + \bar{\nu} \\ p + \bar{\nu} &\longrightarrow n + e^+. \end{aligned} \tag{4.323}$$

From angular momentum considerations in β-decay it was deduced that the neutrino must be spin-1/2 particle. Precise experimental analyses showed that the spin of the neutrino ν is antiparallel and the of the anti-neutrino $\bar{\nu}$ is parallel to its momentum direction. This si the basic phenomenon of parity violation: If parity was conserved, neutrinos as well as anti-neutrinos must exist in nature with both spin directions.

In 1929 Hermann Weyl proposed a two-component equation to describe a massless spin-1/2 particle (H. Weyl, Z. Physik, **56**, 330 (1929)). However, the Weyl equation violates parity and thus was rejected at the time. After the parity violation of the weak interaction was experimentally established in 1957, Landau, Salam and Lee and Young took Weyl's equation and regarded it as the basic equation of motion for the neutrino.

First, consider the Dirac equation for a massless particle

$$i\frac{\partial\psi(x)}{\partial t} = \vec{\alpha}\cdot\vec{p}\;\psi(x) \tag{4.324}$$

This equation does not longer contain the matrix $\beta = \gamma^0$, and the anti-commutation relations $\alpha_i, \alpha_j = 2\delta_{ij}$ can be satisfied by the Pauli matrices. The appearance of $\beta = \gamma^0$ as set of the anti-commuting matrices requires the construction of 4x4 matrices and four-component spinors to describe spin-1/2 particles. Thus, if spin-1/2 particles are massless, this reason is no longer present, and the particle can be described by a **two component amplitude** $\phi^{(+)}(x)$

$$\begin{aligned} i\frac{\partial\phi^{(+)}}{\partial t} &= \vec{\sigma}\cdot\hat{p}\;\phi^{(+)} \\ \frac{\partial\phi^{(+)}}{\partial t} &= -\vec{\sigma}\cdot\hat{\nabla}\;\phi^{(+)}, \end{aligned} \tag{4.325}$$

where σ_i are the Pauli matrices. The plane wave solutions of the Weyl equation are given by

$$\phi^{(+)}(x) = \frac{1}{\sqrt{2E(2\pi)^3}}e^{-ipx}u^{(+)}(p), \tag{4.326}$$

where $p \equiv (p_0, \mathbf{p}) = (E, \mathbf{p})$. The wave functions are normalized such that the norm remains invariant under Lorentz transformations. The two-component spinor $u^{(+)}(p)$ fulfills

$$p_0 u^{(+)}(p) = \vec{\sigma}\cdot\vec{p}\;u^{(+)}(p) \tag{4.327}$$

This equation is of particular interest, since it relates a solution with a specific sign of p_0 to the projection of the spin along the direction of motion.

Applying the helicity operator $\Lambda = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ to (4.327) gives

$$(p_0^2 - \vec{p}^2)u^{(+)}(p) = 0 \tag{4.328}$$

which states that nonvanishing solutions $u^{(+)}(p)$ only exist if $p_0 = \pm|\vec{p}|$.

In the common representation of the Pauli matrices, and with the z-axis in $\vec{p}$ direction, the solution of (4.327) is given by

$$u^{(+)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.329}$$

describing a right-handed massless particle with spin into the direction of motion, i.e.

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}u^{(+)} \equiv \sigma_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.330}$$

We see that for positive energy states ($p_0 = +|\vec{p}|$), the spin is directed along the direction of motion, i.e. we have positive helicity. States with $p_0 = -|\vec{p}|$ have negative helicity.

Thus, if the neutrino is supposed to be a left-handed (negative helicity) particle, one needs to start with the equation

$$i\frac{\partial \phi^{(-)}(x)}{\partial t} = -\vec{\sigma}\cdot\vec{p}\;\phi^{(-)}(x). \tag{4.331}$$

Replacing $\vec{\sigma}$ with $-\vec{\sigma}$ in (4.325) also results in a realization of the commutation relations $\alpha_i, \alpha_j = 2\delta_{ij}$, thus (4.331) and (4.325) are two equally possible equations for massless Dirac particles.

With the ansatz

$$\phi^{(-)}(x) = \frac{1}{\sqrt{2E(2\pi)^3}}e^{-ipx}u^{(-)}(p), \tag{4.332}$$

one finds

$$p_0 u^{(-)}(p) = -\vec{\sigma}\cdot\vec{p}u^{(-)}(p) \tag{4.333}$$

with

$$u^{(-)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.334}$$

Defining a four-vector $\sigma \equiv (\mathbf{1}, \vec{\sigma})$, the Weyl equation for the neutrino can be written in a more compact way

$$\sigma_\mu \nabla^\mu \phi^{(-)} = 0 \tag{4.335}$$

In order to better understand the connection of the two-component solution with the four component one, we solve for the Dirac equation and choose the **chiral representation** of the Dirac matrices, which is given by applying the transformation

$$U = \frac{1}{\sqrt{2}} \left(\mathbf{1} + \gamma_5 \gamma_0 \right) \tag{4.336}$$

to the standard Dirac matrices. (As an aside, the standard representation of the Dirac matrices are Φ particularly useful when considering the nonrelativistic limit.) In the chiral representation we obtain

$$\begin{aligned} \vec{\gamma}^{ch} &= \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \\ \gamma^{0\,ch} &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ \gamma^{5\,ch} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4.337}$$

and thus

$$\vec{\alpha} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}. \tag{4.338}$$

In this representation the Dirac equation together with a two-component wave function

$$\Psi(x) = \begin{pmatrix} \phi^{(+)} \\ \phi^{(-)} \end{pmatrix} \tag{4.339}$$

becomes

$$\begin{aligned} \frac{\partial \phi^{(+)}}{\partial t} &= -i\vec{\sigma} \cdot \vec{\nabla}\, \phi^{(+)} - m\, \phi^{(-)} \\ \frac{\partial \phi^{(-)}}{\partial t} &= -i\vec{\sigma} \cdot \vec{\nabla}\, \phi^{(+)} - m\, \phi^{(+)}. \end{aligned} \tag{4.340}$$

The upper and lower components of $\Psi(x)$ are only coupled through the mass term. In the limit $m \to 0$ one obtains two uncoupled two-component equations as in (4.325).

Using the chiral representation of γ_5, it is easily seen that

$$\Psi^{(+)} = \begin{pmatrix} \phi^{(+)} \\ 0 \end{pmatrix} \tag{4.341}$$

for $m \to 0$ corresponds to right handed neutrinos, since

$$\gamma_5 \Psi^{(+)} = +\Psi^{(+)}. \tag{4.342}$$

Analogously one obtains for

$$\Psi^{(-)} = \begin{pmatrix} 0 \\ \phi^{(-)} \end{pmatrix} \tag{4.343}$$

for $m \to 0$ and

$$\gamma_5 \Psi^{(-)} = -\Psi^{(-)}, \tag{4.344}$$

i.e. $\Psi^{(-)}$ describes left handed neutrinos. Therefore, we can introduce projection operators

$$P_\pm \equiv \frac{1}{2}\left(\mathbf{1} \pm \gamma_5\right) \tag{4.345}$$

projecting the state Ψ on the two different handed neutrino states

$$\begin{aligned} \Psi^{(+)} &= \frac{1}{2}(\mathbf{1}+\gamma_5)\Psi \\ \Psi^{(-)} &= \frac{1}{2}(\mathbf{1}-\gamma_5)\Psi. \end{aligned} \tag{4.346}$$

The above shows that the two-component Weyl theory is equivalent to a four-component Dirac representation. However, in the framework of the Weyl equation, the distinction between particles and anti-particles is superfluous. The two possible states of the neutrino are only characterized by parallel and antiparallel orientation of the spin with respect to the momentum.

Next, we want to convince ourselves that the two-component theory is **not** invariant under parity transformations. Under space inversion one has $\vec{p} \to -\vec{p}$, $\vec{\sigma} \to \vec{\sigma}$, i.e. $\vec{p}$ is a polar vector, $\vec{\sigma}$ is an axial vector, and thus $\vec{\sigma}\cdot\vec{p}$ is a pseudoscalar under inversion. Using (4.342) and (4.344) one has

$$\bar{\Psi}^{(\pm)}\ \gamma_5\ \Psi^{(\pm)} = \pm\bar{\Psi}^{(\pm)}\ \Psi^{(\pm)}. \tag{4.347}$$

The left hand side is a pseudoscalar density, while the right hand side is a scalar density. Thus, helicity eigenstates are not parity eigenstates.