Chapter 6

Spin in Quantum Mechanics

6.1 Spinors and Their Properties

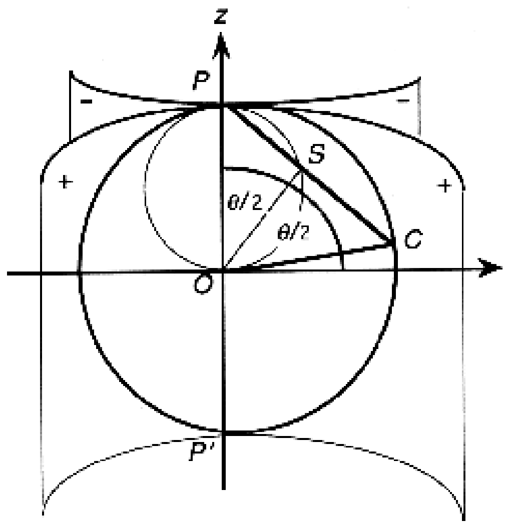
Spinors are mathematical entities, which are useful when describing half-integer spins in the context of rotations of physical systems.

Definition of Spinors:
A mathematical entity S is a spinor if it satisfies the requirement that it changes sign under a 2π rotation:

$$S(\theta + 2\pi) \;=\; -S(\theta) \; . \tag{6.1}$$

Part of this requirement is the assumption that S can have a sign associated with it. As it turns out, spin-$\frac{1}{2}$ objects (primarily electrons and subatomic particles) have wave functions that change sign, if the object is rotated through 2π and which can, therefore, be represented by spinors.

6.1.1 Geometric Visualization of Spinors



A unit circle in the $x - z$ plane is centered at the origin O and intercepting the positive z-axis at P. Consider a ray OC making an angle θ with the z-axis, so that θ is the usual polar angle. Now, bisect the angle so that the point S bisects PC, since $OP = OC$. Thus

$$OS(\theta) \ = \cos(\theta/2) \ ; \quad PS(\theta) \ = \sin(\theta/2) \ . \tag{6.2}$$

OS and PS are taken to be signed quantities, OS is negative for $\pi \ < \theta \ < \ 2\pi$, as is PS for $2\pi \ < \theta \ < /4\pi$.

As the point C goes once around the unit circle, the point S goes from P to O. A second revolution of C brings S back to P. The path of S is a circle of radius $\frac{1}{2}$. This path is traced out in a period of 4π of θ. Considering (6.2), OS and PS change sign when $\theta \longrightarrow \theta + \pi$. Therefore, according to our rule, the pair (OS, PS) forms a spinor associated with C.

In the two-dimensional representation shown in Fig. 6.1, one may view C as marking the tip of the vector $\overrightarrow{OC}$, which has unit magnitude and θ as polar angle. Then OS and PS form the components of a spinor associated with $\overrightarrow{OC}$.

6.1.2 Rotations of Spinors

The properties under rotations of the spinor (OS, PS) associated with C can also be worked out from Fig. 6.1. If C is rotated through an angle β from its position at θ, then one has the following transformations:

$$\begin{aligned} \theta &\longrightarrow \theta + \beta \\ OS &\longrightarrow OS_\beta = \cos(\beta/2)\ OS - \sin(\beta/2)\ PS \\ PS &\longrightarrow PS_\beta = \sin(\beta/2)\ OS + \cos(\beta/2)\ PS\ , \end{aligned} \tag{6.3}$$

which follow from the trigonometric identities for the sum of two angles. The last two equations can be written as

$$\begin{pmatrix} OS(\theta+\beta) \\ PS(\theta+\beta) \end{pmatrix} = \begin{bmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{bmatrix} \begin{pmatrix} OS(\theta) \\ PS(\theta) \end{pmatrix} \tag{6.4}$$

or

$$S(\theta+\beta) = \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} S(\theta)\ , \tag{6.5}$$

where $S(\theta)$ represents the spinor. In this matrix representation, it is clear that the basic spinor property, $S(\theta+2\pi) = -S(\theta)$, is fulfilled. The matrix in (6.5) is the 2×2 matrix that transforms **any** spin-$\frac{1}{2}$ angular momentum eigenstate under rotation by β about the y-axis.

If we want to define rotations of spinors around the z-axis, then we should require that those spinors are eigenfunctions of the angular momentum operator component for this axis,

$$J_3\ S(\theta) = -i\,\frac{\partial S}{\partial \theta} = \pm S(\theta) \tag{6.6}$$

where we have used the definition (4.24) for a generalized angular momentum $\vec{J}$. The linear combination of spinors that satisfy this equation are complex, namely

$$S(\theta) = e^{\pm i\theta/2}\ . \tag{6.7}$$

The matrix form of rotations around the z-axis, which also satisfy (6.6) are

$$S(\theta+\alpha) = \begin{pmatrix} e^{+i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} S(\theta)\ . \tag{6.8}$$

Thus the spinor $S(\theta)$ is complex valued.

6.2 Pauli Matrices and Their Eigenvectors

Pauli matrices are here introduced through their algebraic properties. They describe the simplest, non-trivial spin system, namely spin $\frac{1}{2}$. In cartesian coordinates the Pauli matrices are given by

$$\sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.9}$$

and they are collectively denoted by $\vec{\sigma} \;=\; (\sigma_1, \sigma_2, \sigma_3)$. In terms of Paul matrices, the "spin" matrices $\vec{S}$ are given as

$$\vec{S} \;=\; \frac{1}{2}\,\vec{\sigma}\;. \tag{6.10}$$

Properties of Pauli Matrices are:

$$\sigma_j^2 \;=\; \mathbf{1} \tag{6.11}$$

for $j = 1, 2, 3$. They anticommute

$$[\sigma_i, \sigma_k]_+ \;=\; \sigma_i\sigma_k \;+\; \sigma_k\sigma_i \;=\; 0\;; \quad k \neq i\;. \tag{6.12}$$

For the quantum mechanical description of a spin-$\frac{1}{2}$ particle, one needs obviously the representation of the angular momentum operators $\vec{J}$ with $j = \frac{1}{2}$. The most general form of spin-$\frac{1}{2}$ state would be given by

$$\alpha\,|\,\frac{1}{2}\,,+\,\frac{1}{2}\rangle \;+\; \beta\,|\,\frac{1}{2}\,,-\,\frac{1}{2}\rangle \tag{6.13}$$

or written as column vector

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\;. \tag{6.14}$$

The two basis vectors are then $|\,\frac{1}{2}\,,\pm\,\frac{1}{2}\rangle$ or

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.15}$$

Let us consider

$$S_3 \;=\; \frac{1}{2}\,\sigma_3\;, \tag{6.16}$$

the component of the spin along an arbitrarily chosen 3-axis. We need to find eigenvalues and eigenvectors of σ_3. The eigenvalue equation for σ_3 reads

$$\sigma_3 \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \lambda_\pm \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \tag{6.17}$$

where $\lambda_\pm$ are the eigenvalues. The solutions for the eigenvectors are given by

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.18}$$

with the corresponding eigenvalues $\lambda_+ = +1$ and $\lambda_- = -1$. The eigenvectors $\chi_\pm$ are orthonormal, as required for the eigenvectors of a hermitian operator. The eigenstates $\chi_\pm$ are usually referred to as **Pauli spinors** and χ_+ represents a "spin-up," χ_- a "spin-down" state.

According to our general considerations in Chapter 3, we introduce the ladder operators

$$S_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.19}$$

and it can be easily verified that

$$\begin{aligned} S_+\chi_+ &= |\, 0\rangle \\ S_+\chi_- &= \chi_+ \\ S_-\chi_+ &= \chi_- \\ S_-\chi_- &= |\, 0\rangle \end{aligned} \tag{6.20}$$

Furthermore

$$\begin{aligned} S_1 &= \frac{1}{2}\,(S_+ + S_-) = \frac{1}{2}\,\sigma_1 \\ S_2 &= \frac{1}{2i}\,(S_+ - S_-) = \frac{1}{2}\,\sigma_2 \end{aligned} \tag{6.21}$$

which is exactly the definition given in (6.10). One could have also introduced

$$\sigma_\pm = (\sigma_1 \pm i\sigma_2) \tag{6.22}$$

and then defined the ladder operators as

$$\sigma_{-1} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} ; \quad \sigma_0 = \sigma_3 \ , \quad \sigma_{+1} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} , \tag{6.23}$$

which is often referred to as 'spherical representation.' Here we have

$$\sigma_\pm^2 = 0 \quad \text{and} \quad \sigma_0^2 = \mathbf{1} \ . \tag{6.24}$$

From this follows that higher powers of these spherical-basis matrix elements must be zero. Furthermore, the basis vectors $\chi_\pm$ span the spin-space for spin-$\frac{1}{2}$ particles.

The completeness relation is given by

$$\chi_+\chi_+^\dagger \;+\; \chi_-\chi_-^\dagger \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;=\; \mathbf{1}\ . \tag{6.25}$$

6.3 Finite Rotations and Pauli Matrices

As discussed in the previous chapter, J_3 describes infinitesimal rotations and the J_i are the generators of a Lie algebra. From the generators all finite rotations can be constructed. For spin-$\frac{1}{2}$ states the rotation operator has the following form (cp. 5.95 g)

$$\mathcal{D}^{\frac{1}{2}}(\alpha\beta\gamma) \;=\; e^{-i\frac{\alpha}{2}\sigma_3}\; e^{-i\frac{\beta}{2}\sigma_2}\; e^{-i\frac{\gamma}{2}\sigma_3}\ . \tag{6.26}$$

Here the description of rotations via Euler angles is used.

6.3.1 Rotations About the 3-Axis

Expanding the expression $e^{-i\alpha\sigma_{3/2}}$ and using the properties of the Pauli matrices, namely that for n even, $n = 2m, \sigma_3^{2m} = 1$, and m odd, $n = 2m+1, \sigma_3^{2m+1} = \sigma_3$, one obtains

$$e^{-i\gamma\sigma_{3/2}} \;= \cos(\gamma/2)\ \mathbf{1} \;-\; i\sin(\gamma/2)\ \sigma_3\ . \tag{6.27}$$

By explicitly inserting σ_3 this leads to

$$e^{-i\gamma\sigma_{3/2}} \;=\; \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{+i\gamma/2} \end{pmatrix}\ . \tag{6.28}$$

Thus, matrix elements of rotations around the 3-axis are given by

$$e^{-im\gamma}\ \delta_{m'm}\ . \tag{6.29}$$

6.3.2 Rotations About the 2-Axis

Since σ_2 is not diagonal, the rotation matrix must have a more complicated form. The rotation matrix for a rotation of a spinor was already heuristically derived in 6.1.2 and is given by

$$e^{-i\beta\sigma_2/2} \;=\; \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \;=\; d^{1/2}(\beta)\ . \tag{6.30}$$

The matrix $d^{1/2}(\beta)$ is called the **reduced rotation matrix** for spin $\frac{1}{2}$, reduced in the sense that the rotation is about a single axis (2), rather than about three axes.

Considering (6.30) one can easily see that the transpose gives the same result as the inverse rotation $\beta \longrightarrow -\beta$. For two successive rotations about the 2-axis, one has

$$d^{\frac{1}{2}}(\beta_2)\, d^{\frac{1}{2}}(\beta_1) \;=\; d^{\frac{1}{2}}(\beta_2+\beta_1) \;=\; d^{\frac{1}{2}}(\beta_1+\beta_2) \;=\; d^{\frac{1}{2}}(\beta_1)\, d^{\frac{1}{2}}(\beta_2)\ , \tag{6.31}$$

which corresponds to the fact that successive rotations around the same axis commute.

6.3.3 Spinor Nature of Spin-$\frac{1}{2}$ Rotations

The striking property of both the rotations around the 3- and the 2-axis is the property of the matrices in (6.28) and (6.30), namely that they change sign when their defining angles change by 2π. Thus

$$\begin{aligned} e^{\pm i(\alpha+2\pi)/2} &= -e^{\pm i\alpha/2} \\ d^{1/2}(\beta+2\pi) &= -d^{1/2}(\beta)\ . \end{aligned} \tag{6.32}$$

Thus, rotation matrices for spin-$\frac{1}{2}$ states are spinors because of thin transformation properties.

6.3.4 General Euler-Angle Rotations for Spin-$\frac{1}{2}$

We can now combine the results of our calculations for rotations about a single axis to obtain the Euler-angle rotation matrices for spin-$\frac{1}{2}$. From (6.20) and (6.22), we obtain the full rotation matrix (6.18) explicitly

$$\mathcal{D}^{\frac{1}{2}}(\alpha\beta\gamma) \;=\; \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{+i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{+i\gamma/2} \end{pmatrix} . \tag{6.33}$$

This matrix has the unitary and spinor properties of its component matrices and provides a complete expression for the rotation of a spin-$\frac{1}{2}$ system in terms of Euler angles. Since rotations about **different** axes in general do **not** commute, the order of the matrices in (6.33) is important.

6.4 Spin Space

The eigenvectors of the Pauli matrices provide examples of spinors, they change sign under rotations of 2π. This behavior of the $\chi_\pm$ is apparent from the behavior of the rotation matrix $\mathcal{D}^{\frac{1}{2}}(\alpha\beta\gamma)$ as given in (6.25).

It is useful to use the spin-$\frac{1}{2}$ description as a building block when constructing states and their representations for larger angular momentum numbers. For this purpose, one introduces the spinor space and angular momentum operators in this space.

6.4.1 Spinor Space and Its Matrix Representation

Suppose we have an abstract space whose 'coordinates' are described in terms of the basis vectors χ_+ and χ_-. Technically, this is a Hilbert space.

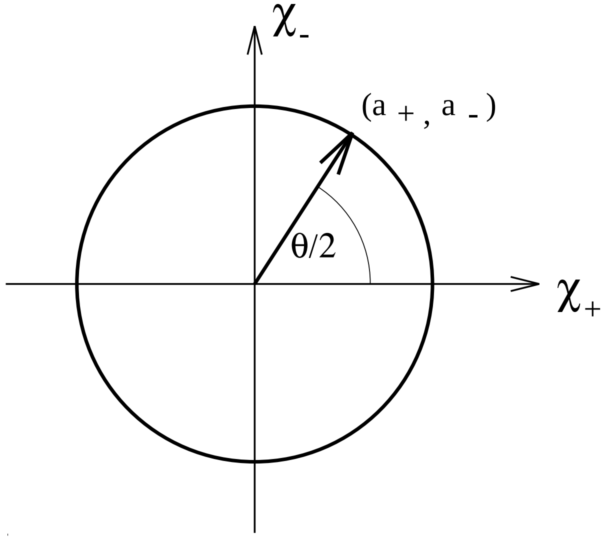


Fig. 6.2 Spinor space for rotations with unit vectors along the axes being χ_+ and χ_-. A representative point in the space undergoes a rotation through $\theta/2$, when the system it describes is rotated by θ.

We want this space to describe rotations for spin-$\frac{1}{2}$ system so the "coordinates" of points in this space, a_+ and a_-, are allowed to be complex and are required to satisfy

$$| a_+ |^2 + | a_- |^2 = 1 . \tag{6.34}$$

The correspondence between the rotation of a spin-$\frac{1}{2}$ system in configuration space and the trajectories of its coordinates in spinor space is to be such that a representative point in the space undergoes a rotation through $\theta/2$ when the system it describes is rotated by θ. In particular, the spinor space coordinates change sign when $\theta = 2\pi$ and return to their original values under a double-angle rotation of $\theta = 4\pi$.

6.4.2 Angular Momentum Operators in Spinor Representation

We define partial differential operators in spinor space (χ_+, χ_-) as

$$\partial_+ \equiv \frac{\partial}{\partial \chi_+} \quad : \quad \partial_- \equiv \frac{\partial}{\partial \chi_-} \, . \tag{6.35}$$

These operators have the usual differentiation properties. For angular momentum operators in the spherical basis, we define

$$\begin{aligned} J_{+1} &= \chi_+ \partial_- \\ J_0 &\equiv \frac{1}{2} \left(\chi_+ \partial_+ - \chi_- \partial_- \right) \\ J_{-1} &\equiv \chi_- \partial_+ \, . \end{aligned} \tag{6.36}$$

To show that the so-defined operators are angular momentum operators, one has to show that they fulfill the commutation relations for angular momentum operators (3.67) - (3.71). They also have the appropriate behavior with respect to χ_+ and χ_-, namely

$$\begin{aligned} J_{\pm 1} \chi_{\mp} &= \chi_{\pm} \\ J_0 \chi_{\pm} &= J_3 \chi_{\pm} = \pm \frac{1}{2} \chi_{\pm} \, . \end{aligned} \tag{6.37}$$

A linear combination of χ_+ and χ_- is usually **not** an eigenstate of J_3. (Here the letter J stands for the generator of the rotation. We have seen in Chapter 3 that in general the eigenvalues can be integers or half-integers.)

6.4.3 Including Spin Space

To include spin in quantum mechanics, we extend the Hilbert space to a **direct product** of an external space with an internal one:

$$|\,\psi\rangle = |\,\psi_{space}\rangle \otimes |\,\chi_{spin}\rangle \equiv |\,\psi_{space} \;|\,\chi_{spin}\rangle \equiv |\,\psi\chi\rangle \, . \tag{6.38}$$

The independence of the two spaces requires

$$[\vec{S}, \vec{L}] \; = \; 0 \; , \tag{6.39}$$

which means that spin operators do not act on the external space (and vice verse).

The *ket* $\mid \chi \rangle$ in (6.38) is the spin-space part of the state vector. It cannot be represented as a continuous function, but instead is represented by $2s+1$ discrete components. To span a spin space of dimension $2s+1$ requires $2s+1$ independent basis vectors. These basis vectors are denoted by $\mid s \; m_s \rangle$, with m_s having all values from s to $-s$ in steps of $1 \; : \; m_s = \{s, s-1, \cdots, -s\}$. These vectors are eigenstates of the spin operators

$$\begin{aligned} \vec{S}^2 \mid s \; m_s \rangle \; &= \; s(s+1) \mid s \; m_s \rangle \; \neq \; S^2 \mid s \; m_s \rangle \\ S_3 \mid s \; m_s \rangle \; &= \; m_s \mid s \; m_s \rangle \; . \end{aligned} \tag{6.40}$$

An **explicit representation** (in the group-theory sense) is given by the set of $2s+1$ numbers

$$\langle \sigma \mid s \; m_s \rangle \; \equiv \; \chi_{s \; m_s}(\sigma) \; = \; \begin{cases} 1, & \text{for } \sigma = m_s \\ 0, & \text{for } \sigma \neq m_s \end{cases} \; . \tag{6.41}$$

Once we have basis vectors, we can expand an arbitrary spin-state vector $\mid \chi_s \rangle$ as linear combination of them

$$\mid \chi_s \rangle \; = \; \sum_{m_s=-s}^{+s} \alpha_{m_s} \mid s \; m_s \rangle \; , \tag{6.42}$$

where α_{m_s} is a number. Whereas the number of components $2s+1$ is finite, the number of $\mid \chi_s \rangle$ states, which can be formed by (6.42) is infinite. The common **spin vector** is a representation of the state vector (6.42) in column vector form:

$$\langle \sigma \mid \chi_s \rangle \; = \; \begin{pmatrix} \alpha_s \\ \alpha_{s-1} \\ \vdots \\ \alpha_{-s} \end{pmatrix} \; . \tag{6.43}$$

In this representation, the basis vectors have the form

$$\mid s, s \rangle \; = \; \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \; ; \; \mid s, s-1 \rangle \; = \; \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \; ; \; \mid s, -s \rangle \; = \; \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \; . \tag{6.44}$$

The operators are now matrices, e.g., the unit operator is given by

$$\mathbf{1}_s \;=\; \sum_{m_s=-s}^{+s} \mid sm_s\rangle\,\langle sm_s\mid = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} . \tag{6.45}$$

The spin states $\mid sm_s\rangle$ combine with the external (orbital angular momentum) states $\mid \ell m_\ell\rangle$ to form states $\mid \ell s; jm_j\rangle$ of total angular momentum $\vec{J} = \vec{L} + \vec{S}$. The direct product of the two spaces,

$$\mid \ell m_\ell\rangle \;\; \mid sm_s\rangle \;\equiv \mid \ell s; m_\ell m_s\rangle \;, \tag{6.46}$$

can be expanded as a direct sum of spaces spanned by basis vectors $\mid \ell s; jm_j\rangle$.

To find the expansion coefficients, take the unit operator in ℓs-space

$$\mathbf{1}_{\ell s} \;=\; \sum_{m_\ell=-\ell}^{\ell}\;\sum_{m_s=-s}^{s} \mid \ell s; m_\ell\; m_s\rangle\,\langle \ell s; m_\ell\; m_s\mid \tag{6.47}$$

and consider

$$\mid \ell s; jm_j\rangle \;\equiv\; \mathbf{1}_{\ell s}\mid \ell s; jm_j\rangle \;=\; \sum_{m_\ell, m_s} \mid \ell s; m_\ell\; m_s\rangle\,\langle \ell s; m_\ell\; m_s\mid \ell s; jm_j\rangle \;. \tag{6.48}$$

The matrix elements $\langle \ell s; m_\ell m_s \mid \ell s; jm_j\rangle$ are *Clebsch-Gordan coefficients.* An explicit representation of the $\mid \ell sj\; m_j\rangle$ in the spin-angle basis $\mid \theta\varphi\sigma\rangle$ is given by the *spin spherical harmonics*

$$\begin{aligned} \mathcal{Y}^{\ell s}_{jm}(\theta,\varphi,\sigma) \;&\equiv\; \langle \theta\varphi\sigma \mid \ell s; jm\rangle \\ &=\; \sum_{m_\ell m_s} Y_{\ell m_\ell}(\theta,\varphi)\;\chi_{sm_s}(\sigma)\;\langle \ell s; m_\ell m_s\mid \ell s; jm_j\rangle \end{aligned} \tag{6.49}$$

where

$$\langle \theta\varphi\sigma \mid \ell s; m_\ell m_s\rangle \;=\; Y_{\ell m_\ell}\;(\theta\varphi)\;\chi_{s\;m_s}(\sigma)\;. \tag{6.50}$$

These 'spin-angle' functions are useful since they are simultaneous eigenfunctions of j, ℓ and s.

6.5 Single Particle States with Spin

6.5.1 Coordinate-Space-Spin Representation

The Hilbert space of the single particle states with spin is given by the spinor (here spin $\frac{1}{2}$):

$$\psi \ = \ \begin{pmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{pmatrix} \tag{6.51}$$

where $\psi_\pm(\vec{x})$ are square integrable functions. The scalar product is defined as

$$\langle \psi \mid \phi \rangle \ = \ \int d^3x \ \{\psi_+^*(\vec{x}) \ \phi_+(\vec{x}) \ + \ \psi_-^*(\vec{x}) \ \phi_-(\vec{x})\} \ , \tag{6.52}$$

and thus the absolute value is given by

$$\| \ \psi \ \|^2 = \int d^3x\{| \ \psi_+(\vec{x}) \ |^2 \ + \ | \ \psi_-(\vec{x}) \ |^2\} \ . \tag{6.53}$$

The action of the $\vec{X}$ and $\vec{P}$ operator are

$$\begin{aligned} X_j\psi \ &= \ \begin{pmatrix} x_j\psi_+(\vec{x}) \\ x_j\psi_-(\vec{x}) \end{pmatrix} \\ P_j\psi \ &= \ \begin{pmatrix} \frac{\hbar}{i} \frac{\partial}{\partial x_j} \ \psi_+(\vec{x}) \\ \frac{\hbar}{i} \frac{\partial}{\partial x_j} \ \psi_-(\vec{x}) \end{pmatrix} \ , \end{aligned} \tag{6.54}$$

the spin operators act as matrix operator on ψ, e.g.,

$$\begin{aligned} \sigma_1\psi \ &= \ \begin{pmatrix} \psi_-(\vec{x}) \\ \psi_+(\vec{x}) \end{pmatrix} \\ \sigma_2\psi \ &= \ \begin{pmatrix} -i\psi_-(\vec{x}) \\ i\psi_+(\vec{x}) \end{pmatrix} \\ \sigma_3\psi \ &= \ \begin{pmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{pmatrix} \end{aligned} \tag{6.55}$$

6.5.2 Helicity Representation

A particle is said to possess **intrinsic spin** if the quantum mechanical states of that particle in its own rest frame are eigenstates of the operator J^2 with the eigenvalues

$s(s+1)$. These states are referred to as $| \vec{p} = 0, \lambda\rangle$, where $\lambda = -s, \cdots, s$ is the eigenvector of the operator J_3 in the rest frame. (The index 3 refers to an appropriately chosen z-direction.)

To define unambiguously a particle state with linear momentum $| \vec{p} |$, and direction $\hat{\eta}(\theta, \varphi)$, we proceed as follows:

1. We specify a "standard state" in a fixed direction (usually chosen to be along the z-axis), and
2. Define all other states relative to the standard state by means of a specific rotational operator.

(The idea behind being similar to the one defining angular momentum states $| jm\rangle$ being obtained from the "standard state" $| jj\rangle$ by applying the lowering operator.)

The standard state is an eigenstate of the momentum operator with components $p_1 = p_2 = 0, p_3 =| \vec{p} |$,

$$\begin{aligned} P_1 \mid p\hat{z}, \lambda\rangle &= 0 = P_2 \mid p\hat{z}, \lambda\rangle \\ P_3 \mid p\hat{z}, \lambda\rangle &= p \mid p\hat{z}, \lambda\rangle \ . \end{aligned} \tag{6.56}$$

Since along the direction of motion (z-axis), there can be no "orbital angular momentum," the spin index λ can be interpreted as the eigenvalue of the total angular momentum $\vec{J}$ along that direction. More formally, since $\vec{J} \cdot \vec{P}$ commutes with $\vec{P}$, the standard state can be chosen as simultaneous eigenstate of these operators. Thus with (6.56), one has

$$\frac{\vec{J} \cdot \vec{P}}{p} \mid p\hat{z}, \lambda\rangle = J_3 \mid p\hat{z}, \lambda\rangle = \lambda \mid p\hat{z}, \lambda\rangle \ . \tag{6.57}$$

Now, a general single particle state with momentum in an arbitrary direction characterized by $\hat{n}(\theta, \varphi)$ is defined via

$$| \vec{p}, \lambda\rangle \equiv | p, \theta, \varphi, \lambda\rangle = U(\varphi, \theta, 0) \mid p\hat{z}, \lambda\rangle \ . \tag{6.58}$$

By construction, the label λ represents the **helicity** of the particle. Since $\vec{J} \cdot \vec{P}$ is invariant under rotations, i.e., $[\vec{J}\cdot\vec{P}, J_k] \; = \; 0$, one has explicitly

$$\begin{aligned}
\frac{\vec{J}\cdot\vec{P}}{p} \;\mid \vec{p},\lambda\rangle \;\; &= \;\; \frac{\vec{J}\cdot\vec{P}}{p}\, U[R(\varphi,\theta,0)] \;\mid p\hat{z},\lambda\rangle \\
&= \;\; U[R]\; U[R]^{-1}\; \frac{\vec{J}\cdot\vec{P}}{p}\, U[R] \;\mid p\hat{z},\lambda\rangle \\
&= \;\; U[R]\; \frac{\vec{J}\cdot\vec{P}}{p} \;\mid p\hat{z},\lambda\rangle \\
&= \;\; U[R]\; \lambda \;\mid p\hat{z},\lambda\rangle \\
&= \;\; \lambda \mid p,\theta,\varphi,\lambda\rangle
\end{aligned} \tag{6.59}$$

Let us turn to states with definite angular momentum (J, M). Then the "standard state" is given by

$$\mid p\hat{z},\lambda\rangle \;=\; \sum_J \;\mid pJ\lambda\lambda\rangle \;, \tag{6.60}$$

and a general state is obtained by rotating the "standard state"

$$\mid p,\theta,\varphi,\lambda\rangle \;=\; \sum_{JM} \;\mid pJM\lambda\rangle\; \mathcal{D}^J(\varphi,\theta,0)^M_\lambda \;. \tag{6.61}$$

The states are composed of states with all values of total angular momentum J.

It should be mentioned that the helicity characterization of states applies equally well for zero mass states (e.g., photon or neutrino states), as for non-zero mass states. In contrast, the static spin, as introduced in 6.6.1, has no meaning for zero-mass states. In a relativistic description of particle states, helicities are usually preferred.

6.6 Isospin

Some elementary particles can behave very similar under certain reactions. In nuclear physics this is the case for protons and neutrons. If one neglects their slight mass difference as well as the Coulomb interaction between the protons with respect to the nuclear interaction, then one can consider protons and neutrons as identical. This suggests to consider both particles as two different charge states of one particle, the "nucleon."

Mathematically one has to enlarge the Hilbert space of the state vector describing a nucleon by a further observable which can have two district eigenvalues and which commutes with all other observables. Further, one would like to introduce operators, which transform formally a proton state into a neutron state with identical space-spin-wave function.

Let us denote with $\mid p\rangle$ a proton state and with $\mid n\rangle$ a neutron state. Then we define the following operators in analogy with the operators for spin-$\frac{1}{2}$ particles:

$$\begin{aligned}
\tau_3 \mid p\rangle &= +\mid p\rangle \\
\tau_3 \mid n\rangle &= -\mid n\rangle \\
\tau_- \mid p\rangle &= \mid n\rangle \\
\tau_- \mid n\rangle &= 0 \\
\tau_+ \mid p\rangle &= 0 \\
\tau_+ \mid n\rangle &= \mid p\rangle \\
\tau_1 &= \frac{1}{2}\left(\tau_+ + \tau_-\right) \\
\tau_2 &= \frac{1}{2i}\left(\tau_+ - \tau_-\right)
\end{aligned} \tag{6.62}$$

One can easily verify that the quantities τ_i behave formally like the Pauli matrices σ_i. Therefore, the three observables $\tau_i/2$ are called isospin operators. The operator of the electric charge is given as

$$q = \frac{1}{2}\left(\tau_3 + 1\right) . \tag{6.63}$$

The general state of a nucleon is now given as

$$\psi \equiv \psi(\vec{x}, m_s, m_t) , \tag{6.64}$$

where m_s gives the m-quantum number of the spin and m_t, the isospin quantum number, gives the charge of the nucleon. m_t can (as m_s) only take the values $\pm\,\frac{1}{2}$, where $+\,\frac{1}{2}$ corresponds to a proton and $-\,\frac{1}{2}$ to a neutron.

Analogously to rotations in spin space, one can define unitary operators in isospin space

$$\tilde{d} = \cos\frac{\varphi}{2} \cdot \mathbf{1} - i\sin\frac{\varphi}{2} \sum_{j=1}^{3} \alpha_j \tau_j \tag{6.65}$$

with α_j, φ real, $\sum_j \alpha_j^2 = 1$, which describe rotations in isospin space. The three operators τ_j behave formally like the components of a vector under rotations

$$\tilde{d}(\vec{\alpha}, \varphi)\, \tau_j\, \tilde{d}^\dagger(\vec{\alpha}, \varphi) = \sum_{k=1}^{3} R(\vec{\alpha}, \varphi)_{jk}\, \tau_k . \tag{6.66}$$

Thus, isospin can be treated formally in the same way as spin.

6.6.1 Nucleon-Nucleon (NN) Scattering

Consider as example the phase shifts in NN scattering. Consider first the quantum numbers that specify the partial wave states:

- orbital angular momentum **L**, L=0,1,2,$\cdots$ $\Longrightarrow$ *not* conserved (due to tensor force)
- total spin **S**=$\frac{1}{2}(\sigma_1 + \sigma_2)$, S=0,1 $\Longrightarrow$ conserved
- total angular momentum **J**=**L**+**S**, J=$|L-S|, \cdots, L+S$ $\Longrightarrow$ conserved by rotational symmetry. Possible values are:

$$J = \begin{cases} L & S=0 \\ |L-1|, L, L+1 & S=1 \end{cases} \tag{6.67}$$

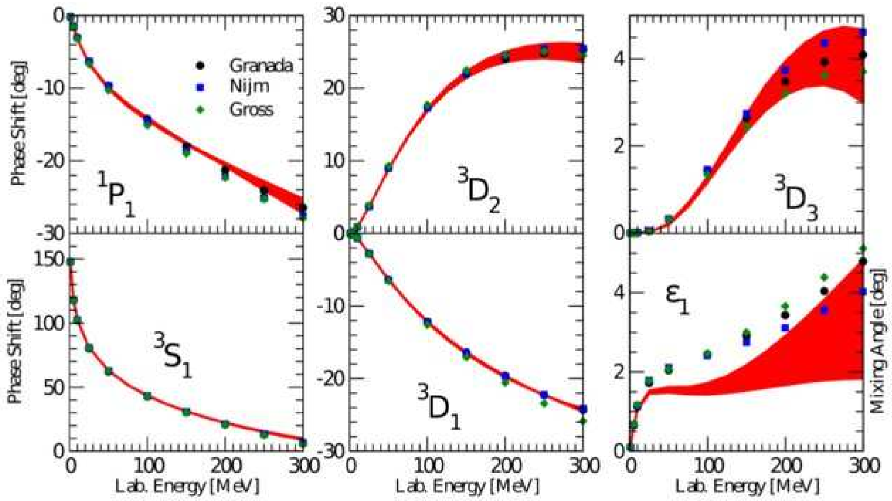
- total isospin T=$\frac{1}{2}(\tau_1 + \tau_2)$ T=0,1 $\Longrightarrow$ conserved

What are the *allowed* partial waves if we account for the Pauli priinciple for fermions, i.e. that the *total* wave function must be totally antisymmetric? The possibilities are spelled out in the following table:

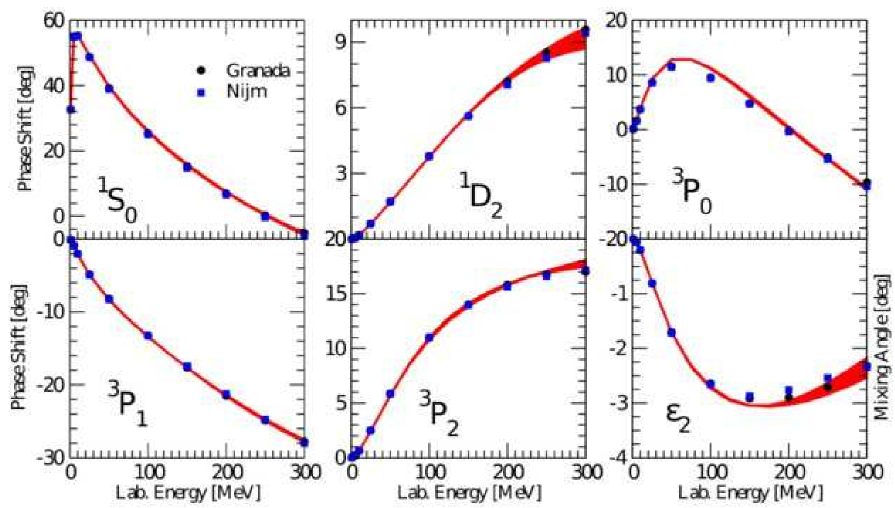
even $L = 0, 2, 4, \cdots$ spatial wf symmetric	$S = 0$ antisymmetric	$\Longrightarrow \quad T = 1$ symmetric
even $L = 0, 2, 4, \cdots$ spatial wf symmetric	$S = 1$ symmetric	$\Longrightarrow \quad T = 0$ antisymmetric
odd $L = 1, 3, 5, \cdots$ spatial wf antisymmetric	$S = 0$ antisymmetric	$\Longrightarrow \quad T = 0$ antisymmetric
odd $L = 1, 3, 5, \cdots$ spatial wf antisymmetric	$S = 1$ symmetric	$\Longrightarrow \quad T = 1$ symmetric

Thus, if L, S, and J are given, then T is completely specified by the Pauli principle.

Using a spectroscopic notation to specify NN scattering partial wave states:
${}^{2S+1}L_J$ with L=0,1,2,3,4, .. $\Longrightarrow$ S,P,D,F, G, ...
the lowest allowd states are therefore:
${}^1S_0$, ${}^3S_1$, ${}^1P_1$, ${}^3P_1$,${}^3P_2$, ${}^1D_2$, ${}^3D_2$, ...
A consequence of the Pauli principle from the table is that if we have neutron-neutron scattering at very low energies, for example, the only available channel is ${}^1S_0$, while ${}^3S_1$ is not allowed.



The lowest NN T=0 neutron-proton phase shifts



The lowest NN T=1 proton-proton phase shifts