Chapter 7

Appendix A: The Mandelstam Variables

In Chapter 2.2 we already encountered different invariant kinematical quantities. We generalize Fig. 2.2.1 insofar that we leave open which particles are incoming and which outgoing. Thus we consider the kinematic situation given in Fig. A.1.

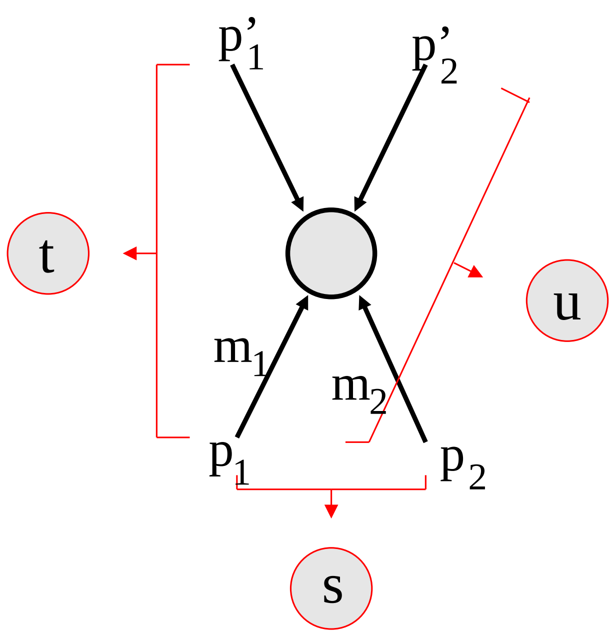

Figure A.1 Definition of s, t, u.

In the discussion of Chapter 2.2, we already defined s and t and found them to be useful variables. Here we define a third, u, although all three, s, t, u, are no longer independent.

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_1' + p_2')^2 = -(p_1 + p_2)(p_1' + p_2') \\
t &= (p_1 + p_1')^2 = (p_2 + p_2')^2 = -(p_1 + p_1')(p_2 + p_2') \\
& = 2(m_1^2 + p_1 p_1') = 2(m_2^2 + p_2 p_2') \\
u &= (p_1 + p_2')^2 = (p_2 + p_1')^2 = -(p_1 + p_2')(p_2 + p_1')
\end{aligned}
\tag{7.1}
$$

The physical significance of these variables can be expressed in two ways:

(a) s is the square of the *c.m.* energy if p_1 and p_2 (or p_1' and p_2') are incoming.

t is the square of the *c.m.* energy if p_1 and p_1' (or p_2 and p_2') are incoming.

u is the square of the *c.m.* energy if p_1 and p_2' (or p_2 and p_1') are incoming.

This is a rather artificial description is defined by another process. The three processes, in which s, t, u are the squared *c.m.* energies are called the "s, t, u channel," respectively.

Example: Let p_1, m_1 describe a pion, and p_2, m_2 a nucleon, then:

$$
\begin{aligned}
s - \text{channel means :} \quad &: \pi + N \longrightarrow \pi + N \\
&: \pi + \bar{N} \longrightarrow \pi + \bar{N} \\
t - \text{channel means :} \quad &: \pi + \pi \longrightarrow N + \bar{N} \\
&: N + \bar{N} \longrightarrow \pi + \pi \\
u - \text{channel means :} \quad &: \pi + N \longrightarrow \pi + N \\
&: \pi + \bar{N} \longrightarrow \pi + \bar{N}
\end{aligned}
\tag{7.2}
$$

(b) If one describes the meaning of s, t, u in a definite process, e.g., the s- channel, then

s is the squared *c.m.* energy.

t is the squared four-momentum transfer. (In particular, it reduces to the squared three-momentum transfer in the Breit system.)

u has no simple physical meaning, since there is no Lorentz system where it reduces to anything obvious.

As s, t, u are not independent, it follows from (7.1) that

$$
s + t + u = 4m_1^2 + 2m_2^2 + 2p_1(p_2 + p_2' + p_1')
$$

$$= \; 2m_1^2 \; + \; 2m_2^2 \; . \tag{7.3}$$

Let us anticipate the notion of a scattering amplitude, i.e., the complex function which completely describes the scattering process. It will be a function of two variables, but can be written as function of s, t, and u having in mind that one of them is redundant, i.e.,

$$T(s,t,u) \;\equiv\; \text{scattering amplitude} \; . \tag{7.4}$$

One can prove, independent of perturbation theory, that this function is an analytic function of any two of the variables if these are considered to be complex. There are then certain domains in the complex st (or su or tu) plane, in which these variables became real and have "physical" meaning. These regions are disconnected and belong to different physical processes. That $T(s,t,u)$ is an analytic function of any two complex variables out of s, t, u means that the "physical scattering amplitude" is the boundary value of that general function when s, t, u take on physical values. In other words, the "physical scattering amplitude" is obtained in any channel from the general function simply by specializing to the "physical values" of s, t, u for that channel.

Here we shall not go into the "analytical structure of the scattering amplitude," but only explain the graphical representation of the variables s, t, u and exhibit their "physical regions."

Let us consider the following triangle,

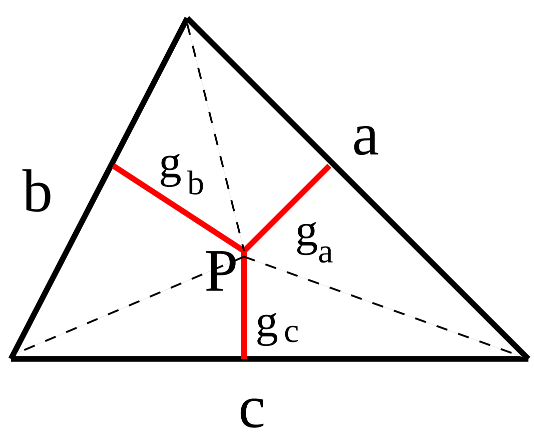

Figure A.2 Geometric Consideration.

and remember from elementary geometry that if from any point p the three distances G_a, G_b, G_c to the sides a, b, c are taken, then

$$aG_a + bG_b + cG_c = ah_a = bh_b = ch_c = 2F , \tag{7.5}$$

where F denotes the surface of the triangle. Here h_a, h_b, h_c are the three heights perpendicular on a, b, c, respectively. Taking ch_c and dividing by c gives

$$\frac{a}{c} G_a + \frac{b}{c} G_b + G_c = h_c . \tag{7.6}$$

If we compare this with (7.6), we see that we can identify

$$\frac{a}{c} G_a := u \; ; \quad \frac{b}{c} G_b := s \; ; \quad G_c = t \; ; \quad h_c = 2m_1^2 + 2m_2^2 \tag{7.7}$$

to have the relation between s, t, u fulfilled. Thus any three coordinate axes intersecting such that they form a triangle with $h_c = 2m_1^2 + 2m_2^2$ can serve to represent s, t, u in a plane. Of course, one chooses particular triangles, where the representation becomes simple. The best choice seems to be $a = b = c$; $h = 2m_1^2 + 2m_2^2$. This is very symmetrical but has one disadvantage: The boundaries of the "physical regions" will be given in form of equations between s and t.

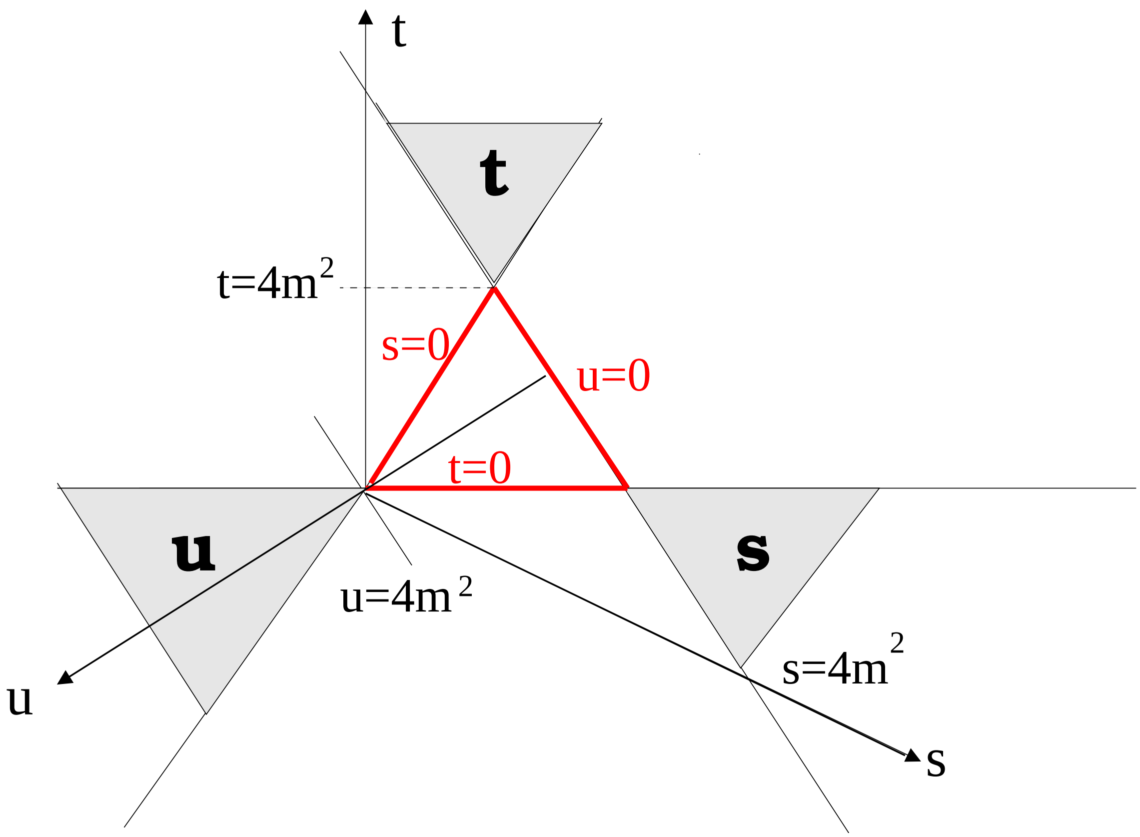

Figure A.3 Physical regions of s, t, u channels in symmetrical representation for $m_1 = m_2$. Every point in the plane satisfies $s + t + u = 4m^2$.

The curves are actually easier to draw in a rectangular coordinate system, and we choose $b = c = \frac{a}{\sqrt{2}} = h = 2m_1^2 + m_2^2$.

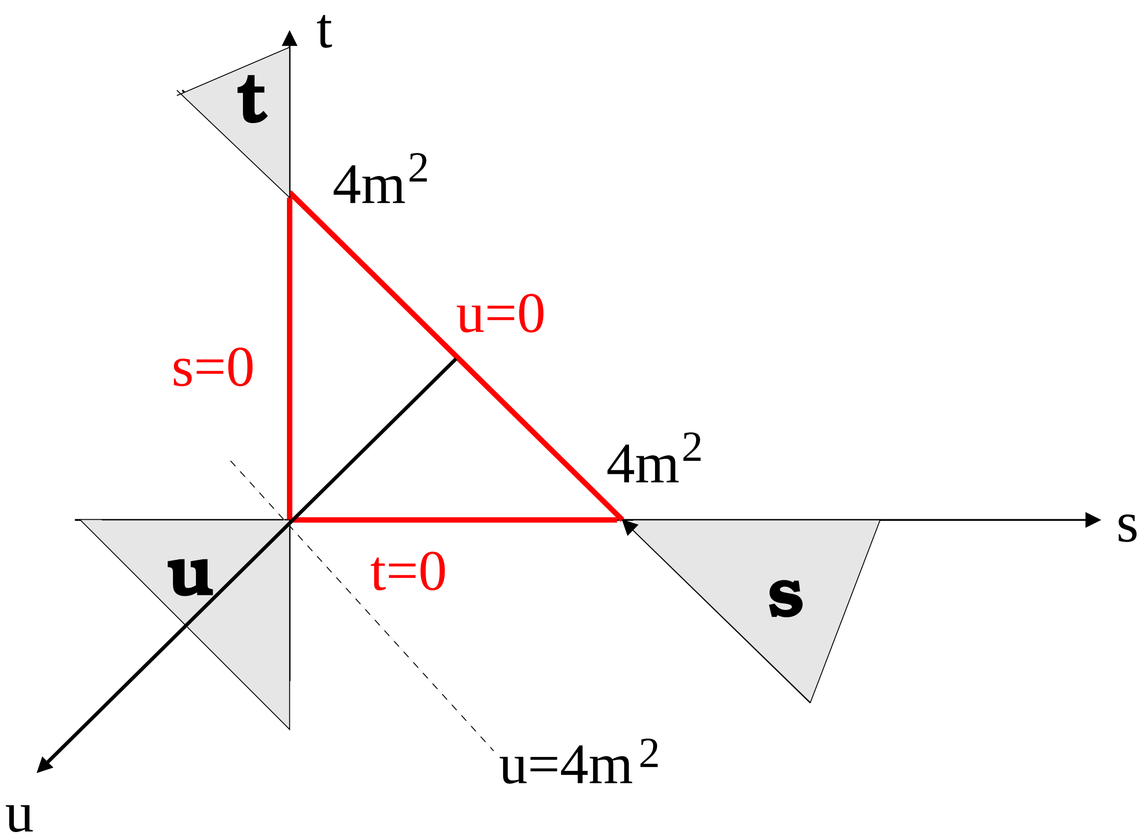

Figure A.4 Physical regions of s, t, u channels in the cartesian st-plane for $m_1 = m_2$. Every point in the plane satisfies $s + t + u = 4m^2$. (Note that the unit along the u axis is smaller by a factor $\frac{1}{\sqrt{2}}$ compared to the t and s axis.)

We now find the "physical region" of s, t, u in the three possible channels of Fig. A.4.

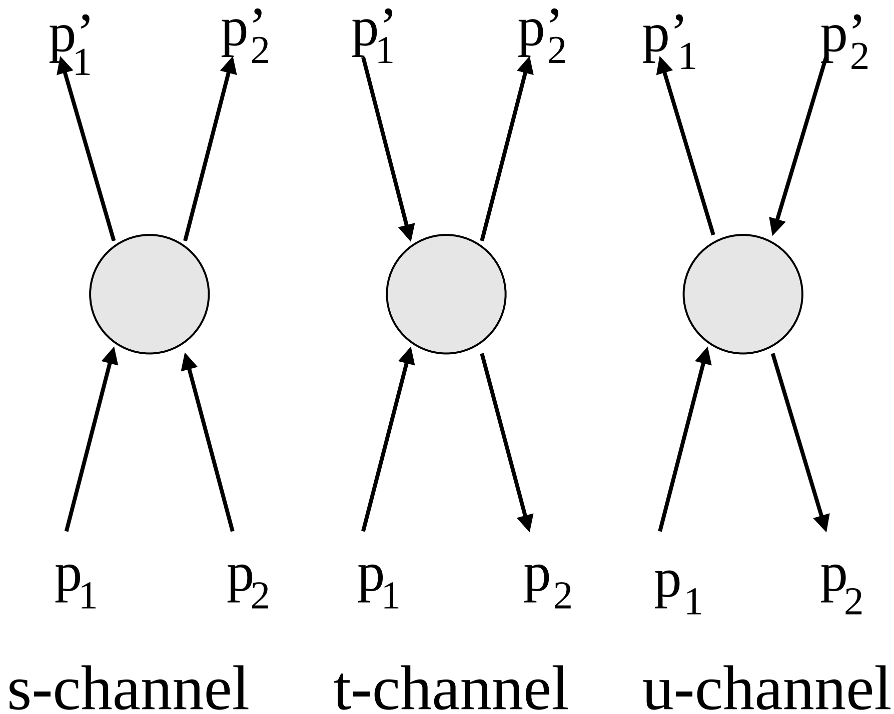

Figure A.5 The three different channels.

For $m_1 \neq m_2$ there is one symmetry, namely that t is the momentum transfer in both, the s and the u channel. It can thus be expected that the physical regions in the s and u channels map on each other if s and u are interchanged. (This is the famous **crossing symmetry**).

If the masses are equal ($m_1 = m_2$), then there is more symmetry:

Going from s to t channel $\rightarrow$ u keeps its meaning.

Going from s to u channel $\rightarrow$ t keeps its meaning.

Going from t to u channel $\rightarrow$ s keeps its meaning.

The physical regions are, therefore, mapped on each other if one

(a) interchanges $s \leftrightarrow t$ and keeps u.

(b) interchanges $s \leftrightarrow u$ and keeps t.

(c) interchanges $t \leftrightarrow u$ and keeps s.

These are in the symmetrical representation of Fig. A.2, the reflections of the entire plane with respect to the three symmetry axes of the basic triangle ABC. In Fig. A.3 the different scale along the axes makes the figure apparently less symmetric, but one can easily translate the physical regions from Fig. A.2 to Fig. A.3. This symmetry allows to discuss the s-channel only. All considerations here are restricted to the case $m_1 = m_2$. We have in the s-channel ($c.m.$ system):

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_1' + p_2')^2 = (2E)^2 = 4(m^2 + q^2) \\ t &= (p_1' - p_1)^2 = (p_2' - p_2)^2 = 2q^2(cos\ \theta_{c.m.} - 1) \end{aligned} \tag{7.8}$$

where q is the $c.m.$ momentum of all four particles. Hence, the "physical region" in the s-channel is given by

$$\begin{aligned} 4m^2 &\leq s \\ t_{min} &\leq t \leq 0 \\ t_{min} &= -4q^2 = 4m^2 - s\ . \end{aligned} \tag{7.9}$$

With $s + t + u = 4m^2$, one finds $s + t_{min} + u = 4m^2 = s + t_{min}$. Hence, the boundary $t_{min} = 4m^2 - s$ is identical with the line $u = 0$. The physical region of the s-channel is, therefore, given by the two conditions

$$\begin{aligned} t &\leq 0 \\ u &\leq 0\ . \end{aligned} \tag{7.10}$$

This region is shown in Figs. A.2 and A.3 shaded and marked by s. The corresponding regions for the two other channels follow from the above symmetry conditions.