Projectile Motion and Range:

You throw a ball into the air with an initial velocity of 24 m/s at an angle of 40^o to the horizontal. You want to find out:

1. the total time the ball is in the air
2. how far the ball travels (horizontal distance)
3. what is the angle under which the ball travels furthest

Picture the problem first:

Equation of motion:

$$\begin{aligned}\vec{r}(t) &= \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 \\ -g \end{pmatrix} t^2\end{aligned}$$

We have given or can deduce:

1. the magnitude of the velocity $|\vec{v}_0| \equiv v_0$
2. angle with horizontal is $\theta = 40^o$
3. $\longrightarrow v_{0x} = v_0 \cos\theta = 18.4m/s$ and $v_{0y} = v_0 \sin\theta = 15.4m/s$.
4. we can choose coordinate system such that $x_0 = 0$ and $y_0 = 0$.
5. ball starts and lands at the same horizontal position $\longrightarrow \Delta y = 0$.

Find the total time from the y-component of the equation of motion:

$$\begin{aligned}\Delta y = v_{0y}t - \frac{1}{2}gt^2 &= t(v_{0y} - \frac{1}{2}gt) \\ \longrightarrow t &= 0 \quad initial \quad condition \\ t &= \frac{2v_{0y}}{g} = 3.1s \end{aligned} \tag{1}$$

Calculate Δx, the range of your throw:

$$\Delta x = v_{0x}t = 57m \tag{2}$$

For calculating the maximum possible range depending on the angle of throwing the ball, we need to set up an equation which relates the range Δx to the angle θ.

We start again from Eq. (2) and insert the expression from Eq. (1):

$$\begin{aligned} \Delta x = v_{0x}t &= v_{0x}v_{0y}\frac{2}{g} \\ &= \frac{2}{g}v_0^2\cos\theta\sin\theta \\ &= \frac{v_0^2}{g}\sin 2\theta \end{aligned} \tag{3}$$

To find the extremum, we differentiate Eq. (3) with respect to θ and consider that the extremum is given by the condition that the first derivative is zero at the extremum.

$$\begin{aligned} \frac{dx}{d\theta} &= 2\frac{v_0^2}{g}\cos 2\theta_m = 0 \\ \longrightarrow \cos 2\theta_m &= 0 \\ \longrightarrow 2\theta_m &= 90^o \\ \theta_m &= 45^0 \end{aligned}$$

Thus, throwing the ball at an angle of $\theta_m = 45^0$ results in a maximum range.