Chapter 7

Combining Angular Momentum Eigenstates

7.1 Addition of Two Angular Momenta

Let $J_k^{(1)}$ and $J_k^{(2)}$ be two sets of angular momentum operators with

$$
\begin{aligned}
\left[J_k^{(1)}, J_\ell^{(1)} \right] &= i \sum_m \varepsilon_{k\ell m} \, J_m^{(1)} \\
\left[J_k^{(2)}, J_\ell^{(2)} \right] &= i \sum_m \varepsilon_{k\ell m} J_m^{(2)} \\
\left[J_k^{(1)}, J_\ell^{(2)} \right] &= 0
\end{aligned}
\tag{7.1}
$$

for $k, \ell, m = 1, 2, 3$. Furthermore, $\mathbf{R}(j_1, j_2)$ is defined as a $(2j_1 + 1) \cdot (2j_2 + 1)$-dimensional space, which is spanned by the common eigenvectors

$$
\mid j_1, m_1; j_2, m_2 \rangle \; = \mid j_1 m_1 \rangle \; \mid j_2 m_2 \rangle
\tag{7.2}
$$

of the set of operators $(\vec{J}^{(1)})^2, (\vec{J}^{(2)})^2, J_3^{(1)}, J_3^{(2)}$. The following relations shall hold for $\nu = 1, 2$:

$$
\begin{aligned}
(\vec{J}^{(\nu)})^2 \mid j_1, m_1; j_2, m_2 \rangle &= j_\nu (j_\nu + 1) \mid j_1, m_1; j_2, m_2 \rangle \\
J_3^{(\nu)} \mid j_1, m_1; j_2, m_2 \rangle &= m_\nu \mid j_1, m_1; j_2, m_2 \rangle
\end{aligned}
\tag{7.3}
$$

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where j_1, j_2 are fixed and m_ν can take the values $j_\nu, j_{\nu-1}, \cdots, -j_\nu$. The states in (7.2) shall be normalized according to

$$\langle j_1, m_1'; j_2, m_2' \mid j_1, m_1; j_2, m_2 \rangle \;=\; \delta_{m_1' m_1} \, \delta_{m_2' m_2} \;. \tag{7.4}$$

In this space we want to consider the operators

$$J_k \;=\; J_k^{(1)} \;+\; J_k^{(2)} \;. \tag{7.5}$$

Considering the commutation relations (7.1), it follows that

$$[J_k, J_\ell] \;=\; i \sum_m \varepsilon_{k\ell m} J_m \;. \tag{7.6}$$

The space in which $\vec{J}$ acts in a **direct product** Hilbert space, and we want to determine in this space $R(j_1, j_2)$ the eigenvalues and common eigenvectors of $\vec{J}$ and J_3. This means we want to find states

$$\mid j_1 j_2 JM \rangle \;=\; \sum_{m_1 m_2} \mid j_1 j_2 m_1 m_2 \rangle \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 JM \rangle \;. \tag{7.7}$$

The coefficients $\langle j_1 j_2 m_1 m_2 \mid j_1 j_2 JM \rangle$ that give the amplitude for each product state in the combined state are called **Clebsch-Gordan** or **vector-coupling** coefficients:

$$\langle j_1 j_2 m_1 m_2 \mid j_1 j_2 JM \rangle \;\equiv\; C(j_1 j_2 J, m_1 m_2 M) \;\equiv\; C^{j_1 \; j_2 \; J}_{m_1 m_2 M}. \tag{7.8}$$

As first step we can immediately determine the eigenvalues of J_3 and their degeneracy. From

$$J_3 \;=\; J_3^{(1)} \;+\; J_3^{(2)} \tag{7.9}$$

follows

$$\begin{aligned}
J_3 \mid j_1 j_2 JM \rangle \;&=\; M \mid j_1 j_2 JM \rangle \\
&=\; \sum_{m_1 m_2} (m_1 + m_2) \mid j_1 j_2 m_1 m_2 \rangle \, C(j_1 j_2 J, m_1 m_2 M) \\
&=\; M \sum_{m_1 m_2} \mid j_1 j_2 m_1 m_2 \rangle \, C(j_1 j_2 J, m_1 m_2 M) \;.
\end{aligned}$$
$$\tag{7.10}$$

Since the basis elements are orthogonal, one has

$$(m_1 + m_2) \, C(j_1 j_2 J, m_1 m_2 M) \;=\; M \, C(j_1 j_2 J, m_1 m_2 M) \;. \tag{7.11}$$

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From this follows that

$$C(j_1 j_2 J, m_1 m_2 M) = 0 \quad \text{if } M \neq m_1 + m_2 . \tag{7.12}$$

Furthermore

$$C(j_1 j_2 J; j_1, J - j_1 J) = 0, \quad \text{unless}$$
$$-j_2 \leq J - j_1 \leq j_2 \quad \text{or} \quad j_1 - j_2 \leq J \leq j_1 + j_2$$
$$\tag{7.13}$$

and

$$C(j_1 j_2 J; J - j_2, j_2 J) = 0, \quad \text{unless}$$
$$-j_1 \leq J - j_2 \leq j_1 \quad \text{or} \quad j_2 - j_1 \leq J \leq j_1 + j_2 . $$
$$\tag{7.14}$$

From (7.13) and (7.14) follows that there are

$$| j_1 - j_2 | \leq J \leq j_1 + j_2 \tag{7.15}$$

allowed values for J.

In general we can say that the eigenvalues of $\vec{J}^{\,2}$ can be numbers $J(J+1)$ with $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. If there is a quantum number J, then there has to be a $(2J+1)$ dimensional space $R(J)$ of eigenvectors $| JM \rangle$ belonging to the operators $\vec{J}^{\,2}$ and $\vec{J}_3$ with $M = J, J-1, \cdots, -J$. From (7.12) followed that $M = m_1 + m_2$, where $-j_1 \leq m_1 \leq j_1$ and $-j_2 \leq m_2 \leq j_2$. Let us consider now the number of states $g(M)$ for different values of M. Without loss of generality we assume $j_1 \geq j_2$.
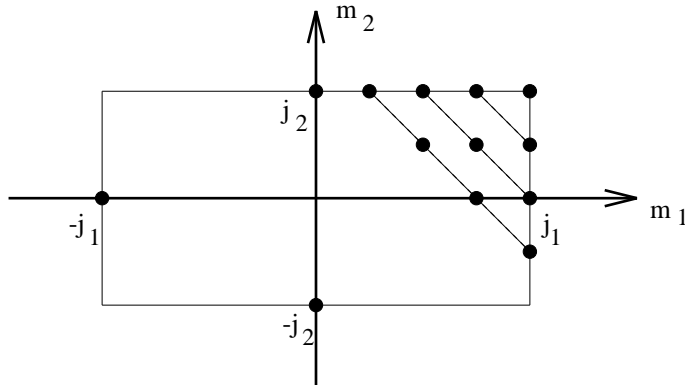
The largest value of M is $j_1 + j_2$, thus one has

$$
\begin{aligned}
g(j_1 + j_2) &= 1 \\
g(j_1 + j_2 - 1) &= 2 \\
g(j_1 + j_2 - 2) &= 3
\end{aligned}
$$

$$(7.16)$$

The number increases by 1 until

$$g(j_1 - j_2) = 2j_2 + 1 \tag{7.17}$$

is reached. Then it will stay the same unit $M = -(j_1 - j_2)$ is reached and will decrease again by **1** from $-(j_1 - j_2) - 1$ or until $g(-j_1 - j_2) = 1$.

In principle, one can have different states belonging to the eigenvalues J and M. We want to determine this number of states $p(J)$. From the number of states belonging to a given M,

$$g(M) = p(J = \mid M \mid) + p(J = \mid M \mid + 1) + p(J = \mid M \mid + 2) + \cdots \tag{7.18}$$

we can conclude by reversing (7.18)

$$
\begin{aligned}
g(M = J) &= p(J) + p(J + 1) + \cdots \\
g(M = J + 1) &= p(J + 1) + p(J + 2) + \cdots
\end{aligned}
$$

$$(7.19)$$

Subtracting both relations leads to

$$p(J) = g(M = J) - g(M = J + 1) . \tag{7.20}$$

Now we can count the states

$$
\begin{aligned}
p(J > j_1 + j_2) &= 0 , \quad \text{since } g(M) = 0 \text{ for } \mid M \mid > j_1 + j_2 \\
p(J = j_1 + j_2) &= g(M = j_1 + j_2) = 1 \\
p(J = j_1 + j_2 - 1) &= g(M = j_1 + j_2 - 1) - g(M = j_1 + j_2) = 1
\end{aligned}
$$

and continue up to

$$p(J = j_1 - j_2) = g(M = j_1 - j_2) - g(M = j_1 - j_2 + 1) = 1 ,$$

and finally

$$p(J < j_1 - j_2) = 0 .$$

Thus the eigenvalues J within the allowed interval are given by

$$J = j_1 + j_2, \ j_1 + j_2 - 1, \ \cdots, j_1 - j_2 \ , \tag{7.21}$$

and they occur exactly only once.

We verify that the number of states $\mid j_1 m_1 \rangle \mid j_2 m_2 \rangle$ is identical with the number of states $\mid j_1 j_2 J M \rangle$, where $\mid j_1 - j_2 \mid \leq J \leq j_1 + j_2$ and $-M \leq J \leq M$:

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1) = (2j+1)(2j_2+1) \ . \tag{7.22}$$

7.2 Construction of the Eigenstates

Since we know which vectors $\mid j_1 j_2 J M \rangle$ exist, we are left with their explicit construction and with the determination of the Clebsch-Gordan coefficients.

In analogy to the determination of single angular momentum states, we start from a state

$$\mid J = j_1 + j_2, \ M = j_1 + j_2 \rangle \ \equiv \ \mid j_1 j_1; j_2 j_2 \rangle \tag{7.23}$$

and apply the ladder operator

$$J_- = (J_1 - iJ_2) = (J_-^{(1)} + J_-^{(2)}) \ . \tag{7.24}$$

In general we have

$$J_\pm \mid j_1 j_2 J M \rangle = \sqrt{J(J+1) - M(M \pm 1)} \ \mid j_1 j_2 J M \pm 1 \rangle \tag{7.25}$$

and the corresponding relations for $J_\pm^{(1)}$ and $J_\pm^{(2)}$. Applying the ladder operators on a state as given in (7.7) leads to

$$\sqrt{J(J+1) - M(M \pm 1)} \mid j_1 j_2; J M \pm 1 \rangle$$
$$= \sum_{m_1 m_2} C(j_1 j_2 J, m_1 m_2 M) \left[\sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)} \ \mid j_1 j_2 \ m_1 \pm 1 \ m_2 \rangle \right.$$
$$\left. + \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)} \ \mid j_1 j_2 \ m_1 m_2 \pm 1 \rangle \right] \ . \tag{7.26}$$

If we project this result on states $\mid j_1 j_2 m_1 m_2 \rangle$, we obtain the following relation for the coefficients:

$$\sqrt{J(J+1) - M(M \pm 1)} \; C(j_1 j_2 J, m_1 m_2 M \pm 1)$$
$$= \sqrt{j_1(j_1+1) - m_1(m_1 \mp 1)} \; C(j_1 j_2 J, m_1 \mp 1, m_2 M)$$
$$+ \sqrt{j_2(j_2+1) - m_2(m_2 \mp 1)} \; C(j_1 j_2 J, m_1 m_2 \mp 1, M) \;. \qquad (7.27)$$

This relation allows to determine all C-G-coefficients up to a number $\lambda(J, j_1 j_2)$, which does not depend on the quantum numbers M, m_1 or m_2. If we set $M = J$ in (7.27), we obtain for the case $M + 1$

$$0 = \sqrt{j_1(j_1+1) - m_1(m_1 - 1)} \; (j_1 j_2 J, (m_1 - 1)(J + 1 - m_1)J)$$
$$+ \sqrt{j_2(j_2+1) - (J + 1 - m_1)(J - m_1)} \; C(j_1 j_2 J, m_1, J - m_1, J) \;, \qquad (7.28)$$

where we used that in this special case $M = J = m_1 - 1 + m_2$. Thus, with $C(j_1 j_2 J, j_1, J - j_1 \; J) \equiv \lambda(J, j_1 j_2) \geq 0$ and real, all other C-G-coefficients can be obtained from the recursion relation (7.28). To determine the constant $\lambda(J, j_1 j_2)$, one uses the normalization condition for the states

$$1 = \langle j_1 j_2 J \mid j_1 j_2 J \rangle = \sum_{m_1 m_2} \mid \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 J J \rangle \mid^2 \;. \qquad (7.29)$$

The remaining phase is chosen so that the C-G-coefficients are real.

The transformation matrix between states, which is represented by C-G-coefficients, is unitary:

$$\langle j_1 j_2 J M \mid j_1 j_2 J' M' \rangle = \sum_{m_1 m_2} \langle j_1 j_2 J M \mid j_1 j_2 m_1 m_2 \rangle \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 J' M' \rangle \qquad (7.30)$$

or

$$\delta_{JJ'} \; \delta_{MM'} = \sum_{m_1 m_2} C(j_1 j_2 J, m_1 m_2 M) \; C(j_1 j_2 J', m_1 m_2 M') \qquad (7.31)$$

and

$$\langle j_1 j_2 m_1 m_2 \mid j_1 j_2 m_1' m_2' \rangle = \sum_{JM} \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 J M \rangle \langle j_1 j_2 J M \mid j_1 j_2 m_1' m_2' \rangle \qquad (7.32)$$

or equivalently

$$\delta_{m_1 m_1'} \; \delta_{m_2 m_2'} = \sum_{JM} C(j_1 j_2 J, m_1 m_2 M) \; C(j_1 j_2 J, m_1' m_2' \; M) \;. \qquad (7.33)$$

From the unitarity and the property (7.12) follow simplified relations

$$\delta_{JJ'} = \sum_{m_1} C(j_1 j_2 J, m_1 \ M - m_1, M) \ C(j_1 j_2 J, m_1 M - m_2 M) \tag{7.34}$$

$$\delta_{m'_1 m_1} = \sum_{J} C(j_1 j_2 J, m_1 \ M - m_1, M) \ C(j_1 j_2 J, m'_1 \ M - m'_1, M) \ . \tag{7.35}$$

7.2.1 Symmetry Properties of Clebsch-Gordan Coefficients

A study of the general expressions for the C.G. coefficients will reveal the following symmetry properties:

$$\begin{aligned}
C(j_1 j_2 j, m_1 m_2 m) &= (-1)^{j_1 + j_2 - j} \ C(j_1 j_2 j, -m_1 - m_2 - m) \\
&= (-1)^{j_1 + j_2 - j} \ C(j_2 j_1 j, m_2 m_1 m) \\
&= (-1)^{j_1 - m_1} \frac{[j]}{[j_2]} \ C(j_1 j j_2, m_1 - m - m_2) \\
&= (-1)^{j_2 + m_2} \frac{[j]}{[j_1]} \ C(j j_2 j_1, -m m_2 - m_1),
\end{aligned} \tag{7.36}$$

where the symbol $[j]$ is defined by

$$[j] := \sqrt{2j + 1} \tag{7.37}$$

The relations (7.36) bring out the symmetry properties of the C.G. coefficients under the permutations of any two columns or the reversal of the sign of the projection quantum numbers. Note that when the third column is permuted with the first or second, there is a reversal of the sign of the projection quantum numbers of the permuted columns. This is essential to preserve the relation $m_1 + m_2 = m$. By using the first symmetry relation one finds

$$C(j_1 j_2 j, 000) = (-1)^{j_1 + j_2 - j} \ C(j_1 j_2 j, 000). \tag{7.38}$$

Thus, one obtains the condition

$$C(j_1 j_2 j, 000) = 0 \tag{7.39}$$

if $j_1 + j_2 - j$ is odd. Moreover, the quantum numbers j_1, j_2, j should all be integers; otherwise the projection quantum numbers can not be zero. The C.G. coefficient of (7.39) is known as *parity* C.G. coefficient, since in physical problems such a coefficient contains the parity selection rule.

7.3 Special Cases

7.3.1 Spin-Orbit Interaction

Angular momentum coupling is often used when calculating with Hamiltonians constructed from subsystem angular momentum operators. One example is the spin-orbit coupling in the H-atom. Consider the Hamiltonian

$$H = \frac{P^2}{2m} + V(r) + W(r)\,\vec{L}\cdot\vec{S} \equiv H_0 + H_{SO}\,\vec{L}\cdot\vec{S}\,, \tag{7.40}$$

in which the spin-independent term is H_O and the term with the factor H_{SO} is the spin-orbit part. The independent spaces are orbital angular momentum for operator $\vec{L}$ and intrinsic spin for $\vec{S}$. Since the two operators commute, one can write $\vec{L}\cdot\vec{S}$ or $\vec{S}\cdot\vec{L}$. We combine the two angular momenta to a total angular momentum

$$\vec{J} = \vec{L} + \vec{S}\,. \tag{7.41}$$

Since the electron carries spin $\frac{1}{2}$, one has

$$|\vec{J}| = |\vec{L}| \pm \frac{1}{2} \quad |\vec{L}| > 0\,. \tag{7.42}$$

The total angular momentum is conserved, thus

$$[H, \vec{J}^{\,2}] = [H, J_3] = 0\,. \tag{7.43}$$

From (7.41) follows

$$\vec{J}^{\,2} = (\vec{L}+\vec{S})^2 = \vec{L}^{\,2} + \vec{S}^{\,2} + 2\vec{L}\cdot\vec{S}\,. \tag{7.44}$$

By solving for the scalar product and inserting the result in (7.40), one obtains

$$H = H_0 + H_{SO}\,\frac{1}{2}\,(\vec{J}^{\,2} - \vec{L}^{\,2} - \vec{S}^{\,2})\,. \tag{7.45}$$

Thus by forming combined angular momentum states (eigenfunctions of $\vec{J}^{\,2}$, J_3 as well as of $\vec{L}^{\,2}$ and $\vec{S}^{\,2}$), H has become diagonal in these angular momenta, and one can directly read off the energy:

$$\begin{aligned}
E_{J\ell} &\equiv \langle \ell s J \mid H \mid \ell s J \rangle \\
&= \langle H_0 \rangle + \langle H_{SO} \rangle \frac{1}{2}\,[J(J+1) - \ell(\ell+1) - S(S+1)]\,. \tag{7.46}
\end{aligned}$$

To be more precise, we form the eigenstates to $\vec{J}^2$ and J_3 as

$$| \ell \frac{1}{2} JM \rangle = \sum_{m_1 m_2} C(\ell \frac{1}{2} J, m_1 m_2 M) | \ell \frac{1}{2} m_1 m_2 \rangle \tag{7.47}$$

or in coordinate and spinor representation

$$\mathcal{Y}_{\ell \frac{1}{2}}^{JM} (\theta, \varphi) = \sum_{m_1 m_2} C(\ell \frac{1}{2} J, m_1 m_2 M) \, Y_{\ell m_1}(\theta, \varphi) \, \chi_{\frac{1}{2} \, m_2} . \tag{7.48}$$

These functions are normalized according to

$$\int d\varphi \, d\cos\theta \, (\mathcal{Y}_{\ell \frac{1}{2}}^{JM})^*(\theta, \varphi) \, \mathcal{Y}_{\ell' \frac{1}{2}}^{J'M'}(\theta, \varphi) = \delta_{JJ'} \, \delta_{MM'} \, \delta_{\ell\ell'} . \tag{7.49}$$

The eigenstates of H then have the following form

$$\psi \equiv R_{\ell J}(r) \, \mathcal{Y}_{\ell \frac{1}{2}}^{JM} \tag{7.50}$$

and

$$\vec{L} \cdot \vec{S} \, \mathcal{Y}_{\ell \frac{1}{2}}^{JM} = \frac{1}{2} \left(J(J+1) - \ell(\ell+1) - S(S+1) \right) \mathcal{Y}_{\ell \frac{1}{2}}^{JM} . \tag{7.51}$$

Introducing the radial momentum P_r as in (3.64), one obtains the radial Schrödinger equation after projecting on the $\mathcal{Y}_{\ell \frac{1}{2}}^{JM}$

$$\left(P_r + \frac{\hbar^2 \, \ell(\ell+1)}{2mr^2} + V(r) + W(r) \frac{1}{2} \left(J(J+1) - \ell(\ell+1) - S(S+1) \right) - E_{J\ell} \right) R_{\ell J}(r) = 0 . \tag{7.52}$$

Since $J = \ell \pm \frac{1}{2}$, each energy level with $\ell \neq 0$ splits into two separate levels. This is in the case of the H-atom called fine structure.

The order of the levels depends on the sign of $W(r)$. A relativistic theory the for H-atom gives

$$W(r) = \frac{\hbar}{4m^2C^2} \frac{1}{r} \frac{dV(r)}{dr} \ . \tag{7.53}$$

7.3.2 Coupling of Two Spin-$\frac{1}{2}$ Particles

Let us define

$$u_{\pm} := |\ \frac{1}{2}, \pm\frac{1}{2}\ \rangle_{(1)} \quad ; \quad v_{\pm} := |\ \frac{1}{2}, \pm\frac{1}{2}\ \rangle_{(2)} \ . \tag{7.54}$$

The states $\chi(j, m) := |\ jm\rangle$ with total spin $j = 0$ and $j = 1$ are obtained in the following way. There is one state

$$\chi(1, 1) = u_+ v_+ \ . \tag{7.55}$$

From this state, one obtains via applying the ladder operator J_- the states

$$\chi(1, 0) = \frac{1}{\sqrt{2}} (u_+ v_- + u_- v_+) \tag{7.56}$$

and

$$\chi(1, -1) = u_- v_- \ . \tag{7.57}$$

These states already have the correct normalization because of

$$\langle u_+ v_+\ |\ u_+ v_+\rangle = \langle u_+\ |\ u_+\rangle \langle v_+\ |\ v_+\rangle = 1 \ . \tag{7.58}$$

One could have obtained (7.56) even without calculation, observing that $\chi(1, 0)$ has to be symmetric in u and v, since $J_- = J_-^{(1)} + J_-^{(2)}$ is symmetric in (1) and (2). The relative phase follows from

$$J_-^{(1)} u_+ = u_- \quad \text{and} \quad J_-^{(2)} v_+ = v_- \ , \tag{7.59}$$

whereas the factor $\frac{1}{\sqrt{2}}$ ensures the correct normalization. The vector $\chi(0, 0)$ with $j = m = 0$ has to be orthogonal to $\chi(1, 0)$ and has to contain the products $u_+ v_1$ and $u_- v_+$. Therefore,

$$\chi(0, 0) = \frac{1}{\sqrt{2}} (u_+ v_- - u_- v_+) \ . \tag{7.60}$$

The phase is for both cases consistent with the fact that the C-G-coefficients have to be positive and real. $\chi(0, 0)$ is an antisymmetric state vector, which means that when coupling two spin-$\frac{1}{2}$ states the singlet state is antisymmetric with respect to the exchange of the particles.

7.4 Properties of Clebsch-Gordan Coefficients

As relatively simple example for the results derived in Section 7.2, we study the coupling of two equal angular momenta $j_1 = j_2 = j$ to a total angular momentum 0. This means we have to calculate C-G-coefficients $C(jj0; m - m0)$. From (7.28) we obtain with $J = M = 0, m_1 = m$ and $m_2 - 1 = -m$

$$
\begin{aligned}
0 &= \sqrt{j(j+1) - m(m+1)}\ C(jj0; m-1, -m+1, 0) \\
&+ \sqrt{j(j+1) - m(m+1)}\ C(jj0; m, -m, 0)\ ,
\end{aligned}
\tag{7.61}
$$

from which follows

$$
C(jj0; m-1; -m+1, 0) = -C(jj0; m, -m, 0)\ .
\tag{7.62}
$$

Since the C-G-coefficients (7.28) are positive and real, one has

$$
C(jj0; j, -j, 0) = a = \lambda(0, jj) > 0\ .
\tag{7.63}
$$

Applying the recursion relation (7.28) $(j - m)$ times starting from (7.63) gives

$$
C(jj0; m, -m, 0) = a\ (-1)^{j-m}\ ,
\tag{7.64}
$$

which gives a change of sign $(j-m)$ times. Here a is a positive constant. The normalization condition (7.29) gives

$$
|a|^2 \sum_{m=-j}^{+j} |(-1)^{j-m}|^2 = 1\ .
\tag{7.65}
$$

Thus

$$
a = \frac{1}{\sqrt{2j+1}}
\tag{7.66}
$$

and

$$
C(jj0; m, -m, 0) = \frac{(-1)^{j-m}}{\sqrt{2j+1}}\ .
\tag{7.67}
$$

From this follows that the vector

$$
\begin{aligned}
|jj00\rangle &= \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{+j} (-1)^{j-m}\ |jjm-m\rangle \\
&= \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{+j} (-1)^{j-m}\ |jm\rangle_{(1)}\ |j-m\rangle_{(2)}
\end{aligned}
\tag{7.68}
$$

is rotationally invariant, i.e., a scalar.

The result given in (7.67) illustrates the following important fact: In the above considered case, one obtains an additional factor (-1) if j and thus m are half-integers. The reason is the positive choice of $\lambda(J, j_1, j_2)$. Apart from this, the addition of two angular momenta is a completely symmetric process. Specifically, the states

$$| j_1 j_2 JM \rangle \quad \text{and} \quad | j_2 j_1 JM \rangle \, , \tag{7.69}$$

which differ only in the order of j_1 and j_2 are identical up to a phase factor. The same has to be valid for

$$C(j_1 j_2 J; m_1 m_2 M) \quad \text{and} \quad C(j_2 j_1 J; m_2 m_1 M) \, . \tag{7.70}$$

Since the C-G-coefficients are real, one has to have

$$C(j_2 j_1 J; m_2 m_1 M) \;=\; (-1)^N \; C(j_1 j_2 J; m_1 m_2 M) \, , \tag{7.71}$$

where N does not depend on M, since the operators $J_\pm$, which combine different M values, are symmetric with respect to interchanging (1) and (2). To determine N, we set $M = J$ and use again that $\lambda(J, j_1 j_2)$ has to be positive. Then $C(j_1 j_2 J, m_1 m_2 J)$ has to be positive for $m_1 = j_1$ as well as $C(j_2 j_1 J : m_2 m_1 J)$ for $m_2 = j_2$.
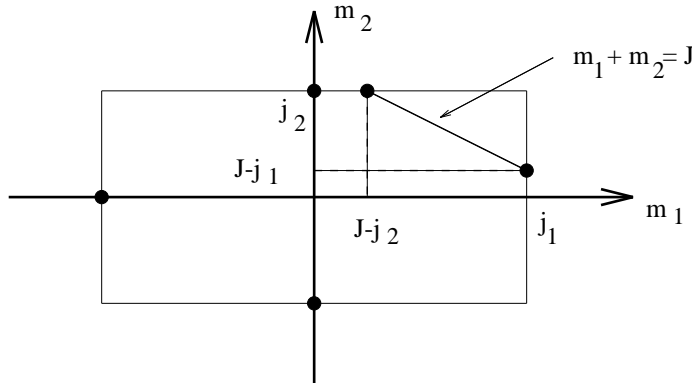
Fig. 7.3 $m_1 -$ m_2 plane, as used for calculating the phase when interchanging j_1 and j_2.

In Fig. 7.3 the state $| j_1 j_2, j_1, J - j_1 \rangle$ is marked as A, and the state $| j_1 j_2, J - j_2, j_2 \rangle$ as B. When going from B to A, the recursion relation (7.28), (7.69) has to be applied

$$j_2 \;-\; (J - j_1) \;=\; j_1 + j_2 - J \quad \text{times} \, .$$

Thus, one obtains

$$C(j_1 j_2 J; J - j_2, j_2, J) = (-1)^{j_1 + j_2 - J} \, C(j_1 j_2 J; j_1, J - j_1, J) . \tag{7.72}$$

According to (7.71) we had

$$C(j_2 j_1 J; j_2, J - j_2, J) = (-1)^N \, C(j_1 j_2 J; J - j_2, j_2 J) . \tag{7.73}$$

Inserting (7.72) into (7.73) gives

$$C(j_2 j_1, J; j_2, J - j_2, J) = (-1)^N \, (-1)^{j_1 + j_2 - J} \, C(j_1 j_2 J; j_1, J - j_1, J) \tag{7.74}$$

where the C-G-coefficients need to be positive. This is fulfilled for

$$N = J - j_1 - j_2 . \tag{7.75}$$

Since N, as mentioned before, is independent of M, it follows for the general case

$$C(j_2 j_1 J; m_2 m_1 M) = (-1)^{J - j_1 - j_2} \, C(j_1 j_2 J; m_1 m_2 M) . \tag{7.76}$$

7.5   Clebsch-Gordan Series

We had constructed

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_j C(j_1 j_2 j; m_1 m_2 m)|jm\rangle \tag{7.77}$$

How does this construction behave under rotation? Both sides of (7.77) are vectors, which need to be rotated by and angle $\omega \equiv (\alpha, \beta, \gamma)$, where latter are the Euler angles. This leads to

$$\sum_{\nu_1, \nu_2} D^{j_1}_{\nu_1 m_1}(\omega) D^{j_2}_{\nu_2 m_2}(\omega)|j_1 \nu_1\rangle |j_2 \nu_2\rangle = \sum_{j\mu} C(j_1 j_2 j; m_1 m_2 m) D^{j}_{\mu m}(\omega)|j\mu\rangle \tag{7.78}$$

The state on the right side of (7.78) must also be

$$|j\mu\rangle = \sum_{mu'_1} C(j_1 j_2 j; \mu'_1 \mu'_2 \mu)|j_1 \mu'_1\rangle |j_2 \mu'_2\rangle \tag{7.79}$$

Inserting this into (7.78) and taking the scalar product with $\langle j_1 \mu'_1| \langle j_2 \mu'_2|$ leads to a relation between the rotation matrices

$$\sum_{\nu_1 \nu_2} D^{j_1}_{\nu_1 m_1}(\omega) D^{j_2}_{\nu_2 m_2}(\omega) \delta_{\mu_1 \nu_1} \delta_{\mu_2 \nu_2} =$$

$$\sum_{j\mu\mu_1} C(j_1 j_2 j; m_1 m_2 m) \, C(j_1 j_2 j; \mu_1' \mu_2' \mu) D_{\mu m}^j(\omega) \delta_{\mu_1 \mu_1'} \delta_{\mu_2 \mu_2'} \qquad (7.80)$$

Since $\mu_2' = \mu - \mu_1'$, together with evaluating the Kronecker symbols, one obtains from (7.78) the so-called *Clebsch-Gordan Series*

$$D_{\mu_1 m_1}^{j_1}(\omega) \, D_{\mu_2 m_2}^{j_2}(\omega) = \sum_j C(j_1 j_2 j; m_1 m_2 m) \, C(j_1 j_2 j; \mu_1 \mu_2 \mu) D_{\mu m}^j(\omega). \qquad (7.81)$$

Without proof, the inverse is given by

$$D_{\mu m}^j(\omega) = \sum_{m_1 \mu_1} C(j_1 j_2 j; m_1 m_2 m) \, C(j_1 j_2 j; \mu_1 \mu_2 \mu) D_{\mu_1 m_1}^{j_1}(\omega) \, D_{\mu_2 m_2}^{j_2}(\omega) \qquad (7.82)$$

Application to wave functions: A rotation is equivalent to unitary transformations. Take e.g. two wave functions,

$$
\begin{aligned}
\psi_{jm}(\vec{r}') &= \sum_{m'} D_{m'm}^j(\omega) \psi_{jm'}(\vec{r}) \\
\psi_{j\mu}(\vec{r}') &= \sum_{\mu'} D_{\mu'\mu}^j(\omega) \psi_{j\mu'}(\vec{r}).
\end{aligned}
\qquad (7.83)
$$

Taking the expectation value leads to

$$
\begin{aligned}
\langle \psi_{j\mu}(\vec{r}') | \psi_{jm}(\vec{r}') \rangle &= \sum_{m'\mu'} (D_{\mu'\mu}^j)^*(\omega) D_{m'm}^j(\omega) \langle \psi_{jm'}(\vec{r}) | \psi_{j\mu'}(\vec{r}) \rangle \\
\delta_{\mu m} &= \sum_{\mu'} (D_{\mu'\mu}^j)^*(\omega) D_{\mu'm}^j(\omega)
\end{aligned}
\qquad (7.84)
$$

7.5.1   Addition theorem for spherical Harmonics

Take two points on a sphere S: $P_1 \equiv (\theta_1 \phi_1)$ and $P_2 \equiv (\theta_2 \phi_2)$. Then consider a system which is rotated by an angle ω leading to a sphere S' with $P_1 \equiv (\theta_1' \phi_1')$ and $P_2 \equiv (\theta_2' \phi_2')$. We want to show that the quantity

$$I := \sum_m (Y_l^m)^*(\theta_1 \phi_1) \, Y_l^m(\theta_2 \phi_2) \qquad (7.85)$$

is invariant under rotation of the coordinate system. Start with

$$I = \sum_m (Y_l^m)^*(\theta_1' \phi_1') \, Y_l^m(\theta_2' \phi_2')$$

$$\begin{aligned}
&= \sum_{mm_1m_2} (D^l_{m_1m})^*(\omega)D^l_{m_2m}(\omega)(Y^{m_1}_l)^*(\theta_1\phi_1)\ Y^{m_2}_l(\theta_2\phi_2) \\
&= \sum_{m_1m_2} (Y^{m_1}_l)^*(\theta_1\phi_1)\ Y^{m_2}_l(\theta_2\phi_2) \tag{7.86}
\end{aligned}$$

For the last equality the orthogonality of the D matrices (e.g. (7.85) was used.

Let us choose a specific coordinate system S_0, where P_1 is parallel to the z-axis, and P_2 is located in the x-z-plane: the coordinates ins S_0 are (00) and $(\theta 0)$. Then

$$\begin{aligned}
I &= \sum_m (Y^m_l)^*(00)\ Y^m_l(\theta 0) \\
&= \sum_m \sqrt{\frac{2l+1}{4\pi}}\delta_{m0}\ Y^m_l(\theta 0) \\
&= \sqrt{\frac{2l+1}{4\pi}}Y^0_l(\theta 0) \\
&= \frac{2l+1}{4\pi}P_l(\cos\theta). \tag{7.87}
\end{aligned}$$

7.5.2 Coupling Rule for spherical Harmonics

Consider a rotation of a frame from S to S_0 by an angle ω.
Coordinates in S: $P_1(\theta_1\phi_1)$ and $P_2(\theta_2\phi_2)$
coordinates in S_0: P_1 parallel z-axis, P_2 in x-z plane,
rotation in Euler angles $(\alpha\beta\gamma) \equiv (\phi\theta 0)$.

How does $Y^m_l((\theta_2\phi_2)$ associated with P_2 transform under this rotation? Consider

$$\begin{aligned}
Y^0_l(\theta 0) &= \sum_m D^l_{m0}(\phi_1\theta_1 0)Y^m_l(\theta_2\phi_2) \\
&= \sqrt{\frac{2l+1}{4\pi}}\sum_m (Y^m_l)^*(\theta_1\phi_1)Y^m_l(\theta_2\phi_2), \tag{7.88}
\end{aligned}$$

where the last identity is given by (7.87). This gives a simple representation for

$$D^l_{m0}(\phi\theta 0) = \sqrt{\frac{2l+1}{4\pi}}(Y^m_l)^*(\theta\phi). \tag{7.89}$$

This relation is very useful in connecting rotation matrices for integer j with spherical Harmonics.

Consider the Clebsch-Gordan Series

$$D^{l_1}_{m_1 0}(\phi\theta 0)D^{l_2}_{m_2 0}(\phi\theta 0) = \sum_l C(l_1 l_2 l; m_1 m_2 m) \; C(l_1 l_2 l; 000)D^l_{m0}(\phi\theta 0). \tag{7.90}$$

Replacing the rotation matrices with spherical Harmonics according to (7.89), taking the complex conjugate and having in mind that C.G. coefficients are real, leads to

$$Y^{m_1}_{l_1}(\theta\phi) \; Y^{m_2}_{l_2}(\theta\phi) = \sum_l \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \; C(l_1 l_2 l; m_1 m_2 m) \; C(l_1 l_2 m; 000) \; Y^m_l(\theta\phi), \tag{7.91}$$

which is the coupling rule for spherical Harmonics with the same argument. The parity C.G. indicates that this product is non-vanishing only if $l_1 + l_2 - l$ is even.

The above allows an easy evaluation of integrals involving three spherical Harmonics:

$$\int d\Omega (Y^{m_3}_{l_3})^*(\theta\phi) \; Y^{m_2}_{l_2}(\theta\phi) \; Y^{m_1}_{l_1}(\theta\phi)$$

$$= \sum_l \frac{\hat{l}_1 \hat{l}_2}{\sqrt{4\pi}\hat{l}} C(l_1 l_2 l; m_1 m_2 m) \; C(l_1 l_2 l; 000) \int d\Omega (Y^{m_3}_{l_3})^*(\theta\phi) \; Y^m_l(\theta\phi)$$

$$= \frac{\hat{l}_1 \hat{l}_2}{\sqrt{4\pi}\hat{l}_3} \; C(l_1 l_2 l_3; m_1 m_2 m_3) \; C(l_1 l_2 l_3; 000), \tag{7.92}$$

where $\hat{l} \equiv \sqrt{2l+1}$.

7.6 Wigner's $3 - j$ Coefficients

In our considerations of combining two angular momenta, we so far treated the third angular momentum, the sum $\mid JM\rangle$, in a special way. A more symmetric treatment in terms of $3 - j$ coefficients considers the three angular momenta on equal terms.

Suppose that we combine two angular momentum states j_1 and j_2 to form a third state, $\mid j_3, -m_3\rangle$, then we combine this state with one of the same j_3 but opposite 3-projection, m_3, so that the total projection is zero. If couple this to a state with total $J = 0$, then we have an isotropic quantity, a scalar, formed by coupling three angular momenta to zero. The corresponding coupling coefficient was invented by Wigner and is called $3 - j$ coefficient. Its symmetries under permutation of arguments are simpler than those of the C-G-coefficient.

In deriving the $3-j$ coefficient we need to perform the usual angular momentum coupling. The first combination is, according to (7.7)

$$| (j_1 j_2) j_3, -m_3 \rangle \;=\; \sum_{m_1 m_2} \; | j_1 j_2 m_1 m_2 \rangle \, \langle j_1 j_2 m_1 m_2 \mid j_1 j_2, j_3 - m_3 \rangle \;. \tag{7.93}$$

The second condition, which produces the state $(0,0)$, is

$$| (j_1 j_2 j_3) 00 \rangle \;=\; \sum_{m_3} \; | (j_1 j_2) j_3 - m_3, m_3 \rangle \, \langle j_3 j_3, -m_3 m_3 \mid j_3 j_3 00 \rangle \;. \tag{7.94}$$

The C-G-coefficient for coupling to equal angular momentum to total angular momentum zero is given by (7.67) and is in our case $(-1)^{j_3 - m_3}/\sqrt{2 j_3 + 1}$. Using this in (7.94) and combining this with (7.93), we obtain the expression for three angular momenta coupled to zero

$$| (j_1 j_2 j_3), 00 \rangle \;\sim\; \sum_{m_1 m_2 m_3} \; | j_1 m_1 \rangle \; | j_2 m_2 \rangle \; | j_3 m_3 \rangle \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{7.95}$$

in which the proportionality constant is just a phase. With the phase $(-1)^{j_1 - j_2 - m_3}$, this leads to the definition of the **Wigner $3 - j$ coefficient**

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \;=\; \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2 j_3 + 1}} \, C(j_1 j_2 j_3, m_1 m_2 - m_3) \;. \tag{7.96}$$

The $3 - j$ coefficient is zero unless

$$m_1 + m_2 + m_3 \;=\; 0 \;. \tag{7.97}$$

The triangle condition $\mid j_1 - j_2 \mid \; \leq \; j_3 \; \leq \; j_1 + j_2$ of (7.15) has, of course, also to be fulfilled. Because of (7.76), one has

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} \;=\; (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \;. \tag{7.98}$$

This can be verified since with (7.96)

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \frac{(-1)^{j_2-j_1-m_3}}{\sqrt{2j_3+1}} C(j_2 j_1 j_3; m_2 m_1 - m_3)$$

$$= \frac{(-1)^{(j_2-j_1-m_3)+(j_3-j_1-j_2)}}{\sqrt{2j_3+1}} C(j_1 j_2 j_3; m_1 m_2 - m_3)$$

$$= (-1)^{j_3-2j_1} (-1)^{-j_1+j_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= (-1)^{j_3+j_2+j_1} (-1)^{4j_1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

(7.99)

The factor $(-1)^{4j_1} = 1$ even for half-integer j_1. Furthermore, one can show that the $3-j$ coefficients are invariant under cyclic permutation, i.e.,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} \qquad (7.100)$$

and that they fulfill the relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \qquad (7.101)$$

We can rewrite the basic formula (7.7) for combining two angular momenta using the $3-j$ notation:

$$| j_1 j_2 JM \rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} | j_1 j_2 m_1 m_2 \rangle \, \delta_{m_1+m_2,M}$$

$$\times \ (-1)^{j_1-j_2-M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix}. \qquad (7.102)$$

The unitarity relation analogous to (7.33) reads

$$\sum_{j_3} | 2j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m'_1 m_1} \delta_{m'_2 m_2} \delta_{m_1+m_2,-m_3}$$

(7.103)

and the one corresponding to (7.31)

$$\sum_{m_1 m_2} (2j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{J_3' j_3} \, \delta_{m_3' m_3} \; .$$

$$(7.104)$$

Thus, in both sums the orthogonality conditions for the $3 - j$ coefficients require weight factors of $2j_3 + 1$.

If one of the angular momenta in the $3 - j$ coefficients is zero, the expression simplifies and is in analogy with (7.67) given by

$$\begin{pmatrix} j_1 & 0 & j_3 \\ m_1 & 0 & m_3 \end{pmatrix} = \delta_{j_1 j_3} \, \delta_{m_1 - m_3} \, \frac{(-1)^{j_1 - m_1}}{\sqrt{2j + 1}} \; . \qquad (7.105)$$

This is more complicated than (7.67), but it is simpler under the exchange of j_1 and j_3.

7.7   Cartesian and Spherical Tensors

For theoretical and practical purposes, it is important to analyze the classification of tensors of different ranks with respect to their behavior under rotations. A tensor of rank K can be defined as a quantity with 3^k components

$$t_{n_1 n_2 \cdots n_k} \quad , \quad n_i = 1, 2, 3 \qquad (7.106)$$

if it transforms under rotations $x_{n'}' = \sum_{n=1}^{3} R_{n'n} \, x_n$ as

$$\begin{aligned}
t_{n_1' n_2' \cdots n_k'} &= \sum_{n_1 \cdots n_k} R_{n_1' n_1} \; \cdots \; R_{n_k' n_k} \, t_{n_1 \cdots n_k} \\
&= \sum_{n_1 \cdots n_k} t_{n_1 \cdots n_k} \, R^{-1}_{n_1 n_1'} \; \cdots \; R^{-1}_{n_k n_k'} \; .
\end{aligned}$$

$$(7.107)$$

In other words, the tensor components $t_{n_1 \, \cdots \, n_k}$, which span the space, are at the same time a basis for the representation of the rotation group, which is defined as

$$(R_{n'n}) \longmapsto (R_{n_1' n_1} \; \cdots \; R_{n_k' n_k}) \; . \qquad (7.108)$$

The $(t_{n_1 \cdots n_k})$ represents a k-fold product representation of $SO(3)$ and are called **Cartesian tensors**.

In Chapter 5 we constructed via the angular momentum eigenvectors $\mid jm \rangle$ all representations of $SO(3)$. Thus there has to be a connection to the above introduced quantities. According to (5.196) we had found for the transformation under rotation of a state $\mid jm \rangle$

$$
\begin{aligned}
\mid jm' \rangle' &= U(\alpha\beta\gamma) \mid jm \rangle \\
&= \sum_{m=-j}^{+j} \mid jm \rangle \langle jm \mid U(\alpha\beta\gamma) \mid jm' \rangle' ,
\end{aligned}
$$

(7.109)

where the rotation is parameterized with Euler angles. The matrix elements have the form (cp. 5.197)

$$
\begin{aligned}
\mathcal{D}^j_{mm'}(\alpha\beta\gamma) &= \langle jm \mid e^{-i\alpha J_3} \, e^{-i\beta J_2} \, e^{-i\gamma J_3} \mid jm' \rangle \\
&= e^{i\alpha m - i\gamma m'} \, \langle jm \mid e^{i\beta J_2} \mid jm' \rangle \\
&= e^{i\alpha m - i\gamma m'} \, d^j_{mm'}(\beta) .
\end{aligned}
$$

(7.110)

Thus we have the explicit transformation

$$
\mid jm' \rangle' = \sum_{m=-j}^{+j} \mid jm \rangle \, \mathcal{D}^j_{mm'}(\alpha\beta\gamma) .
$$

(7.111)

For $j = \frac{1}{2}$ the $\mathcal{D}$-function is explicitly given in (6.33). The set of vectors $\mid jm \rangle$ for fixed j can be combined analogously to tensors to a multi-component mathematical object, whose components transform under rotations according to (7.111):

$$
\mid jm' \rangle' = \sum_{m} \mid jm \rangle \, \mathcal{D}^j_{mm'}(\mathbf{R})
$$

(7.112)

with $\mathcal{D}^j_{mm'}(\mathbf{R}) = \langle jm \mid U(R) \mid jm' \rangle$.

We **define**: A quantity with $(2k + 1)$ components

$$
t_q^{(k)} \; ; \quad q = -k, -k+1, \, \cdots + k
$$

is called **spherical or irreducible tensor** of rank k, if its components transform as

$$
t_{q'}^{(k)} = \sum_{q=-k}^{+k} t_q^{(k)} \, \mathcal{D}^k_{qq'}(R^{-1}) .
$$

(7.113)

Let us compare the simplest Cartesian and spherical tensors and order according to the number N of components of the tensors:

$$
N = 3^k \quad \text{for cartesian tensors of rank } k
$$

$$
N = 2k + 1 \quad \text{for spherical tensors of rank } k
$$

- $N = 1$: Scalar quantities. Here Cartesian and spherical tensors coincide , scalars are invariant quantities.

- $N = 2$: This case only occurs for spherical tensors and leads to the spinor representation $k = \frac{1}{2}$.

- $N = 3$: A spherical tensor with three components corresponds to an angular momentum $j = 1$. A Cartesian tensor with $N = 3$ is a vector. Both quantities are irreducible representations of the rotation group, since they are in irreducible spaces: The spherical tensor via its definition (7.113), and the vector, since via $\vec{x}\,' = \mathbf{R}\vec{x}$ the rotations get defined. Thus, both quantities have to transform equivalently. And we have to show that we can uniquely associate with each vector $\vec{v}$ a tensor $v_q^{(1)}$.

To illustrate this connection, we consider a special case rotations around the 3-axis. The components of $v_q^{(1)}$ transform with the matrix

$$\mathcal{D}_{qq'}^{(1)}(-\theta, 00) \;=\; \langle jq \mid e^{i\theta J_3} \mid jq'\rangle \;=\; e^{i\theta J_3}\,\delta_{qq'}$$

$$= \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}, \tag{7.114}$$

so that we have for the components

$$v_j^{(1)'} \;=\; \begin{cases} e^{i\theta} & v_1^{(1)} \\ & v_0^{(1)} \\ e^{-i\theta} & v_{-1}^{(1)} \end{cases} . \tag{7.115}$$

On the other hand, the components of the rotation matrix in Cartesian coordinates is given as

$$R_{n'n}(\theta) \;=\; \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.116}$$

thus

$$\begin{aligned} v_1' &= \cos\theta\, v_1 - \sin\theta\, v_2 \\ v_2' &= \sin\theta\, v_1 + \cos\theta\, v_2 \\ v_3' &= v_3 . \end{aligned} \tag{7.117}$$

Comparing (7.115) and (7.117), we see that v_3 **transforms like** $v_0^{(1)}$. If one defines

$$v_\pm \;=\; \mp\, \frac{1}{\sqrt{2}}\, (v_1 \pm iv_2) \tag{7.118}$$

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one finds from (7.117) that

$$
\begin{aligned}
v'_+ &= e^{i\theta} v_+ \\
v'_- &= e^{-i\theta} v_- \ .
\end{aligned}
\tag{7.119}
$$

A comparison with (7.115) shows that $v_\pm$ **transforms as** $v_\pm^{(1)}$.

In summary:
By defining

$$
\begin{aligned}
v_{\pm 1}^{(1)} &\equiv v_\pm = \mp \frac{1}{\sqrt{2}} \left(v_1 \pm i v_2 \right) \\
v_0^{(1)} &= v_3
\end{aligned}
\tag{7.120}
$$

we can associate with each Cartesian vector $\vec{v}$ a spherical tensor with angular momentum $j = 1$. The sign in (7.118) to chosen so that when introducing spherical coordinates, one has

$$
x_\pm = r \sqrt{\frac{4\pi}{3}} \, Y_{1\pm 1}(\theta, \varphi) \ .
\tag{7.121}
$$

Thus we explicitly showed that each Cartesian vector $\vec{v}$ transforms with respect to rotations like a quantity with angular momentum $j = 1$.

Let us now consider a Cartesian tensor of rank 2 with the components $t_{k\ell}, k, \ell = 1, 2, 3$. It has nine components, as many as a spherical tensor with $k = 4$. But $t_{k\ell}$ and $t_q^{(4)}$ are completely independent and have no connection with each other!

According to the general tensor rule (7.103), $t_{k\ell}$ transforms like the product of two systems with $j_1 = j_2 = 1$. We have already shown explicitly that two angular momenta $j_1 = j_2 = 1$ can be coupled to total angular momentum $0, 1, 2$. One can show this explicitly in the following way:

1. $tr(t) = \sum_{k=1}^3 t_{kk} := t^{(0)}$ is invariant and corresponds to $j = 0$.

2. The antisymmetric part $t_{k\ell}^{(1)} := \frac{1}{2} \left(t_{k\ell} - t_{\ell k} \right)$ has three independent components. One can define an axial vector (pseudovector) via $a_n := \sum_{k\ell} \varepsilon_{nk\ell} \, t_{k\ell}^{(1)}$. Then a_n transforms like an angular momentum state with $j = 1$.

3. The remaining part is the symmetric, trace-free quantity $t_{k\ell} := \frac{1}{2}(t_{k\ell} + t_{\ell k}) - \frac{1}{3}\delta_{k\ell} \, tr \, t$, which has five components and transforms like an object with $j = 2$.

Therefore, we can rewrite a tensor of rank 2 as

$$t_{k\ell} = t_{k\ell}^{(2)} + t_{k\ell}^{(1)} + \frac{1}{3}\,\delta_{k\ell}t^{(0)} \ . \tag{7.122}$$

It is no accident that the parts with $j = 0$ and $j = 2$ correspond to symmetric tensors, while $t_{k\ell}^{(1)}$ is antisymmetric. This corresponds to the rule for the interchange of two angular momenta in the state $\mid j_1 j_2 JM\rangle$, since the factor $(-1)^{J-j_1-j_2} = (-1)^{J-2}(j_1 = j_2 = 1)$ gives a change of sign only for $J = 1$. In general, any tensor $t_{n_1 \ \dots \ n_k}$ can be decomposed by the techniques of symmetrization, antisymmetrization and trace-operation into its irreducible parts. For the lowest rank tensors these results are summarized in the following table:

N	Cartesian Tensor	Spherical Tensor
1	scalar S	$t_0^{(0)} \Leftrightarrow \mid 0,0\rangle$
2	–	$t_{(q)}^{(1/2)} \Leftrightarrow \mid \frac{1}{2},m\rangle$, spinor
3	3-vector $\vec{v}$	$t_{(q)}^{(1)} \Leftrightarrow \mid 1,m\rangle;\ Y_{1m}$
4	–	$t_{(q)}^{(3/2)} \Leftrightarrow \mid \frac{3}{2},m\rangle$
5	symmetric tensor with trace 0 $\frac{1}{2}\,(t_{k\ell} + t_{\ell k}) - \frac{1}{3}\,\delta_{k\ell}\,tr(t)$	$t_{(q)}^{(2)} \Leftrightarrow \mid 2,m\rangle$

The observables of classical physics can be characterized according to this scheme. However, those observables only correspond to spherical tensors with **integer** j. For $j = 0$ and $j = 1$ we already encountered numerous examples. An example for $j = 2$ is the quadrupole moment of a classical charge distribution

$$Q_{k\ell} = \int d^3r\ \rho(r)\left[x_k x_\ell - \frac{1}{3}\delta_{k\ell}\,r^2\right] \ . \tag{7.123}$$

This tensor is obviously symmetric and $tr\ Q_{kk} = 0$.

7.8 Tensor Operators in Quantum Mechanics

We apply the considerations of the previous Section now to quantum mechanical observables. By writing

$$T' = U^{-1}(R)\ T\ U(R) \tag{7.124}$$

we arrive at the following transformations for

1. Cartesian tensors

$$
\begin{aligned}
U^{-1}(R) \, T_{n'_1 \, \cdots \, n'_k} \, U(R) \;&=\; T'_{n'_1 \, \cdots \, n'_k} \\
&=\; \sum_{n_1 \cdots n_k} T_{n_1 \cdots n_k} \, R^{-1}_{n_1 n'_1} \; \cdots \; R^{-1}_{n_k n'_k} \,.
\end{aligned}
\tag{7.125}
$$

2. Spherical tensors

$$
U^{-1}(R) \, T^{(k)}_{q'} \, U(R) \;=\; T^{(k)\prime}_{q'} \;=\; \sum_{q=-k}^{+k} T^{(k)}_{q} \, \mathcal{D}^{(k)}_{qq'} \, (U(R^{-1})) \,.
\tag{7.126}
$$

Considering infinitesimal rotations around an axis $\vec{n}$, we obtain with

$$
\begin{aligned}
U^{-1}(\theta) \;=\; \mathbf{1} + i\theta \, \vec{n} \cdot \vec{J} \;\Longrightarrow\; \mathcal{D}^{(k)}_{qq'} \;&=\; \langle kq \,|\, \mathbf{1} \,+\, i\theta \, \vec{n} \cdot \vec{J} \,|\, kq' \rangle \\
&=\; \delta_{qq'} \,+\, i\langle kq \,|\, \theta \, \vec{n} \cdot \vec{J} \,|\, kq' \rangle
\end{aligned}
\tag{7.127}
$$

the result

$$
[\vec{n} \cdot \vec{J}, \, T^{(k)}_{q'}] \;=\; \sum_{q} T^{(k)}_{q} \, \langle kq \,|\, \vec{n} \cdot \vec{J} \,|\, kq' \rangle \,.
\tag{7.128}
$$

For a rotation around the 3-axis, we recover the familiar relation

$$
[J_3, T^{(k)}_{q'}] \;=\; q' \, T^{(k)}_{q'} \,,
\tag{7.129}
$$

which is analogous to

$$
J_3 \,|\, kq \rangle \;=\; q \,|\, kq \rangle \,.
\tag{7.130}
$$

Defining the commutator with the tensor operators corresponds thus to applying the operator on states

$$
[J_3, T^{(k)}_{q}] \;\longrightarrow\; J_3 \,|\, kq \rangle \,.
\tag{7.131}
$$

Since (7.128) contains the scalar product $\vec{n} \cdot \vec{J}$, this correspondence holds also for J_2 and J_1 and $J_\pm$. We obtain

$$
[J_\pm, T^{(k)}_{q}] \;=\; \sqrt{k(k+1) - q(q \pm 1)} \; T^{(k)}_{q\pm1} \,.
\tag{7.132}
$$

The following table summarizes the results of this Section:

	Cartesian Tensor Rank k	Cartesian Tensor Operator	Spherical Tensor Rank k	Spherical Tensor Operator	Angular Momentum States
	$t_{n_1 \cdots n_k}$	$T_{n_1 \cdots n_k}$	$t_q^{(k)}$	$T_q^{(k)}$	$\mid kq \rangle$
El	c-numbers	Operators in Hilbert Space	c-numbers	Operators in Hilbert Space	Hilbert Space Vectors
Trf.	$R^{-1}_{n'_1 n_1} \cdots R^{-1}_{n'_1 n_k}$		Rotation Matrices	$\mathcal{D}^{(k)}_{q'q}(R^{-1})$	$\mathcal{D}^{(k)}_{q'q}(R)$
Ex: C. Sp.	$\vec{x}$	$\vec{X}$	$x_\pm = \mp\frac{1}{\sqrt{2}}(x_1 \pm ix_2)$ $x_0 = x_3$	$X_\pm = \mp\frac{1}{\sqrt{2}}(X_1 \pm iX_2)$ $X_0 = X_3$	$\mid 1, \pm1 \rangle$ $\mid 1, 0 \rangle$

Note: The Hilbert space vector $\mid \vec{x} \rangle = \mid x_1 x_2 x_3 \rangle$ is not contained in any of the categories in the table, since it transforms as

$$U(R) \mid \vec{x} \rangle = \mid R\vec{x} \rangle . \tag{7.133}$$

The state $\mid \vec{x} \rangle$ is a superposition of infinitely many angular momentum eigenstates:

$$
\begin{aligned}
\mid \vec{x} \rangle &= \sum_{n\ell m} \mid E_{n,\ell}, \ell, m\rangle\langle E_{n\ell}, \ell, m \mid \vec{x} \rangle \\
&= \sum_{n\ell m} \mid E_{n\ell}, \ell, m\rangle \frac{U_{n\ell}(r)}{r} \mathcal{Y}^*_{\ell m}(\Omega) \tag{7.134}
\end{aligned}
$$

where one has to sum (integrate) over the entire spectrum of the Hamiltonian.

7.9 Wigner-Eckart Theorem

7.9.1 Qualitative

The Wigner-Eckart Theorem states that the matrix elements of an irreducible tensor operator between any well defined angular momentum states can be factored into two parts:

1. one part depending on the magnetic quantum numbers

2. another part completely independent of the magnetic quantum numbers.

That means that (1) contains the entire geometry or the symmetry properties of the system, and (2) contains the dynamics of the physical process. Formally this is written as

$$\langle j_f m_f | T_k^\mu | j_i m_i \rangle = C(j_i k j_f; m_i \mu m_f) \, \langle j_f \| T_k \| j_i \rangle. \tag{7.135}$$

This means the entire dependence of the matrix element on the magnetic quantum numbers can be factored out as C.G. coefficient, and a 'reduced' matrix element independent of the projection quantum numbers (double bar matrix element).

Remarks:

(a) The literature gives different definitions, sometimes containing a factor $\hat{j}_f = \sqrt{2j_f + 1}$.

(b) the C.G. coefficient depends on the coordinate system that is used to evaluate the matrix element and implies the conservation of angular momentum. If this factorization is possible in one coordinate system, then it is possible in all coordinate systems obtained by rotation from the original coordinate system.

Consider matrix elements in a rotated coordinate system $(\hat{r}')$:

$$
\begin{aligned}
&\langle \psi_{j_f m_f}(\hat{r}') | T_k^\mu(\hat{r}') | \psi_{j_i m_i}(\hat{r}') \rangle \\
&= \sum_{m_f' \mu' m_i'} (D_{m_f' m_f}^{j_f})^*(\omega) \, D_{\mu' \mu}^k(\omega) D_{m_i' m_i}^{j_i}(\omega) \langle \psi_{j_f m_f'}(\hat{r}') | T_k^{\mu'}(\hat{r}') | \psi_{j_i m_i'}(\hat{r}') \rangle \\
&= \sum_{m_f' \mu' m_i'} (D_{m_f' m_f}^{j_f})^*(\omega) \, D_{\mu' \mu}^k(\omega) D_{m_i' m_i}^{j_i}(\omega) \, C(j_i k j_f; m_i' \mu' m_f') \langle j_f \| T_k \| j_i \rangle \\
&= \sum_J C(j_i k J; m_i \mu M) \sum_{m_f'} (D_{m_f' m_f}^{j_f})^*(\omega) D_{m_f' M}^J(\omega) \langle j_f \| T_k \| j_i \rangle \delta_{j_f J} \\
&= C(j_i k j_f; m_i \mu m_f) \, \langle j_f \| T_k \| j_i \rangle. \tag{7.136}
\end{aligned}
$$

For the second equation the coupling of two rotation matrices,

$$D^k_{\mu'\mu}(\omega)D^{j_i}_{m'_i m_i}(\omega) = \sum_J C(j_i k J; m_i \mu M) \, C(j_i k J; m'_i \mu' M') D^J_{M'M}(\omega) \qquad (7.137)$$

was used and summed over m'_i and μ', remembering that $m'_f - m'_i + \mu' = M'$. For the last equality the summation over J was replaced by j_f and the orthogonality of the D matrices gives $\delta_{m_f m}$.

The consideration in (7.136) is not a proof of the Wigner-Eckart theorem, rather a consistency check.

7.9.2 Proof of the Wigner-Eckart Theorem

Following Wigner as layed out in Brink-Satchler (1962).

Define an irreducible tensor operator of rank k via

$$
\begin{aligned}
\mathcal{Q} &\equiv \langle \psi_{j_f m_f}(\hat{r}) | T^\mu_k(\hat{r}) | \psi_{j_i m_i}(\hat{r}) \rangle \\
&= \int d\Omega \, \psi^*_{j_f m_f}(\hat{r}) \, T^\mu_k(\hat{r}) \, \psi_{j_i m_i}(\hat{r}).
\end{aligned}
\qquad (7.138)
$$

One can carry out the angular integration either by rotating the functions in a fixed coordinate system or by rotating the coordinate system and keep the functions fixed. Let us do the latter and rotate the coordinate system through Euler angles such that $\hat{r}$ goes from (00) to $(\theta\phi)$. This leads to

$$
\begin{aligned}
\mathcal{Q} &= \int d\Omega \sum_{m'_f \mu' m'_i} (D^{j_f}_{m'_f m_f})^*(\Omega) \, D^k_{\mu'\mu}(\Omega) D^{j_i}_{m'_i m_i}(\Omega) \, \psi^*_{j_f m_f}(00) \, T^\mu_k(00) \, \psi_{j_i m_i}(00) \\
&= \sum_{m'_f \mu' m'_i} \sum_J C(j_i k J; m_i \mu M) \, C(j_i k J; m'_i \mu' M') \, \psi^*_{j_f m_f}(00) \, T^\mu_k(00) \, \psi_{j_i m_i}(00) \\
&\quad \times \int d\Omega (D^{j_f}_{m'_f m_f})^*(\Omega) D^J_{M'M}(\Omega)
\end{aligned}
\qquad (7.139)
$$

With

$$\int d\Omega (D^{j_f}_{m'_f m_f})^*(\Omega) D^J_{M'M}(\Omega) = \frac{4\pi}{2j_f + 1}\delta_{j_f J}\delta_{m'_f M'}\delta_{m_f M} \qquad (7.140)$$

Summing over J and m'_f gives

$$\mathcal{Q} = C(j_i k j_f; m_i \mu M) \, \delta_{m_f M} \left[\frac{4\pi}{2j_f + 1} \sum_{m'_i \mu'} C(j_i k j_f; m'_i \mu' m'_f) \, \psi^*_{j_f m_f}(00) \, T^\mu_k(00) \, \psi_{j_i m_i}(00) \right]$$

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$$= C(j_i k j_f; m_i \mu M) \, \delta_{m_f M} \, \langle j_f \| T_k \| j_i \rangle, \tag{7.141}$$

where the term in the bracket in (7.141) is defined as 'reduced' matrix element.

Example:
Calculate the reduced matrix element for the case of the spherical harmonics:

$$\langle l_f \| Y_l \| l_i \rangle = \frac{4\pi}{2j_f + 1} \sum_{m_i' m'} C(l_i l l_f; m_i' m' m_f') \, (Y_{l_f}^{m_f'})^*(00) Y_l^{m'}(00) Y_{l_i}^{m_i'}(00)$$

$$= \left(\frac{(2l_i + 1)(2l + 1)}{4\pi(2l_f + 1)} \right)^{1/2} C(l_i l l_f, 000), \tag{7.142}$$

where $Y_l^m(00) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m0}$ was used.

7.10   Applications

An important application of the Wigner-Eckart Theorem are the so-called **selection rules**. From (7.141) and the properties of the $C - G$ coefficients follows that

$$\langle \beta, j'm' \mid T_k^q \mid \alpha, jm \rangle = 0 \tag{7.143}$$

only if

$$m' = m + q \tag{7.144}$$

and

$$j' = k + j, k + j - 1, \cdots, \mid k - j \mid . \tag{7.145}$$

A very important application of this result are the selection rules for electromagnetic transitions. If the wave length is **large** compared to the size of the system under consideration, the radiation probability is given by the square of the matrix elements of the **electric dipole operator** $e\vec{Q}$:

$$\langle E', j'm' \mid e \, \vec{Q} \cdot \vec{\varepsilon} \mid Ejm \rangle , \tag{7.146}$$

where $\vec{\varepsilon}$ describe the polarization vector of the electric field (and is thus a c-number). Only the matrix elements

$$\langle E', j'm' \mid \vec{Q} \mid E, jm \rangle \tag{7.147}$$

determine the transition probability. Since $\vec{Q}$ corresponds to a spherical operator with $k = 1$, the allowed transitions must fulfill

$$j' = j+1, j, j-1$$

or

$$\Delta j = j' - j = \pm 1, 0 \tag{7.148}$$

which constitutes the so-called $E1$-selection rule.

The selection rule (7.144), m' $m+q$ (here $q = 1$) determines the direction of polarization of the emitted light and can be observed if the m-degeneracy is removed (as in the Zeeman-effect). If $\vec{\varepsilon}$ is parallel to the magnetic field, one obtains in (7.146)

$$\vec{\varepsilon} \cdot \vec{Q} = Q_3 = Q_0 . \tag{7.149}$$

Because of $q = 0$ follows

$$m' = m \quad \text{or} \quad \Delta m = 0 . \tag{7.150}$$

This means: The $\Delta m = 0$ transitions lead to quanta which are polarized parallel to the magnetic field. Due to the transversality of the electromagnetic waves, those quanta are emitted perpendicular to the magnetic field. If $\vec{\varepsilon}$ is perpendicular to the magnetic field:

$$\vec{\varepsilon} \cdot \vec{Q} = \varepsilon_1 Q_1 + \varepsilon_2 Q_2 = \alpha Q_{+1} + \beta Q_{-1} . \tag{7.151}$$

Here we use the relation (7.120) for the transition from Cartesian to spherical vector components. Then the selection rule reads

$$m' = m \pm 1 \quad \text{or} \quad \Delta m = m' - m = \pm 1 . \tag{7.152}$$

Thus $\Delta m = \pm 1$ transition lead to polarization perpendicular to the magnetic field, i.e., the quanta can also be emitted in the direction of the magnetic field. In this direction they are either right ($\Delta m = -1$) or left ($\Delta m = +1$) circular polarized, if m denotes the initial and m' the final state.

Apart from selection rules, the Wigner-Eckart Theorem can be used to make predictions about **relative magnitudes of matrix elements**, especially of intensities. In relative magnitudes the reduced matrix element of (7.136) cancels. We consider as example the **magnetic moment**. We consider a system of electrons (or nucleons). The magnetic moment is given by a vector operator $\vec{\mu}$, which is in general composed of the angular momenta $\vec{L}_{(i)}$ and the spins $\vec{S}_{(i)}$:

$$\vec{\mu} = \sum_i \mu_{(i)}^\ell \vec{L}_{(i)} + \sum_i \mu_{(i)}^s \vec{S}_{(i)} , \tag{7.153}$$

where the parameters $\mu_{(i)}^{\ell}$ and $\mu_{(i)}^{s}$ are proportional to the magnetons of each particle. The entire system is assumed to be in a state $\mid E_n, jm\rangle$. The observable values of the magnetic moment are given by the expectation values

$$\langle E_n jm \mid \vec{\mu} \mid E_n jm\rangle , \tag{7.154}$$

and consists of $(2j + 1)$ numbers. According to the Wigner-Eckart Theorem, all these numbers can be expressed by a single number, the reduced matrix element
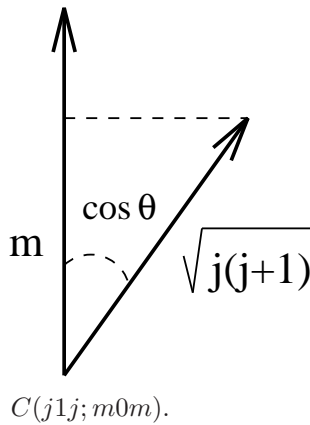
$$\langle E_n j \parallel \mu \parallel E_n j\rangle , \tag{7.155}$$

(independent of m_1 and thus a number) and well-known C-G-coefficients. In this sense, one can refer to **a** magnetic moment of the state. Let us consider a specific expectation value, μ_3 $(m = 0)$. With (7.136) one obtains

$$\langle E_n jm \mid \mu_3 \mid E_n jm\rangle = (-1) \frac{C(1jj; 0mm)}{\sqrt{2j+1}} \langle E_n jm \parallel \mu \parallel E_n jm\rangle . \tag{7.156}$$

The C-G-coefficients are given by

$$C(j1j; m0m) = -C(1jj; 0mm) = \frac{m}{\sqrt{j(j+1)}} := \cos\theta . \tag{7.157}$$

Fig. 7.4 C-G-coefficient $C(j1j; m0m)$.

We can define **the** magnetic moment by

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$$\mu(E_n j) \;:=\; \langle E_n jj \mid \mu_3 \mid E_n jj \rangle \;=\; -\sqrt{\frac{j}{(j+1)(2j+1)}} \; \langle E_n j \parallel \mu \parallel E_n j \rangle \;. \qquad (7.158)$$

Then we obtain with (7.157)

$$\langle E_n jm \mid \mu_3 \mid E_n jm \rangle \;=\; \mu(E_n j) \; \cos\theta \; \sqrt{\frac{j+1}{j}} \;. \qquad (7.159)$$

From this result and Fig. 7.4, one can see the geometric character of the $C - G$ coefficients. They express the dependence of the magnetic moment from the direction of $\vec{\mu}$ in space with respect to the states. Thus, the Wigner-Eckart Theorem allows a group theoretical calculation of matrix elements as function of the geometrical parameters of the problem, and one can calculate ratios of matrix elements without needing further dynamical information. The theorem does **not** give any tools for the calculation of absolute magnitudes.

As example, we calculate the **reduced matrix elements** for the **anomalous Zeeman Effect**. First, we need to calculate the splitting of the atomic levels according to the interaction Hamiltonian

$$H_{Zeeman} \;=\; \mu_B \; B(L_3 + 2S_3) \;, \qquad (7.160)$$

where the constant magnetic field B points in z-direction. The splitting of the energy levels is calculated via the matrix elements

$$\langle E_{n\ell j}, \ell, jm \mid H_{Zeeman} \mid E_{n\ell j}, jm \rangle \;. \qquad (7.161)$$

Since H_{Zeeman} only contains the 3 components of $\vec{L}$ and $\vec{S}$, this matrix is already diagonal, and the splitting of the energy levels is given by

$$\Delta E_{n\ell jm} \;:=\; \langle E_{n\ell j}, \ell, jm \mid H_{Zeeman} \mid E_{n\ell j}, \ell, jm \rangle \;. \qquad (7.162)$$

We need to calculate the matrix elements

$$\langle E_{n\ell j}, \ell, jm' \mid \vec{L} + 2\vec{S} \mid E_{n\ell j}, \ell, jm \rangle \ . \tag{7.163}$$

According to the Wigner-Eckart Theorem, they are (expressed in spherical tensor components) proportional to

$$\langle E_{n\ell j}, \ell, jm' \mid \vec{J} \mid E_{n\ell j}, \ell, jm \rangle \ . \tag{7.164}$$

This can be seen most easily when considering the 3 components

$$
\begin{aligned}
\langle \gamma, jm' \mid L_3 + 2S_3 \mid \gamma, jm \rangle &= \langle \gamma, jm' \mid L_0 + 2S_0 \mid \gamma, jm \rangle = C(j1j; m0m') \, F(\gamma, j) \\
\langle \gamma, jm' \mid J_3 \mid \gamma, jm \rangle &= \langle \gamma, jm' \mid J_0 \mid \gamma, jm \rangle = C(j1j; m0m') \, \tilde{F}(\gamma, j)
\end{aligned}
\tag{7.165}
$$

Both quantities are proportional to the same C-G-coefficient. Thus we set

$$\langle \cdots \mid \vec{L} + 2\vec{S} \mid \cdots \rangle \equiv g \, \langle \cdots \mid \vec{J} \mid \cdots \rangle \ . \tag{7.166}$$

This factor g is known as **Landé g-Factor**. In the present case, we can determine g without calculating the dynamics of the problem. We first use that

$$\vec{L} + 2\vec{S} = \vec{J} + \vec{S} \tag{7.167}$$

and set

$$g = 1 + \alpha \tag{7.168}$$

where α is defined via

$$\langle \cdots \mid \vec{S} \mid \cdots \rangle = \alpha \, \langle \cdots \mid \vec{J} \mid \cdots \rangle \ . \tag{7.169}$$

To calculate α we calculate the matrix elements of $\vec{J} \cdot \vec{S}$ in two different ways. First by using $\vec{J} = \vec{L} + \vec{S}$,

$$\vec{J} \cdot \vec{S} = \vec{L} \cdot \vec{S} + \vec{S}^{\,2} = \frac{1}{2} \, (\vec{J}^{\,2} - \vec{L}^{\,2} + \vec{S}^{\,2}) \ , \tag{7.170}$$

i.e.,

$$\langle E_{n\ell j} \ell jm' \mid \vec{J} \cdot \vec{S} \mid E_{n\ell j} \ell jm \rangle = \frac{1}{2} \, (j(j+1) - \ell(\ell+1) + s(s+1)) \, \delta_{mm'} \ . \tag{7.171}$$

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On the other hand, we have ($E := E_{n\ell j}$)

$$\langle E, \ell jm' \mid \vec{J} \cdot \vec{S} \mid E, \ell jm \rangle = \sum_{m''} \langle E, \ell jm' \mid \vec{J} \mid E, \ell jm'' \rangle \langle E, \ell jm'' \mid \vec{S} \mid E, \ell jm \rangle \quad (7.172)$$

since the states $\mid E_{n\ell j}, \ell jm \rangle$ are a complete set in the considered subspace. With (7.169) follows

$$\begin{aligned} \langle E, \ell jm' \mid \vec{J} \cdot \vec{S} \mid E, \ell jm \rangle &= \alpha \sum_{m''} \langle E, \ell jm' \mid \vec{J} \mid E, \ell jm'' \rangle \langle E, \ell jm'' \mid \vec{J} \mid E, \ell jm \rangle \\ &= \alpha \langle E, \ell jm' \mid \vec{J}^{\,2} \mid E, \ell jm \rangle . \end{aligned}$$
$$(7.173)$$

A comparison of (7.173) with (7.171) gives

$$\alpha = \frac{1}{2j(j+1)} \left(j(j+1) + s(s+1) - \ell(\ell-1) \right) \quad (7.174)$$

and

$$g = 1 + \frac{1}{2j(j+1)} \left(j(j+1) + s(s+1) - \ell(\ell+1) \right) . \quad (7.175)$$

In all formulas we have $s(s+1) = \frac{3}{4}$; however, they are written in a more general way since they are valid for many-electron systems as long as one has LS-**coupling**. The Landé factor depends on j and ℓ, not on $E_{n\ell j}$. One can thus write

$$\Delta E_{n\ell jm} = g(j,\ell) \, \mu_B B m . \quad (7.176)$$

The energy splitting is proportional to m, but of different magnitude for the different terms. This leads to the complicated level scheme of the anomalous Zeeman Effect.