Chapter 3

The 3N Bound State

3.1 The Faddeev Equations

A safe though not necessarily the only way to arrive at a rigorous solution of the 3N Schrödinger equation is the Faddeev scheme: L.D. Faddeev, Sov. Phys. JETP 12 (1961) 1014; see also W. Glöckle, The Quantum Mechanical Few-Body Problem, Springer Verlag 1983

In a three-body system there are three different two-body subsystems. The idea is to sum up first the pair forces in each two-body subsystem to inifite order, and then in a second step among all three particles. We start with the Schrödinger equation for a three-body system

$$(H_0 \; + \; \sum_{i=1}^{3} \; V_i) \; \Psi \; = \; E \; \Psi \tag{3.1}$$

and use the notation

$$V_1 \equiv V_{23}, V_2 \equiv V_{13} \cdots \text{etc.}, \tag{3.2}$$

To characterize the interactions in the two-body subsystems. H_0 is the kinetic energy of the relative motion for three particles. We rewrite Eq. (3.1) into an integral equation

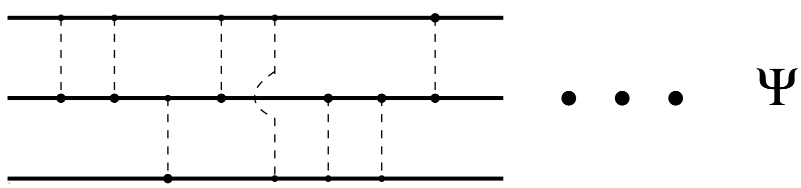
$$\Psi \; = \; \frac{1}{E - H_0} \; \sum_i \; V_i \; \Psi \tag{3.3}$$

(there is no $i\epsilon$ needed since $E < 0$ for a bound state)

Let us iterate this equation many times in order to see the physics:

$$\Psi \;=\; G_0 \sum V_i \Psi \;=\; G_0 \sum_i V_i G_0 \sum_j V_j G_0 \sum_k V_k \ldots \Psi \tag{3.4}$$

Graphically one typical term out of the very many can be



We see a sequence of pair forces with free 3N propagations G_0 in between. Consecutive pair forces can be within the same pair or between different pairs. The idea now is to sum up all forces within each pair to infinite order first. This is achieved according to Faddeev by decomposing Ψ into 3 components, nowadays called Faddeev components:

$$\Psi = \sum \psi_i \tag{3.5}$$

with

$$\psi_i \equiv \; G_0 \; V_i \; \Psi \tag{3.6}$$

We see that ψ_i is that part of Ψ which has V_i as the last interaction to the left. Let us insert that decomposition for Ψ on the right hand side:

$$\psi_i \;=\; G_0 \; V_i \sum_j \psi_j \;=\; G_0 \; V_i \; \psi_i \;+\; G_0 \; V_i \sum_{j \neq i} \psi_j \tag{3.7}$$

The first term is responsible for a renewed interaction V_i, whereas in the second term the next interaction is $V_j \neq V_i$. Bringing the first term to the left side gives

$$(1 \;-\; G_0 \; V_i) \; \psi_i \;=\; G_0 \; V_i \sum_{j \neq i} \psi_j \; . \tag{3.8}$$

Then we invert

$$\psi_i \;=\; (1 \;-\; G_0 \; V_i)^{-1} \; G_0 \; V_i \sum_{j \neq i} \psi_j \; . \tag{3.9}$$

What happens in the kernel?

$$\begin{aligned} & (1 \;-\; G_0 \; V_i)^{-1} \; G_0 \; V_i \;=\; (1 \;+\; G_0 V_i \;+\; G_0 V_i G_0 V_i \;+\; \ldots) \; G_0 V_i \qquad (3.10) \\ = \;\; & G_0 \; (V_i \;+\; V_i G_0 V_i \;+\; V_i G_0 V_i G_0 V_i \;+\; \ldots) \\ \equiv \;\; & G_0 \; t_i \end{aligned}$$

We recover the NN t-matrix for the pair "i". This t_i obviously sums up V_i to infinite order. As we already know, it obeys the LSE

$$t_i \ = \ V_i \ + \ V_i G_0 t_i \tag{3.11}$$

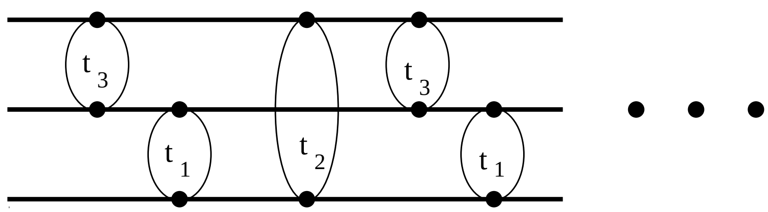
or

$$(1 - V_i G_0) t_i \ = \ V_i \ .$$

Thus we end up with

$$\psi_i \ = \ G_0 t_i \sum_{j \neq i} \psi_j \tag{3.12}$$

which is a set of 3 coupled equations, the Faddeev equations. If we iterate that set many times we find the typical processes



which is a sequence of t-operations between different pairs and free propagations in between.

If we choose the case of three identical fermions, then Ψ is totally antisymmetric.

As a consequence ψ_1,ψ_2, and ψ_3 are identical in their functional form, only the particles are permuted. This is easily shown as follows:

From Eq. (3.6) we have

$$\psi_2 \ \equiv \ G_0 V_2 \Psi \ = \ P_{12} P_{23} G_0 V_1 \Psi \ \equiv \ P_{12} P_{23} \psi_1 \tag{3.13}$$

where P_{ij} permutes particles i and j. Similarly,

$$\psi_3 \ \equiv \ G_0 V_3 \Psi \ = \ P_{13} P_{23} G_0 V_1 \Psi \ \equiv \ P_{13} P_{23} \psi_1 \ . \tag{3.14}$$

Therefore, only one of the Faddeev components is sufficient. Choosing ψ_1 and using Eqs. (3.9) and (3.11), we obtain

$$\psi_1 \ = \ G_0 t_1 \ (P_{12} P_{23} \ + \ P_{13} P_{23}) \ \psi_1 \tag{3.15}$$

Let us drop the index 1 and introduce the permutation operator

$$P \;\equiv\; P_{12}P_{23} \;+\; P_{13}P_{23} \tag{3.16}$$

then

$$\psi = \; G_0 \; t \; P \; \psi \tag{3.17}$$

is the one Faddeev equation for three identical particles. This is of course also valid for bosons. The total wave function is then given by

$$\Psi = (1 + P) \; \psi \tag{3.18}$$

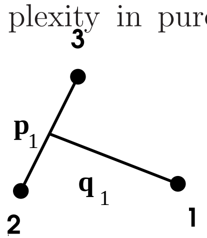
In order that Ψ is totally antisymmetric one has to require that the Faddeev component ψ is antisymmetric in the pair "1" $\equiv$ (23).

For instance, using the notation $\psi_i \;\equiv\; \psi(1,23)$,

$$\begin{aligned} P_{13}\Psi \;&=\; P_{13}\;(1 \;+\; P_{12}P_{23} \;+\; P_{13}P_{23})\;\psi(1,23) \\ &=\; \psi(3,21) \;+\; P_{13}\psi(2,31) \;+\; P_{13}\psi(3,12) \\ &=\; \psi(3,21) \;+\; \psi(2,13) \;+\; \psi(1,32) \\ &=\; -\psi(3,12) \;-\; \psi(2,31) \;-\; \psi(1,23) \\ &=\; -(1 \;+\; P)\;\psi(1,23) \;=\; -\Psi \end{aligned} \tag{3.19}$$

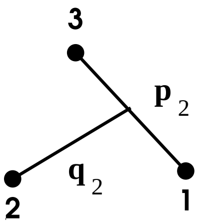
3.2 Momentum Space Representation

Let us first disregard spin and isospin degrees of freedom, in order to see the complexity in pure momentum space alone. It is natural to work with Jacobi momenta



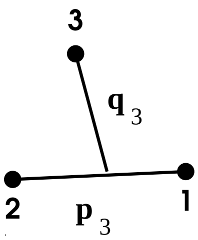
$$\begin{aligned} \vec{p}_1 \;&=\; \frac{1}{2}\;(\vec{k}_2 - \vec{k}_3) \\ \vec{q}_1 \;&=\; \frac{2}{3}\;(\vec{k}_1 - \frac{1}{2}\;(\vec{k}_2 \;+\; \vec{k}_3)\;), \end{aligned} \tag{3.20}$$

which corresponds to $\psi_1(1,23)$. There are three choices corresponding to the three two-body subsystems:



$$\vec{p}_2 = \frac{1}{2}\,(\vec{k}_3 - \vec{k}_1) \tag{3.21}$$
$$\vec{q}_2 = \frac{2}{3}\,(\vec{k}_2 - \frac{1}{2}\,(\vec{k}_3 + \vec{k}_1)\,)$$

and



$$\vec{p}_3 = \frac{1}{2}\,(\vec{k}_1 - \vec{k}_2) \tag{3.22}$$
$$\vec{q}_3 = \frac{2}{3}\,(\vec{k}_3 - \frac{1}{2}\,(\vec{k}_1 + \vec{k}_2)\,)$$

One can express $\vec{p}_2$, $\vec{q}_2$ linearly by $\vec{p}_1$ and $\vec{q}_1$, and this is true for all choices of indices. For instance

$$\vec{p}_2(\vec{p}_1,\vec{q}_1) \equiv \vec{p}_2 = -\frac{1}{2}\,\vec{p}_1 - \frac{3}{4}\,\vec{q}_1 \tag{3.23}$$
$$\vec{q}_2(\vec{p}_1,\vec{q}_1) \equiv \vec{q}_2 = \vec{p}_1 - \frac{1}{2}\,\vec{q}_1$$

and

$$\vec{p}_3(\vec{p}_1,\vec{q}_1) \equiv \vec{p}_3 = -\frac{1}{2}\,\vec{p}_1 + \frac{3}{4}\,\vec{q}_1 \tag{3.24}$$
$$\vec{q}_3(\vec{p}_1,\vec{q}_1) \equiv \vec{q}_3 = -\vec{p}_1 - \frac{1}{2}\,\vec{q}_1$$

The inverse relations are given by:

$$\vec{p}_1 = -\frac{1}{2}\,\vec{p}_2 + \frac{3}{4}\,\vec{q}_2 \tag{3.25}$$
$$\vec{q}_1 = -\vec{p}_2 - \frac{1}{2}\,\vec{q}_2$$

and

$$\vec{p}_1 = -\frac{1}{2}\,\vec{p}_3 - \frac{3}{4}\,\vec{q}_3 \qquad (3.26)$$
$$\vec{q}_1 = \vec{p}_3 - \frac{1}{2}\,\vec{q}_3$$

All remaining relation can be obtained by suitable permutations.

Now we introduce states

$$|\,\vec{p}_i\;\vec{q}_i > \equiv |\,\vec{p}_i > |\,\vec{q}_i > \qquad (3.27)$$

which are assumed to be normalized as in chapter 1:

$$\langle \vec{p}_i^{\,\prime}\;\vec{q}_i^{\,\prime}\,|\,\vec{p}_i\;\vec{q}_i\rangle = \delta(\vec{p}_i^{\,\prime}-\vec{p}_i)\delta(\vec{q}_i^{\,\prime}-\vec{q}_i) \qquad (3.28)$$

One and the same state of relative motion can be presented in three ways:

$$|\,\vec{p}_1\;\vec{q}_1 >_1 = |\,\vec{p}_2\;\vec{q}_2 >_2 = |\,\vec{p}_3\;\vec{q}_3 >_3 \qquad (3.29)$$

where $\vec{p}_2$, $\vec{q}_2$ or $\vec{p}_3$, $\vec{q}_3$ are given by (2.32-2.33) and (2.34-2.35), respectively. We also added subscripts to unambiguously fix the meaning of the momentum quantum numbers. The index 1 means, that $\vec{p}_1$ is the relative momentum in the subsystem (23) and $\vec{q}_1$ the relative momentum of particle 1 with respect to that subsystem, and correspondingly for the subscripts 2 and 3. That subscript notation is very convenient if one has to permute particles. Consider

$$P_{12}P_{23}\;\;|\,\vec{p}\vec{q}\,\rangle_1\;. \qquad (3.30)$$

The momentum quantum numbers will not change, but the particles will be cyclically permuted and the result can be written

$$P_{12}P_{23}\;\;|\,\vec{p}\vec{q}\,\rangle_1 \;=\; P_{12}P_{23}(1,23) \;=\; (2,31) \;= |\,\vec{p},\vec{q}\,\rangle_2\;. \qquad (3.31)$$

In the ket $|\,\rangle_1$ the momentum $\vec{p}$ describes the pair (23) and in the ket $|\,\rangle_2$ the pair (31), etc.

Let us now consider the momentum space representation of the Faddeev equation

$$\psi = G_0\;t\;P\;\psi \qquad (3.32)$$

One has

$$\begin{aligned} <\vec{p}\;\vec{q}\,|\,\psi> &= <\vec{p}\;\vec{q}\,|\,G_0\;t\;P\,|\,\psi> \qquad (3.33)\\ &= \int d^3p''\;d^3q''\int d^3p'\;d^3q'\;\;<\vec{p}\;\vec{q}\,|\,G_0\;t\,|\,\vec{p}^{\,\prime}\;\vec{q}^{\,\prime}> \\ &\times\;<\vec{p}^{\,\prime}\;\vec{q}^{\,\prime}\,|\,P\,|\,\vec{p}^{\,\prime\prime}\;\vec{q}^{\,\prime\prime}><\vec{p}^{\,\prime\prime}\;\vec{q}^{\,\prime\prime}\,|\,\psi> \end{aligned}$$

Let us first consider $G_0 \equiv \frac{1}{E-H_0}$. The momenta in a three-body system are given by $\vec{k}_i$, and the total momentum is $\vec{K} = \sum_i \vec{k}_i$. Thus

$$H_0 \;=\; \sum_i \frac{k_i^2}{2m} \;=\; \frac{K^2}{2M} \;+\; \frac{p_\ell^2}{2\mu_\ell} \;+\; \frac{q_\ell^2}{2M_\ell} \tag{3.34}$$

with $\ell = 1, 2, 3$. In Eq. (3.34)

$$M \;=\; 3m \;; \quad M_\ell \;=\; \frac{2}{3}\, m \;; \quad \mu_\ell \;=\; \frac{1}{2}\, m \;, \tag{3.35}$$

where m is the mass of a single particle (e.g., the nuclear mass).

A complete set of states in a three-particle Hilbertspace is given by

$$\int d^3k_1 \int d^3k_2 \int d^3k_3 \;\; |\, \vec{k}_1\, \vec{k}_2\, \vec{k}_3\rangle\, \langle \vec{k}_3\, \vec{k}_2\, \vec{k}_1\, | = \; \mathbf{1} \;. \tag{3.36}$$

A change of basis to Jacobi momenta is given by

$$\langle \vec{k}_1\vec{k}_2\vec{k}_3 \mid \vec{p}_k\; \vec{q}_k K\rangle = \; \delta\left(\vec{p}_k - \; \frac{1}{2}\, (\vec{k}_\ell - \vec{k}_m)\right) \delta\left(\vec{q}_k - \; \frac{2}{3}\left[\vec{k}_k - \; \frac{1}{2}(\vec{k}_\ell + \vec{k}_m)\right]\right) \\ \delta(\vec{K} - \vec{k}_k - \vec{k}_\ell - \vec{k}_m) \;, \tag{3.37}$$

where $(k\ell m)$ are cyclic permutations of (123).

Thus a different complete set of states is given by

$$\int \; d^3p_\ell\; d^3q_\ell\; d^3K \;\; | \;\; \vec{p}_\ell\; \vec{q}_\ell\; \vec{K}\rangle\; \langle \vec{K}\; \vec{q}_\ell\; \vec{p}_\ell\, | = \; \mathbf{1} \;. \tag{3.38}$$

If the total momentum of the three-body system is conserved, i.e., $K^2 = 0$, we obtain for the non-relativistic kinetic energy

$$H_0 \;=\; \frac{p_\ell^2}{m} \;+\; \frac{q_\ell^2}{\frac{4}{3}\; m} \;=\; \frac{p_\ell^2}{m} \;+\; \frac{3}{4m}\, q_\ell^2 \;, \tag{3.39}$$

where $\ell = 1, 2, 3$. It follows that

$$< \vec{p}\; \vec{q}\,|\, G_0\; t\,|\, \vec{p}\,'\; \vec{q}\,' > = \; \frac{1}{E - \frac{p^2}{m} - \frac{3}{4m} q^2} \;\; < \vec{p}\; \vec{q}\,|\; t\,|\, \vec{p}\,'\; \vec{q}\,' > \tag{3.40}$$

The t-operator is driven by V which acts only on the states $|\; \vec{p} >$ in the two-body subsystem, thus

$$\langle \vec{p}\; \vec{q}\,|\; V\; |\, \vec{p}'\; \vec{q}'\rangle \;=\; \delta(\vec{q} - \vec{q}')\; \langle \vec{p}\; \vec{q}\,|\; V\; |\, \vec{p}'\; \vec{q}'\rangle \tag{3.41}$$

The third noninteracting particle described by $\vec{q}$ is often called the spectator particle. Let us now consider the LSE for $t(E)$:

$$\begin{aligned}
\langle \vec{p}\,\vec{q}\,|\, t(E)\,|\,\vec{p}\,'\,\vec{q}\,'\rangle &= \langle \vec{p}\,\vec{q}\,|\,V\,|\,\vec{p}\,'\,\vec{q}\,'\rangle + \;<\vec{p}\,\vec{q}\,|\,VG_0t(E)\,|\,\vec{p}\,'\,\vec{q}\,'\rangle \qquad (3.42)\\
&= \delta(\vec{q}-\vec{q}\,')\,\langle\vec{p}\,|\,V\,|\,\vec{p}\,'\rangle\\
&+ \int d^3p\,''\,d^3\vec{q}\,''\langle\vec{p}\,\vec{q}\,|\,V\,|\,\vec{p}\,''\,\vec{q}\,''\rangle\,\frac{1}{E+i\epsilon\;-\;\frac{p\,''^2}{m}\;-\;\frac{3}{4m}\,q''^2}\\
&\quad\langle\vec{p}\,''\vec{q}\,''\,|\,t(E)\,|\,\vec{p}\,'\vec{q}\,'\rangle\\
&= \delta(\vec{q}-\vec{q}\,')\,\langle(\vec{p}\,|\,V\,|\,\vec{p}\,'\rangle\\
&+ \int d^3p''\,\langle\vec{p}\,|\,V\,|\,\vec{p}\,''\rangle\,\frac{1}{E+i\epsilon\;-\;\frac{p''^2}{m}\;-\;\frac{3}{4m}\,q''^2}\,\langle\vec{p}\,''\vec{q}\,'\,|\,t(E)\,|\,\vec{p}\,'\vec{q}\,'\rangle
\end{aligned}$$

The solution is

$$<\vec{p}\,\vec{q}\,|\,t(E)\,|\,\vec{p}'\,\vec{q}'> = \;\delta(\vec{q}-\vec{q}')\;\;<\vec{p}\,|\,\hat{t}(E-\frac{3}{4m}q^2)\,|\,\vec{p}'> \qquad (3.43)$$

where $\hat{t}$ is the true two-nucleon t-operator at the subsystem energy $E-\frac{3}{4m}\,q^2$. This is called a complete off-the-energy shell t-matrix, since its two-body subsystem energy is independent of the initial and final momenta $\vec{p}'$ and $\vec{p}$, respectively. For on-the-energy shell NN scattering one has $|\,\vec{p}\,| = |\,\vec{p}'\,| = \sqrt{m\cdot \text{two-body energy}}$. This off-shell $\hat{t}$-matrix obeys the LSE

$$\begin{aligned}
<\vec{p}\,|\,\hat{t}(z)\,|\,\vec{p}'> &= \;<\vec{p}\,|\,V\,|\,\vec{p}'> \qquad (3.44)\\
&+ \int d^3p''\;<\vec{p}\,|\,V\,|\,\vec{p}''>\;\frac{1}{z-\frac{\vec{p}''^2}{m}}\;<\vec{p}''\,|\,\hat{t}(E)\,|\,\vec{p}'>
\end{aligned}$$

where $z=E-\frac{3}{4m}\vec{q}^{\,2}$. Since for bound states $E<0, z$ ranges from E towards increasingly negative energies for increasing q-values.

Now back to the Faddeev equation

$$\begin{aligned}
\langle\vec{q}\,\vec{p}\,|\,\psi\rangle &= \int d^3p' \int d^3p''\,d^3q''\,\frac{1}{E-\frac{p^2}{m}-\frac{3}{4m}q^2} \qquad (3.45)\\
&\quad\langle\vec{p}\,|\,\hat{t}(E-\frac{3}{4m}q^2)\,|\,\vec{p}\,'\rangle\langle\vec{p}\,'\vec{q}\,|\,P\,|\,\vec{p}\,''\,\vec{q}\,''\rangle\langle\vec{p}\,''\vec{q}\,''\,|\,\psi\rangle
\end{aligned}$$

We face now the central feature of the three-body system, the transitions between different two-body subsystems generated by the permutation operator $P=P_{12}P_{23}+P_{13}P_{23}$.

Let us first consider

$$\begin{aligned}
P_{12}P_{23}\,|\,\vec{p}\vec{q}\,\rangle_1 &= P_{12}P_{23}(1,23) = (2,31) = |\,\vec{p}\vec{q}\,\rangle_2 \qquad (3.46)\\
P_{13}P_{23}\,|\,\vec{p}\vec{q}\,\rangle_1 &= P_{13}P_{23}(1,23) = (3,12) = |\,\vec{p}\vec{q}\,\rangle_3
\end{aligned}$$

The state $|\,\vec{p}\vec{q}\rangle_2$ is of type $|\,\rangle_2$. Since the operators have to be evaluated in the same type of basis vectors, one has to consider $\vec{q}_1(\vec{p}_2,\vec{q}_2)$ and $\vec{p}_1(\vec{p}_2,\vec{q}_2)$ in order to evaluate the matrix elements. Using Eqs. (3.25), we obtain

$$|\,\vec{p}\vec{q}\,\rangle_2 \;=|\left(-\,\frac{1}{2}\,\vec{p}\,+\,\frac{3}{4}\,\vec{q}\right)\;\left(-\vec{p}\,-\,\frac{1}{2}\,\vec{q}\right)\rangle \tag{3.47}$$

in coordinates of type 1. Similarly, we obtain

$$|\,\vec{p}\vec{q}\,\rangle_3 \;=|\left(-\,\frac{1}{2}\,\vec{p}\,-\,\frac{3}{4}\,\vec{q}\right)\left(\vec{p}\,-\,\frac{1}{2}\,\vec{q}\right)\rangle\;. \tag{3.48}$$

Now we are prepared to evaluate the matrix element

$$\begin{aligned}
\langle\vec{p}\,'\vec{q}\,|\,P\,|\,\vec{p}\,''\vec{q}\,''\rangle &= {}_1\langle\vec{p}\,'\vec{q}\,|\,P\,|\,\vec{p}\,''\vec{q}\,''\rangle_1 \qquad (3.49)\\
&= {}_1\langle\vec{p}\,'\vec{q}\,\vec{p}\,''\vec{q}\,''\rangle_2 \;+\; {}_1\langle\vec{p}\,'\vec{q}\,|\,\vec{p}\,''\vec{q}\,''\rangle_3\\
&= \langle\vec{p}\,'\vec{q}\,|\left(-\,\frac{1}{2}\,\vec{p}\,''+\,\frac{3}{4}\,\vec{q}\,''\right);\left(-\vec{p}\,''\,-\,\frac{1}{2}\,\vec{q}\,''\right)\rangle\\
&\quad +\langle\vec{p}\,'\vec{q}\,|\left(-\,\frac{1}{2}\,p''\,-\,\frac{3}{4}\,q''\right);\left(\vec{p}\,''\,-\,\frac{1}{2}\vec{q}\,''\right)\rangle\\
&= \delta\left(\vec{p}\,'+\,\frac{1}{2}\,\vec{p}\,''\,-\,\frac{3}{4}\,\vec{q}\,''\right)\delta\left(\vec{q}+\vec{p}\,''+\frac{1}{2}\,\vec{q}\,''\right)\\
&\quad +\delta\left(\vec{p}\,'\,+\,\frac{1}{2}\,\vec{p}\,''+\frac{3}{4}\,\vec{q}\,''\right)\delta\left(\vec{q}\,-\,\vec{p}\,''+\,\frac{1}{2}\,\vec{q}\,''\right)
\end{aligned}$$

The two δ-functions eliminate two of the integrations in Eq. (3.32). Eliminating the $\vec{p}\,''$-dependence in the δ function gives for the matrix elements of the permutation operator

$$\begin{aligned}
\langle\vec{p}\,'\vec{q}\,|\,P\,|\,\vec{p}\,''\vec{q}\,''\rangle &= \delta\left(\vec{p}\,'\,-\,\frac{1}{2}\,\vec{q}-\vec{q}\,''\right)\;\delta\left(\vec{q}\,+\,\vec{p}\,''\,+\,\frac{1}{2}\,\vec{q}\,''\right) \qquad (3.50)\\
&\quad+\;\delta\left(\vec{p}\,'\,+\,\frac{1}{2}\,\vec{q}\,+\,\vec{q}\,''\right)\;\delta\left(\vec{q}\,-\,\vec{p}\,''\,+\,\frac{1}{2}\,\vec{q}\,''\right)
\end{aligned}$$

Now the Faddeev equation can be explicitly written as

$$\begin{aligned}
\langle\vec{p}\vec{q}\,|\,\psi\rangle &= \frac{1}{E-\frac{p^2}{m}\,-\,\frac{3}{4m}\,q^2}\int\,d^3\vec{q}\,''\\
&\times\;\Bigg\{\langle\vec{p}\,|\,\hat{t}\,\left(E\,-\,\frac{3}{4m}\,q^2\right)\;|-\,\frac{1}{2}\,\vec{q}-\vec{q}\,''\rangle\,\langle\vec{q}\,+\,\frac{1}{2}\,\vec{q}\,'';\vec{q}\,''\,|\,\psi\rangle\\
&+\;\langle\vec{p}\,|\,\hat{t}\,\left(E\,-\,\frac{3}{4m}\,q^2\right)|\;\frac{1}{2}\,\vec{q}+\vec{q}\,''\rangle\,\langle-\,\vec{q}\,'\,-\,\frac{1}{2}\,\vec{q}\,'';\vec{q}\,''\,|\,\psi\rangle\Bigg\}\;. \qquad (3.51)
\end{aligned}$$

We see that the permutation operator screws up the arguments of the $\hat{t}$ and ψ amplitudes. The Faddeev equation is a three-dimentional integral equation for six variables, apparently not a trivial task. However, with modern computers and computational tools, it is possible to solve Eq. (3.51), as has been demonstrated recently for the bosonic case by Elster, Schadow, Nogga, Glöckle (to be published in Few Body Systems).

3.3 The Faddeev Equation in Momentum Space and Partial Wave Projected

In the application to nuclear physics one makes use of the short range nature of the nuclear forces, which leads to the fact, that $\hat{t}$ acts mainly in s-waves (including the d-wave part induced by the tensor force). Thus a partial wave representation is used in almost all practical applications [1].

The NN subsystem basis

$$\mid p(ls)jm\ tm_t > \tag{3.52}$$

is now enriched by the motion of the third particle with respect to that subsystem as described by

$$\mid q\ (\lambda\frac{1}{2})\ JM\ \frac{1}{2}\nu > \tag{3.53}$$

where λ is the orbital angular momentum and J the total angular momentum of that third particle. One then couples j and J to the total conserved angular momentum $\mathcal{J}$ and t with $1/2$ to the total isospin T. This leads to the basis states

$$\begin{aligned} & \mid pq\ (ls)\ j\ (\lambda\frac{1}{2})\ J\ \mathcal{J}M\ (t\frac{1}{2})T\ M_T > \qquad (3.54) \\ \equiv\quad & \sum_\mu C(jJ\ \mathcal{J}, \mu\ M-\mu)\ \sum_n C(t\frac{1}{2}T, \nu\ M_T-\nu) \\ & \mid p\ (ls)\ j\mu\ t\nu > \mid q\ (\lambda\frac{1}{2})\ J\ M-\mu\ \frac{1}{2}\ M_T-\nu > \end{aligned}$$

For convenience we abbreviate all the discrete quantum numbers by α and write $\mid pq\alpha >$. That basis is complete:

$$\sum_\alpha \int_0^\infty dp\ p^2 \int_0^\infty dq\ q^2\ \mid pq\alpha >< pq\alpha \mid =\ \mathbf{1} \tag{3.55}$$

[1] We follow the presentation described in detail in W. Glöckle, The Quantum-Mechanical Few-Body Problem, Springer Verlag 1983 and W. Glöckle, Nucl. Phys. A381 (1982) 343.

If we require $l + s + t = odd$, $\mid pq\alpha >$ is antisymmetric in the two-body subsystem and thus suitable to expand the Faddeev component ψ:

$$\mid \psi > = \sum_\alpha \int_0^\infty dp\ p^2 \int_0^\infty dq\ q^2\ \mid pq\alpha >< pq\alpha \mid \psi > \tag{3.56}$$

The derivation of presenting the Faddeev equation in the basis of Eq. (3.55) is briefly sketched below.

$$\begin{aligned}
< p\ q\ \alpha \mid \psi > &= < p\ q\ \alpha \mid G_0 tP \mid \psi > \qquad (3.57)\\
&= \frac{1}{E - \frac{p^2}{m} - \frac{3}{4m}q^2}\ < p\ q\ \alpha \mid tP \mid \psi > \\
&= \sum_{\alpha'} \int dp'\ p'^2 \int dq'\ q'^2 < pq\alpha|t|p'q'\alpha' >< p'q'\alpha'|P|\psi > \qquad (3.58)
\end{aligned}$$

The t-matrix is diagonal in the quantum numbers of the spectator particle

$$< p\ q\ \alpha \mid t(E) \mid p'\ q'\ \alpha' > = \frac{\delta(q-q')}{qq'}\ \delta_{\lambda\lambda'}\delta_{JJ'}\ \hat{t}_{ll'}^{sjt}(p, p', E - \frac{3}{4m}q^2) \tag{3.59}$$

Let α' be equal α except for a possible change of l to l' in case of coupled two-body channels. Then

$$\begin{aligned}
< p\ q\ \alpha|\psi > &= \frac{1}{E - \frac{p^2}{m} - \frac{3}{4m}q^2} \sum_{l'} \int_0^\infty dp'\ p'^2\ \hat{t}_{ll'}^{sjt}(p, p', E - \frac{3}{4m}q^2) \qquad (3.60)\\
&\quad \times < p'\ q\ \overline{\alpha} \mid P \mid \psi > \\
&= \frac{1}{E - \frac{p^2}{m} - \frac{3}{4m}q^2} \sum_{l'} \int_0^\infty dp'\ p'^2\ \hat{t}_{ll'}^{sjt}(p, p', E - \frac{3}{4m}q^2) \qquad (3.61)\\
&\quad \sum_{\alpha''} \int_0^\infty dp''\ p''^2 \int_0^\infty dq''\ q''^2 < p'q\overline{\alpha}|P|p''q''\alpha'' >< p''q''\alpha''|\psi >
\end{aligned}$$

Again the evaluation of the permutation operator is at the very heart of the 3-body problem. The result is

$$< p\ q\ \alpha \mid P \mid p'\ q'\ \alpha' > = \int_{-1}^{-1} dx\ G_{\alpha\alpha'}(q\ q'\ x)\ \frac{\delta(p - \pi_1)}{\pi_1^{l+2}}\ \frac{\delta(p' - \pi_2)}{\pi_2^{l'+2}} \tag{3.62}$$

with

$$\pi_1 = \sqrt{\frac{1}{4}q^2 + q'^{\,2} + qq'x} \tag{3.63}$$

$$\pi_2 = \sqrt{q^2 + \frac{1}{4}q'^{\,2} + qq'x} \tag{3.64}$$

and $G_{\alpha\alpha'}(qq'x)$ a purely geometrical quantity. We end up with

$$< p\ q\ \alpha \mid \psi > \;=\; \frac{1}{E - \frac{p^2}{m} - \frac{3}{4m}q^2} \sum_{l'} \sum_{\alpha''} \int_0^\infty dq''\ q''^{\,2} \tag{3.65}$$
$$\times \int_{-1}^{1} dx\ \frac{\hat{t}^{sjt}_{ll'}(p, \pi_1, E - \frac{3}{4m}q^2)}{\pi_1^{l'}}\ G_{\overline{\alpha}\alpha''}(q, q'', x)\ \frac{< \pi_2\ q''\ \alpha'' \mid \psi >}{\pi_2^{l''}}$$

This is a set of an infinite number of coupled equations for amplitudes in two variables. If one assumes that the NN t-matrix acts only in very few partial waves, say s-waves only, then the number of coupled equations is correspondingly small. Let us regard the so called five channel case:

α	l	s	j	λ	J	$\mathcal{J}^\pi = 1/2^+$
1	0	0	0	0	1/2	
2	0	1	1	0	1/2	
3	2	1	1	0	1/2	
4	0	1	1	2	3/2	
5	2	1	1	2	3/2	

We see that in the two-body subsystem the NN forces act only in the states ${}^1S_0$ and ${}^3S_0 -{}^3 S_1$. This restriction gives already most of the triton binding energy.

How does one solve that coupled set? Apparently the skew arguments in ψ under the integral require an interpolation. We use Spline interpolation
(W. Glöckle et al, Z. Phys. A 305 (1982) 217) of the form:

$$f(x) \;\simeq\; \sum_k S_k(x)\ f(x_k) \tag{3.66}$$

where $S_k(x)$ are given Spline elements and $\{x_n\}$ a suitable set of grid points. Also the $\hat{t}$-matrix has to be interpolated. Finally the q''-integration can be discretized by Gaussian quadrature, for instance. Thus choosing two sets of grid points in p and q, we are lead to

$$\psi_\alpha(p_k\ q_l) \;=\; \frac{1}{E - \frac{p_k^2}{m} - \frac{3}{4m}q_l^2} \tag{3.67}$$
$$\times \sum_{l'} \sum_{\alpha''} \sum_n \omega_n\ q_n^2 \int_{-1}^{1} dx \sum_i t^{sjt}_{ll'}(p_k,\ p_i,\ E - \frac{3}{4m}q_l^2)$$
$$\times \frac{S_i(\pi_1)}{\pi_1^{l'}}\ G_{\overline{\alpha}\alpha''}(q_l,\ q_m,\ x) \sum_m S_m(\pi_2)\ \psi_{\alpha''}(p_m\ q_n)$$
$$\equiv \sum_{mn} \sum_{\alpha''} K_{\alpha\alpha''}\ (kl, mn)\ \psi_{\alpha''}(p_m q_n) \tag{3.68}$$

What is the dimension N of the kernel?

$$N \;=\; N_p \; N_q \; N_\alpha \tag{3.69}$$

Typical numbers are $N_p = 34$
$N_q = 20$

It is easy to count the number of discrete $\alpha' s$ assuming that $\hat{t}$ acts up to

$$j_{max} = 1, 2, 3, 4, \qquad\qquad \text{etc.} \tag{3.70}$$

It results in

j_{max}	N_α
1	5
2	18
3	26
4	34
5	42

For realistic calculations in the 3N system 34 channels are required and thus $N \sim 25000$

This leads to a sizable matrix K and iteration techniques with simple matrix multiplications are required. We use a Lanczo's type technique, which is very fast and economic.

3.4 Theoretical Triton Properties

We display now triton binding energies for various NN forces allowing the NN forces to act only up to different total two-body angular momenta j_{max}.

				j_{max}	
Potential	1	2	3	4	5
Paris	7.30	7.38	7.44	7.46	7.46
Nijm78	7.49	7.54	7.62	7.63	-
Nijm II(np)	7.65	7.75	-	7.89	-
AV14	7.45	7.58	7.67	7.68	-
Bonn B	8.17	8.10	8.13	8.14	8.14
Ruhrpot	7.59	7.56	7.62	7.64	-

Obviously $j_{max} = 4$ is sufficient in relation to the discrepancy to the experimental value of -8.48 MeV.
The most modern NN forces distinguish between np and nn forces, which can be incorporated into the Faddeev equations and using the isospin formalism. It can easily be shown that it is sufficient to choose in the state $^1S_0$ the effective t-matrix

$$t_{eff} \;=\; \frac{2}{3}\, t_{nn}^{t=1} \;+\; \frac{1}{3} t_{np}^{t=1} \tag{3.71}$$

For a derivation see
H. Witala et al, Phys. Rev. C43 (1991) 1619

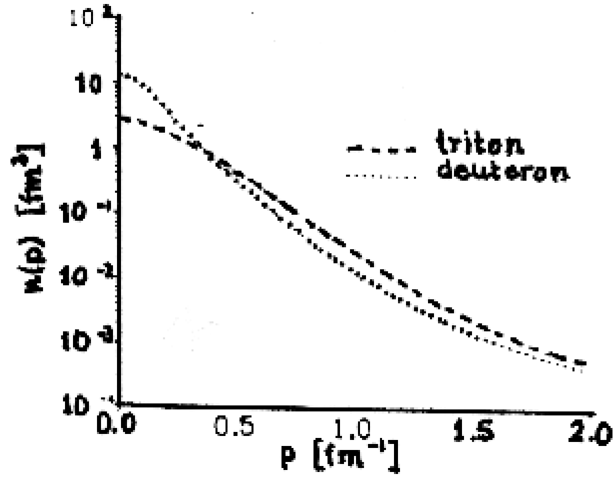
There is a tiny admixture of $T = 3/2$ states, which can be neglected for present day purposes. See the same reference.

We display now the theoretical ^{3}H binding energies using CIB in the state $^1S_0$ (the t_{eff} of Eq. (3.71) for the most recent NN forces.

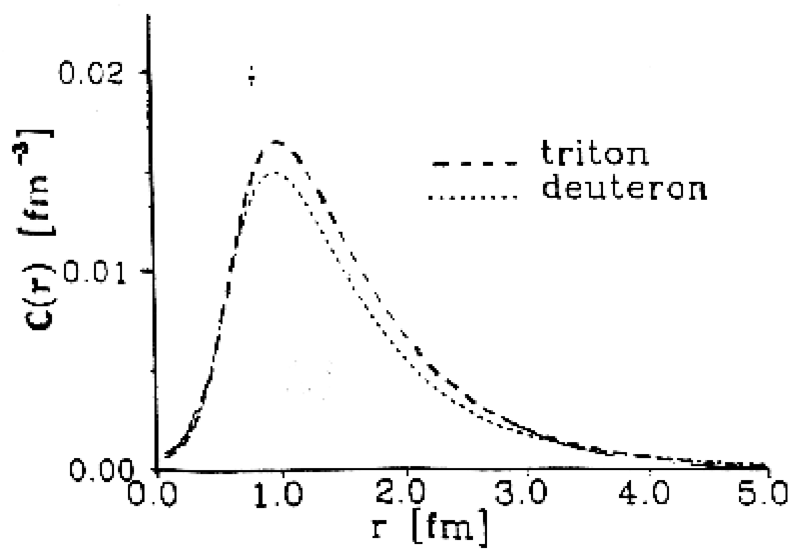
Potential	$E_{^3H}$ ($j_{max} = 4$)
Nijm93 (cd)	7.66
Nijm I (cd)	7.73
Njm II (cd)	7.64
AV18 (cd)	7.65
Bonn (cd)	8.00

While the purely local potentials Nijm II, Nijm 93 and AV18 yield essentially the same binding energies, the weakly nonlocal force Nijm I has about 100 keV more and the highly nonlocal one, CD Bonn, about 350 keV. The reason for the stronger attraction resulting from nonlocalities has not yet been clearly worked out and in any case nonlocalities deserve more attention in the future.
Right now one ends up with 500 to 800 keV underbinding out of 8.48 MeV. This is significant, but a nuclear binding energy is a difference of two big numbers, the negative potential energy and the positive kinetic energy. The potential energy ranges between -45 MeV to -55 MeV in the triton, depending on the NN force, thus the missing binding energy is of the order of 1% of the potential energy. In other words a change of the potential energy in the Hamiltonian of the order of 1 % would be sufficient to cure that binding energy defect. We shall come back below to typical contributions of present day three-nucleon forces.

A comparison of the momentum distribution $n(k)$ for the deuteron and ^3H: gives

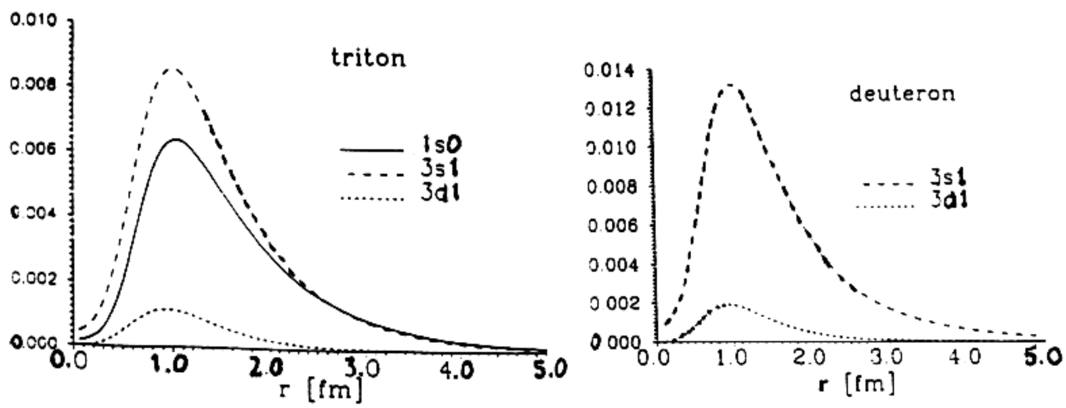


One recognizes a shift of strength to higher components in ^3H in comparison to the deuteron. Of interest is also the NN correlation function $C(r)$ to find two nucleons at a certain distance r. This is compared for the deuteron and ^3H:



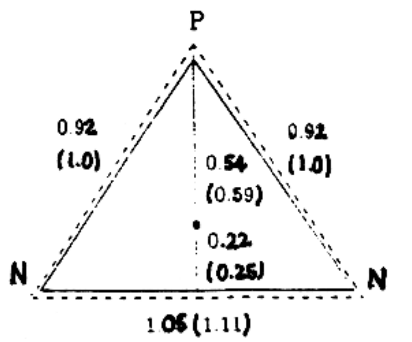
Both $C's$ are normalized in the same manner: $4\pi \int_0^\infty dr\ r^2 C(r) = 1$. The fact that the maximum around 1 fm is higher for ^{3}H than for the deuteron is mainly a consequence of the fact that the separation energy in ^{3}H is larger and thus $C(r)$ drops faster for large $r's$. Interesting however is the fact that both $C's$ are very close to each other for $r \leq 1$ fm. The strong short range repulsion of the NN force dominates totally and the presence or absence of a third particle does not matter.

One can also consider state dependent correlation functions, which describe the probability to find two nucleons at a certain distance r under the additional condition to be in a certain state $(ls)j$. The stronger ones are displayed and compared to the ones of the deuteron:



Since two nucleons in ^{3}H can also occupy the state $^1S_0$ the probabilities in the $^3S_1$ and $^3D_1$ states are reduced in relation to the deuteron, but otherwise they look quite similar. Correlation functions in p and d-states are much smaller, by far more than a factor 10.

Finally one can ask, which are the most probable sites for the three nucleons in ^{3}H. The result is:



The most probable sites of protons and neutrons in ^{3}H for the Paris (dashed) and the Bonn B (solid) potentials. Distances are given in fm for Bonn B and Paris (in parenthesis).

We see a nearly equilateral triangle with the pair distances between the identical neutrons slightly larger than between the proton and the neutrons. The pair distances are about 1 fm and depend somewhat on the NN force as shown by the two examples. If one goes away from that most probable sites the probabilities drop quickly.
This picture of the ^{3}H wave function shows granularity: the nucleons are rather well separated.

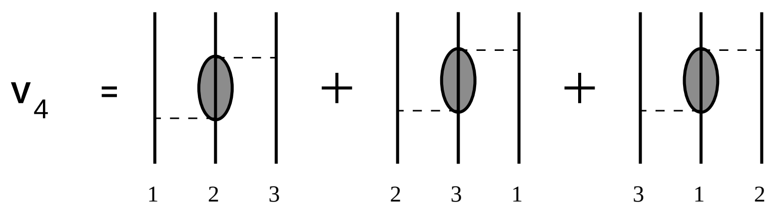
3.5 Inclusion of Three-Nucleon Forces

There is a quite rich literature on the interesting issue of three-nucleon forces, see for example

- M.R. Robilotta, Few-Body Systems, Suppl. 2 (1987) 35
- B.H.J. McKellar, Lecture Notes in Physics 260 (1986) 7
- S.A. Coon, Few-Body Systems, Suppl. 1 (1986) 92
- P.U. Sauer et al, Europhysics News 15 (1984) 5
- R.B. Wiringa, Phys. Rev. C43 (1991) 1585

- D. Plümper et al, Phys. Rev. C49 (1994) 2370
- and more recently in the context of chiral perturbation theory U. van Kolck, Phys. Rev. C 49 (1994) 2932

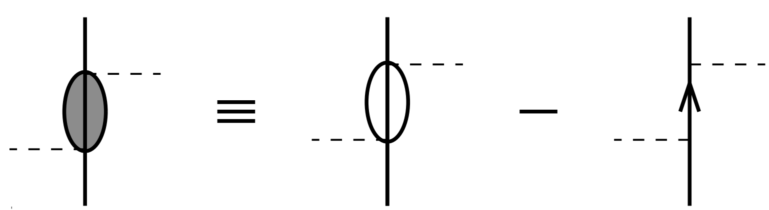
Here these physical questions should not be discussed. They are of course intimately connected to the NN force problem itself, which we also did not treat. Instead we concentrate on the technical challenge, how to incorporate a given 3NF into the Faddeev scheme and its numerical realization. One of possible 3NF mechanisms, which has the longest range and should exist, is the $\pi - \pi$ exchange:



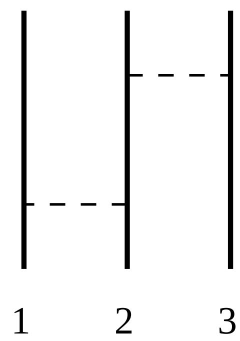
$$= \quad V_4^{(2)} \qquad V_4^{(3)} \qquad V_4^{(1)}$$

$$= \sum_i V_4^{(i)}$$

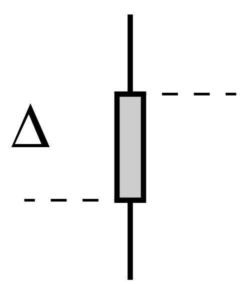
where the blob indicates the complete off- the mass-shell $\pi - N$ scattering amplitude minus the forward propagating nucleon part of the fermion propagator:



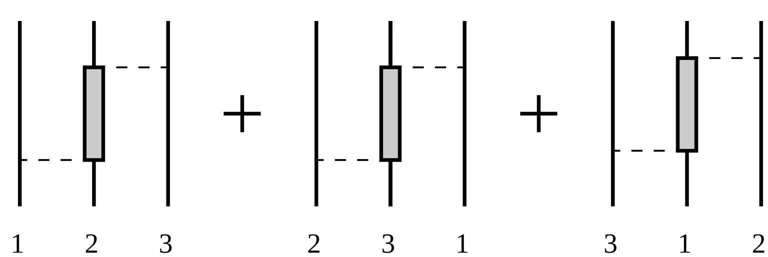
That intermediate state with a nucleon inserted into any of the 3 processes above leads for instance to



which is just a sequence of pair interactions with a free 3N propagator in between: $V_{12}\ G_0\ V_{23}$. That is already included in the Schödinger equation with NN forces only. Therefore that term in the expressions V_4 has to be subtracted from the full $\pi - N$ amplitude. One often mentioned contribution to the $\pi - N$ amplitude, which does not include the nucleon propagator, is



with an intermediate Δ (excited nucleon). This leads to the Fujuta-Miyazwa force



J. Fujita et al, Prog. Theor. Phys. 17 (1957) 360
Low energy theorems and current algebra considerations lead to additional contributions (see Ref. above) to the $\pi - N$ amplitude. It results an often used 3NF model, the Tucson-Melbourne 3NF with $\pi - \pi$ exchange:

$$\begin{aligned} V_4^{(1)} &= \frac{1}{(2\pi)^6} \frac{g_{\pi NN}^2}{4m^2} \frac{F_{\pi NN}^2(\vec{Q}^{\,2})}{\vec{Q}^{\,2}+m_\pi^2} \frac{F_{\pi NN}^2(\vec{Q}'^{\,2})}{\vec{Q}'^{\,2}+m_\pi^2} \vec{\sigma}_2 \cdot \vec{Q}\; \vec{\sigma}_3 \cdot \vec{Q}\,' \\ & \Big\{ \vec{\tau}_2 \cdot \vec{\tau}_3 \; [a \;+\; b\vec{Q}\vec{Q}\,' \;+\; c\,(\vec{Q}^{\,2} \;+\; \vec{Q}'^2\,)\,] \\ & + \; \vec{\tau}_3 \times \vec{\tau}_2 \cdot \vec{\tau}_1 \; \vec{\sigma}_1 \cdot (\vec{Q} \times \vec{Q}\,')\; d \Big\} \end{aligned} \tag{3.72}$$

The superscript (1) indicates the process in the diagram above, where nucleon 1 is in the middle. The remaining two pieces result from cyclical and anticyclical permutations

$$V_4 \;=\; V_4^{(1)} \;+\; P_{12}P_{23}V_4^{(1)}P_{13}P_{23} \;+\; P_{13}P_{23}V_4^{(1)}P_{12}P_{23} \tag{3.73}$$

Further, $F_{\pi NN}(\vec{Q})$ is a strong formfactor, $\vec{Q}$ and $\vec{Q}'$ pion momenta and a, b, c, d constants provided by theory and adjustment to experiment (see Ref above).

The expression for $V_4^{(1)}$ can easily be transformed into configuration space

$$\begin{aligned} V_4^{(1)} \;&=\; (\frac{g_{\pi NN} m_\pi}{2m})^2 \, \frac{1}{(4\pi)^2} \, \Big[\quad \vec{\tau}_2 \cdot \vec{\tau}_3 \, \Big(\\ & \vec{\sigma}_2 \cdot \vec{\nabla}_2 \; \vec{\sigma}_3 \cdot \vec{\nabla}_3 \; \Big\{ \; (a - 2m_\pi^2 c)\; Z_1(x_{12}) \; Z_1(x_{13}) \\ + \quad & c\,[Z_0(x_{12}) \; Z_1(x_{13}) \;+\; Z_1(x_{12}) Z_0(x_{13})\,] \\ + \quad & b\vec{\nabla}_2 \cdot \vec{\nabla}_3 Z_1(x_{12}) Z_1(x_{13}) \; \Big\} \Big) \\ + \quad & \vec{\tau}_3 \times \vec{\tau}_2 \cdot \vec{\tau}_1 \; \vec{\sigma}_2 \cdot \vec{\nabla}_2 \; \vec{\sigma}_3 \cdot \vec{\nabla}_3 \; \vec{\sigma}_1 \cdot \vec{\nabla}_2 \times \nabla_3 Z_1(x_{12}) Z_1(x_{13}) \; d \qquad \Big] \end{aligned} \tag{3.74}$$

where

$$Z_n(x_{ij}) \;=\; \frac{4\pi}{m_\pi} \int \frac{d\vec{Q}}{(2\pi)^3} \; e^{i\; (\vec{x}_i - \vec{x}_j)\cdot\vec{Q}} \; \frac{F_{\pi NN}^2(\vec{Q})}{(\vec{Q}^{\,2}+m_\pi^2)^n} \tag{3.75}$$

If we disregard the spin- and isospin dependencies and consider only the a-term, we just encounter a product of two regularized Yukawa interactions for the nucleons 12 and 13, which is clearly a 3NF. The full force however is more complex and includes more terms and spin- and isospin dependencies.

The Tucson-Melbourne $\pi - \pi$ exchange model has been enriched (see Ref. above) by the $\pi - \rho$ and $\rho - \rho$ exchanges, which because of the spin 1 of the ρ-meson includes new types of spin-dependencies.

How does one incorporate such a force into the Faddeev equation?

Let us start again from the Schrödinger equation

$$(H_0 \ + \ \sum \ V_i \ + \ \sum \ V_4^{(i)}) \ \Psi \ = \ E \ \Psi \tag{3.76}$$

where we assumed a decomposition of the 3NF V_4 into 3 pieces. On physical grounds (identity of the nucleons) $V_4^{(i)}$ has to be symmetrical under exchange of the nucleons jk with $j \neq i \neq k$, like the NN force $V_i \equiv V_{jk}$. Thus it appears natural to join V_i and $V_4^{(i)}$ and we can follow the derivation of the Faddeev equation given above:

$$\Psi \ = \ G_0 \ \sum_i \ (V_i \ + \ V_4^{(i)}) \ \Psi \equiv \ \sum_i \ \psi_i \tag{3.77}$$

and

$$\psi_i = G_0(V_i + V_4^{(i)})\Psi = G_0(V_i + V_4^{(i)}) \sum_j \psi_j \tag{3.78}$$

Proceeding similar to the derivation of the Faddeev equation without 3NF:

$$(1 - G_0 V_i)\psi_i = G_0 V_i \sum_{j \neq i} \psi_j + \ G_0 V_4^{(i)} \sum_j \psi_j, \tag{3.79}$$

from which follows

$$\psi_i = (1 - G_0 V_i)^{-1} G_0 V_i \sum_{j \neq i} \psi_j + (1 - G_0 V_i)^{-1} G_0 V_4^{(i)} \sum_j \psi_j \tag{3.80}$$

A consequence of the LSE for t_i is that

$$(1 \ - \ G_0 \ V_i)^{-1} \ = \ 1 \ + \ G_0 \ t_i \tag{3.81}$$

thus

$$\psi_i \ = \ G_0 t_i \ \sum_{j \neq i} \ \psi_j \ + \ (1 \ + \ G_0 t_i) \ G_0 \ V_4^{(i)} \ \sum_j \ \psi_j \tag{3.82}$$

For identical particles again one amplitude, say $\psi \equiv \psi$, is sufficient and we get dropping the index 1,

$$\psi \ = \ G_0 t P \psi \ + \ (1 \ + \ G_0 t) \ G_0 V_4^{(1)} \ (1 \ + \ P) \ \psi \tag{3.83}$$

The partial wave representation of the additional term is highly nontrivial due to $V_4^{(1)}$, and we refer to

- W. Glöckle, Lecture Notes in Physics 273 (1987) 3
- S. A. Coon et al, Phys. Rev C23 (1981) 1790

for a thorough layout.

In addition the permutation operator P standing between $V_4^{(1)}$ and ψ is better treated in a different manner than in the first term. Looking back to Eq. (3.50) one sees that the two δ-functions can be put into a form, that the momenta to the right can be fixed in terms of the momenta to the left. A partial wave representation in that form is also required if one evaluates the total state Ψ. Its realization can be found in

- D. Hüber et al, Few-Body Systems 16, 165 (1994)
- D.Hüber, H.Witala, A.Nogga, W.Glöckle, and H.Kamada, Few-Body Systems **22**, 107 (1997).

Eq. (3.83) has been solved in

- A. Stadler et al, Phys. Rev. C44, 2319 (1991).

Here are some results from that article:

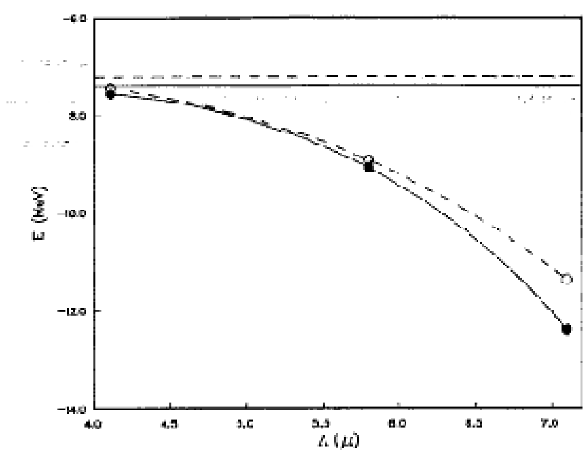


FIG. 1. Triton binding energy E obtained from 18-channel calculations vs the cutoff parameter Λ in the πNN form factor of the Tucson-Melbourne three-nucleon force. The horizontal lines correspond to results without three-nucleon force. The full circles are binding energies calculated actually with the Paris potential and the Tucson-Melbourne three-nucleon force; the empty circles are binding energies actually calculated with the RSC potential and the Tucson-Melbourne three-nucleon force. The solid and dashed curves through the circles are drawn to guide the eye.

We see an uncomfortably strong dependence on the cut-off parameter in the πNN form-factor, chosen as

$$F_{\pi NN}(\vec{Q}^{\,2}) \;=\; \left(\frac{\Lambda^2 \;-\; m_\pi^2}{\Lambda^2 \;+\; \vec{Q}^{\,2}}\right) \tag{3.84}$$

In addition one can include the $\pi - \rho$ and $\rho - \rho$ exchange 3NF's and finds (A. Stadler et al, Phys. Rev. C 51 (1995) 2896):

	no 3NF	$\pi\pi$	$\pi\pi + \pi\rho$	$\pi\pi + \pi\rho + \rho\rho$
RSC	-7.229	-8.904	-8.438	-8.439
Paris	-7.381	-9.060	-8.486	-8.486
Nijm 78	-7.537	-9.347	-8.692	-8.692
OBEPQ	-8.315	-11.056	-9.639	-9.636

These results are based on 'recommended' values for cut-off parameters of various strong meson-nucleon form factors. There remains still, even including $\pi - \rho$ in addition to $\pi - \pi$, an uncomfortably large cut-off dependence. Nevertheless the numbers indicate, that this sort of 3NF's have a good chance to provide the right amount of binding energy. The overbinding in the case of the OBEPQ-potential, however, also tells, that consistency of NN and 3NF's is absolutely necessary. This is still an unsettled question and much more theoretical work is needed.

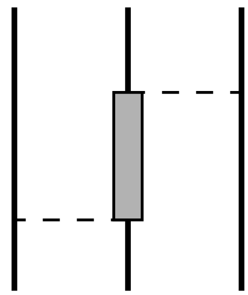
Another approach should be mentioned here, where the nucleons in the triton are allowed to be part time in the excited state of a Δ. Thus the ^{3}H state is of the form

$$|\Psi >= |NNN > + |NN\Delta > + |N\Delta\Delta > + |\Delta\Delta\Delta > . \tag{3.85}$$

Insertion into the Schrödinger equation yields a coupled set of equations for the 4 components, which are driven by various transition potentials between nucleons and $\Delta' s$. The most complete investigation is carried out by

- A. Picklsimer et al, Phys. Rev. C46, 1178 (1992) and References therein,
- Ch. Hajduk et al, Nucl. Phys. A405 (1983) 581; Nucl. Phys. A405 (1983) 605 (earlier and less complete work on the topic)

The main message is, that the attraction delivered by the Δ-mediated 3NF



is essentially canceled by additional repulsive parts and one ends up close to the NN force picture only. Also the $|\ \Delta\,\Delta\,\Delta >$ part turned out to be very small, while the $|\ \Delta\,\Delta\,N >$ part is not.
What is needed is insight into consistency between NN and 3NF's, what is known up to now are first trials only.

3.6 Appendix: The Permutation Group

Permutation: Interchange of 2 items in a group.

Example: Take 3 items: G_1, G_2, G_3

$\Rightarrow 3! = 6$ possible permutations of the ordering of these items:

e:	$G_1G_2G_3$	$\rightarrow$	$G_1G_2G_3$	(no change)
p:	$G_1G_2G_3$	$\rightarrow$	$G_2G_3G_1$	(first to end $\rightarrow$ everything else 1 up)
q:	$G_1G_2G_3$	$\rightarrow$	$G_3G_1G_2$	(last to front)
r:	$G_1G_2G_3$	$\rightarrow$	$G_1G_3G_2$	(first alone, interchange 2 and 3)
s:	$G_1G_2G_3$	$\rightarrow$	$G_3G_2G_1$	(interchange 1 and 3)
t:	$G_1G_2G_3$	$\rightarrow$	$G_2G_1G_3$	(interchange 1 and 2)

This group is called **permutation group** or the **symmetric group** S_3. The general permutation group is S_N and has $N!$ elements.

Standard representation of S_3:

$$e \equiv \begin{pmatrix} G_1 & G_2 & G_3 \\ G_1 & G_2 & G_3 \end{pmatrix} \quad p \equiv \begin{pmatrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_1 \end{pmatrix} \quad q \equiv \begin{pmatrix} G_1 & G_2 & G_3 \\ G_3 & G_1 & G_2 \end{pmatrix}$$

$$r \equiv \begin{pmatrix} G_1 & G_2 & G_3 \\ G_1 & G_3 & G_2 \end{pmatrix} \quad s \equiv \begin{pmatrix} G_1 & G_2 & G_3 \\ G_3 & G_2 & G_1 \end{pmatrix} \quad t \equiv \begin{pmatrix} G_1 & G_2 & G_3 \\ G_2 & G_1 & G_3 \end{pmatrix}$$

Note: The ordering of the items in e.g., the first row is unimportant as long as the type of permutation is preserved, i.e., and interchanges element 2 and 3, leaves one alone $\rightarrow$

$$r = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_1 & G_3 & G_2 \end{pmatrix} \;\; and \;\; r = \begin{pmatrix} G_2 & G_1 & G_3 \\ G_2 & G_3 & G_1 \end{pmatrix}$$

Consider 'multiplication' of permutations:

$$p \cdot r = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_1 \end{pmatrix} \cdot \begin{pmatrix} G_2 & G_3 & G_1 \\ G_2 & G_1 & G_3 \end{pmatrix} = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_2 & G_1 & G_3 \end{pmatrix} = t$$

$\Rightarrow$ set of 6 elements in S_3 is **NOT** independent.

$$p \cdot p = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_1 \end{pmatrix} \cdot \begin{pmatrix} G_2 & G_3 & G_1 \\ G_3 & G_1 & G_2 \end{pmatrix} \begin{pmatrix} G_1 & G_2 & G_3 \\ G_3 & G_1 & G_2 \end{pmatrix} = q$$

$$p \cdot p \cdot r = q \cdot r = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_3 & G_1 & G_2 \end{pmatrix} \cdot \begin{pmatrix} G_3 & G_1 & G_2 \\ G_3 & G_2 & G_1 \end{pmatrix} = s$$

$\Rightarrow$ S_3 has 3 independent elements: e.g., e, p, r.

Check which subsection forms subgroup:

$e, p, q = p^2$ form subgroup of S_3.

Consider following permutations:

$$P_{12}P_{23} \equiv \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ - & - & - \\ 2 & 3 & 1 \end{pmatrix} \quad \Rightarrow \quad P_{12}P_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \equiv p$$

$$P_{13}P_{23} \equiv \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ - & - & - \\ 3 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad P_{13}P_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Application:

Consider a 3-particle wave function

$$\psi = \psi_1 + \psi_2 + \psi_3 = \sum_i \psi_i$$

with **Faddeev components**

$$\psi_1 \equiv \psi(1,23)$$

$$\begin{aligned} \psi_2 \equiv \psi(2,31) = p\psi(1,23) &= P_{12}P_{23}\psi(1,23) \\ &= P_{12}P_{23}\psi_1 \end{aligned} \tag{3.86}$$

$$\psi_3 \equiv \psi(3,12) = q\psi(1,23) = P_{13}P_{23}\psi_1$$

$$\begin{aligned} \Rightarrow \psi = \psi_1 + \psi_2 + \psi_3 &= (1 + P_{12}P_{23} + P_{13}P_{23})\psi_1 \\ &= (1+P)\psi_1 \end{aligned} \tag{3.87}$$

with

$$P = P_{12}P_{23} + P_{13}P_{23}$$

Faddeev components fulfill:

$$\psi_i = G_0 V_i \psi = G_0 V_i(\psi_1 + \psi_2 + \psi_3)$$

$$\Rightarrow \quad \psi_1 = G_0 V_1 \psi$$

$$\psi_2 = G_0 V_2 \psi = P_{12}P_{23}\ G_0 V_1 \psi = P_{12}P_{23}\psi_1$$

here

$$V_2 \equiv V_{13} = P_{12}P_{23}V_{23}$$

$$\psi_3 = G_0 V_3 \psi = P_{13}P_{23}\ G_0 V_1 \psi = P_{13}P_{23}\psi_1$$

here

$$V_3 \equiv V_{12} = p_{13}P_{23}V_{23}$$

with

$$(1 - G_0V_1)^{-1}G_0V_1 = t_1$$

we get

$$\begin{aligned}
\Rightarrow \quad \psi_1 = G_0V_1(\psi_1 + \psi_2 + \psi_3) &= \underbrace{(1 - G_0V_1)^{-1}G_0V_1}_{t_1}(\psi_3 + \psi_3) \qquad (3.88)\\
&= G_0 t_1(\psi_2 + \psi_3)\\
&= G_0 t_1(P_{12}P_{23} + P_{13}P_{23})\psi_1\\
\psi_1 &= G_0 t_1\ P\psi_1
\end{aligned}$$

Show antisymmetry:

$$\begin{aligned}
P_{13}\psi &= P_{13}(1 + P_{12}P_{23} + P_{13}P_{23})\psi(1,23) \qquad (3.89)\\
&= P_{13}\ \psi(1,23) + P_{13}\ \psi(2,31) + P_{13}\ \psi(3,12)\\
&= \psi(3,21) + \psi(2,13) + \psi(1,32)\\
&= -\psi(3,12) - \psi(2,31) - \psi(1,23) =\\
&= -(1+P)\ \psi(1,23)\\
&= = -\psi
\end{aligned}$$