The physics behind path integrals in quantum mechanics

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We present a simple pedagogical discussion of the physics of path integrals in nonrelativistic quantum mechanics. Rather than show that the path integral gives the standard results of quantum mechanics, as is usually done in the literature, we instead discuss why it does so. Specifically, we discuss why the phase factor associated with each path is given by the classical action. We construct the path integral using Huygens's principle. Finally, we show that the existence of many paths is a consequence of the spreading of the quantum-mechanical wave function.

I. INTRODUCTION

The Feynman path integral formulation of quantum mechanics[1,2] provides a global alternative to the local differential equation Schrödinger approach. As such it is perhaps the only really new development in quantum mechanics since its founding in the 1920s. While there are many demonstrations in the literature that the path integral formulation yields the usual results of quantum mechanics and is therefore equivalent to it, there appears to be little discussion in the literature as to why this in fact happens. Feynman in his original papers provides a lot of motivation to support his formulation of quantum mechanics, but since that motivation is somewhat intuitive it can leave students of his work at a loss to understand just exactly what is going on. This purely pedagogical paper is written for such students.

Feynman's global approach to quantum mechanics is based on the use of the finite time propagator

$$G(q',t';q,t) = \langle q,t\,|q',t'\rangle, \tag{1.1}$$

which measures the overlap of eigenstates of the position operator at different times. The finite time propagator thus studies the whole global motion of the system as one entity instead of building up the finite time motion by iterating the local differential motion through small time intervals. The utility of the propagator stems from the superposition principle which enables us to construct position eigenstates at arbitrary times. Specifically, since the set of position eigenstates at a given time is complete we can use the propagator to construct position eigenstates at a later time via

$$|q',t'\rangle = \int dq\, G(q',t';q,t)|q,t\rangle. \tag{1.2}$$

Projecting onto the bra vector $\langle\psi|$ then yields

$$\psi(q',t') = \int dq\, G(q',t';q,t)\psi(q,t). \tag{1.3}$$

Thus a knowledge of the propagator enables us to determine the time development of the wave function without any recourse to differential equations. The standard procedure for actually evaluating the propagator is as follows. We first translate the state $|q',t'\rangle$ to the time t by means of the quantum-mechanical Hamiltonian operator $\hat{H}$. This enables us to rewrite the propagator as

$$G(q',t';q,t) = \langle q,t\,|e^{-(i/\hbar)(t'-t)\hat{H}}|q',t\rangle. \tag{1.4}$$

We next insert a complete set of eigenstates $|n\rangle$ of the operator $\hat{H}$ with eigenvalues E_n to obtain

$$G(q',t';q,t) = \sum_n \psi_n^*(q)\psi_n(q')e^{-(i/\hbar)(t'-t)E_n}. \tag{1.5}$$

Thus the propagator can be constructed explicitly once we have found the eigenspectrum of the Hamiltonian.

The Feynman path integral formulation of quantum mechanics enables us to evaluate the propagator by an entirely different procedure than the standard one given above. This alternative prescription makes no reference to quantum-mechanical operators at all but rather expresses the propagator entirely in terms of classical c-number quantities via the path integral formula which asserts that

$$G(q',t';q,t) = \sum_{\text{paths}} e^{(i/\hbar)S_{CL}}. \tag{1.6}$$

In this formula the sum is to be taken over classical paths only, i.e., only over trajectories in which momentum and position are both simultaneously specified. The quantity S_{CL} is then the value of $\int L\,dt$ integrated over each such trajectory.[3] Equation (1.6) sums over all available classical paths and not only over the path that minimizes the classical action between the end points (q,t) and (q',t'). The actual formal sum over paths is constructed via the following limiting procedure. We first divide the time interval t'-t into N intermediate segments each of width ϵ by introducing $N-1$ intermediate times $t_i = \epsilon i + t$, $i = 1,\ldots,N-1$. For each t_i we next introduce an assigned coordinate q_i. We construct the stationary classical path between q_i and q_{i+1} by minimizing the classical action between momentarily fixed end points (q_i,t_i) and (q_{i+1},t_{i+1}), and then evaluate $S_{CL}(q_{i+1},t_{i+1};q_i,t_i)$ for that particular path. Analogously, we next evaluate $S_{CL}(q_{i+2},t_{i+2};q_{i+1},t_{i+1})$ for the path which minimizes the classical action between end points (q_{i+1},t_{i+1}) and (q_{i+2},t_{i+2}). For our given set of assigned coordinates q_i we can now form a complete path $(q,q_1,\ldots,q_i,\ldots,q_{N-1},q')$, which is not necessarily stationary between its end points (q,t) and (q',t'), and associate with it a quantity

$$\begin{aligned} S_{CL}(q',\ldots,q_i,\ldots,q) = {} & S_{CL}(q',t';q_{N-1},t_{N-1}) \\ & + \sum_{i=1}^{N-2} S_{CL}(q_{i+1},t_{i+1};q_i,t_i) \\ & + S_{CL}(q_1,t_1;q,t). \end{aligned} \tag{1.7}$$

Finally, we generate a whole infinite set of complete paths by allowing our assigned coordinates q_i to vary over all space. The sum on paths is then understood as the limit of a

multiple integral, viz.,

$$G(q',t';q,t) = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} A \int_{-\infty}^{\infty} dq_1 \cdots \int_{-\infty}^{\infty} dq_{N-1}\, e^{(i/\hbar)S_{CL}(q',\ldots,q_i,\ldots,q)}. \quad (1.8)$$

Here A is an appropriate normalization factor whose value depends on the particular problem.[1,2] (An example of A will be given below in a simple case.) As we can thus see the path integral formula only contains two features which depart from pure classical mechanics. Specifically, it involves classically nonstationary paths, and also it gives to each one of them a phase $iS_{CL}/\hbar$ proportional to the associated classical action for that path. It is then remarkable that the interference of the contributions of all of these paths with their own particular phases gives the exact quantum-mechanical amplitude.

In trying to understand why this in fact happens various questions immediately come to mind. The first is why does each phase factor in the path integral depend on the time integral of the Lagrangian rather than on that of the Hamiltonian, since in the conventional treatment of Eq. (1.4) it is the Hamiltonian that propagates in time. Second, even if it does depend on the action why is it the classical as opposed to the quantum-mechanical one. Third, why is the only apparent dependence on quantum mechanics at all in the seemingly innocuous factor of $\hbar$ in each phase. And finally, where do all the various paths come from, why in fact are there paths at all, and how do we know how to identify the actual set of paths required for Eq. (1.6)? In this paper we shall provide a pedagogical discussion of all of these points.

To understand the appearance of S_{CL} in Eq. (1.6) we begin in Sec. II with a discussion of the role of the classical action in pure classical particle mechanics and discuss its connection with Hamilton–Jacobi theory. This enables us to make a connection in Sec. III with the geometrical optics limit of classical wave theory. Using quantum-mechanical wave particle duality we then find that at short wavelengths the phase of the quantum-mechanical wave function is none other than $iS_{CL}/\hbar$. In Sec. IV we show that this same phase factor is also obtained in the limit of small $\hbar$. Thus a quantum-mechanical phase which is dependent on the classical action is natural in quantum mechanics in two quite distinct limits, small λ or small $\hbar$. Both of these limits thus give us an intuitive understanding of the phase factors that appear in Eq. (1.6). However, neither limit is useful for actually deriving Eq. (1.6) as it is difficult to generalize them to all λ or to all $\hbar$. Consequently in Sec. V we discuss yet another limit, namely quantum mechanics for small times, where again the same phase factor emerges. In this limit we find that only one path contributes to the propagator, the path which actually minimizes the classical equations of motion. However, this is now a limit which can be generalized, and in Sec. VI we use Huygens's principle to show how and why the generalization to finite times then leads to the complete path integral formula of Eq. (1.8). Thus the existence of many paths is a consequence of extending quantum-mechanical propagation to finite times. Moreover, because of the connection with Huygens's principle we obtain a method for actually constructing the network of paths required for the path integration. In Sec. VII we complete the discussion of why the c-number path integral is equivalent to ordinary quantum mechanics by showing that the existence of all the various paths is a consequence of the spreading of the quantum-mechanical wave function.

II. ACTION IN CLASSICAL MECHANICS AND HAMILTON-JACOBI THEORY

In order to see how the classical action appears in the path integral formula we begin by reviewing the role that the action plays in classical mechanics. Apart from its well-known role of providing a variational framework for a least-action principle, the action also has some less familiar uses in classical mechanics. Our interest here is in its role as a generator of canonical transformations and in its connection with Hamilton–Jacobi theory. We shall follow the treatment of Ref. 4.

In a canonical transformation we transform from one set of coordinates q_i and momenta p_i to a new set Q_i, P_i. The Hamiltonians associated with the two sets are H and K, respectively, so that

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}, \quad \dot{Q}_i = \frac{\partial K}{\partial P_i}. \quad (2.1)$$

Both sets of variables are canonical if they satisfy Hamilton's principle with variations

$$\delta \int dt \left(\sum p_i \dot{q}_i - H(q,p,t) \right) = 0, \qquad \delta \int dt \left(\sum P_i \dot{Q}_i - K(Q,P,t) \right) = 0. \quad (2.2)$$

For variations which vanish at the end points the two integrands can at most differ by a total time derivative of some arbitrary function. We are interested here in the case where this function is expressed in terms of the convenient independent set of variables q_i and P_i. Then it is conventional to define a generating function $F(q_i,P_i,t)$ such that

$$\begin{aligned} &\sum_i p_i \dot{q}_i - H \\ &= \sum_i P_i \dot{Q}_i - K + \frac{d}{dt}\left(F(q,P,t) - \sum_i P_i Q_i \right) \\ &= -\sum_i \dot{P}_i Q_i - K + \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F}{\partial P_i}\dot{P}_i. \end{aligned} \quad (2.3)$$

Thus the dependent variables are given by

$$p_i = \frac{\partial F}{\partial q_i}, \quad Q_i = \frac{\partial F}{\partial P_i}, \quad (2.4)$$

while the new Hamiltonian is

$$K = H + \frac{\partial F}{\partial t}. \quad (2.5)$$

Canonical transformations like this are changes in the classical variables which leave invariant quantities such as Poisson bracket relations, and their utility stems from the fact that if K should take on a simple form (such as being independent of some of the Q_i or P_i) it becomes possible to solve the equations of motion of Eqs. (2.1) in a simple manner.

The simplest possible situation would be one in which K is zero since then the new variables P_i and Q_i take on com-

pletely time independent values, α_i and β_i say. At first sight it would appear unlikely that there could exist any such $F(q,P,t)$ which could possibly do this. However, there is one and it is constructed as follows. When K is zero Eq. (2.5) takes the form

$$\frac{\partial F}{\partial t} + H\left(q_i, \frac{\partial F}{\partial q_i}, t\right) = 0 \tag{2.6}$$

according to Eq. (2.4). This equation is known as the Hamilton–Jacobi equation. We solve it for F by evaluating the total time derivative

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_i \left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial P_i}\dot{P}_i\right). \tag{2.7}$$

With $\dot{P}_i$ being zero Eq. (2.7) yields

$$\frac{dF}{dt} = \sum_i p_i \dot{q}_i - H(q_i,p_i,t) = L(q_i,\dot{q}_i,t). \tag{2.8}$$

Thus $F(q_i,\alpha_i,t)$ is none other than the classical action, so that the classical action satisfies the Hamilton–Jacobi equations[5]

$$p_i = \frac{\partial S}{\partial q_i}, \quad \frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0 \tag{2.9}$$

and generates canonical transformations to a variable independent Hamiltonian.

Some simplification is possible for systems whose potential V only depends on the coordinates. Then the Lagrangian takes the form (for N independent coordinates)

$$L = \sum_i \frac{m}{2}\dot{q}_i^2 - V(q_1,\ldots,q_N), \tag{2.10}$$

so that the momenta are given by $p_i = m\dot{q}_i$, and the Euler–Lagrange equations of motion are

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i}. \tag{2.11}$$

The equations of motion may be integrated analytically to yield

$$\sum_i \frac{p_i^2}{2m} = \sum_i \frac{m}{2}\dot{q}_i^2 = -V(q_1,\ldots,q_N) + E, \tag{2.12}$$

where the energy E is a constant of the motion. In this case we may set

$$S(q,\alpha,t) = W(q,\alpha) - Et, \tag{2.13}$$

where

$$p_i = \frac{\partial W}{\partial q_i}, \quad H\left(q_i, \frac{\partial W}{\partial q_i}, t\right) = E. \tag{2.14}$$

The quantity $W(q,\alpha)$ that we have introduced is sometimes known as Hamilton's characteristic function. For a single particle in a potential $V(x,y,z)$ it satifies

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] + V(x,y,z) = E \tag{2.15}$$

with the particle momentum being given by

$$|\mathbf{p}| = |\overline{\nabla} W| = [2m(E-V)]^{1/2}. \tag{2.16}$$

In general it is not always possible to completely solve for the motion analytically by integrating Eq. (2.12). However, for a one-dimensional system it can be done readily and yields

$$t = \int_0^x dx \left(\frac{m}{2(E-V)}\right)^{1/2} = \int_0^x dx \frac{\partial p}{\partial E}, \tag{2.17}$$

where we have set $x(t=0) = 0$ as a convenient initial condition. By means of Eq. (2.12) we can also evaluate the stationary classical action along the stationary classical path of Eq. (2.17) to obtain

$$S = \int_0^t dt\,[m\dot{x}^2 - E\,] = \int_0^x p\,dx - Et. \tag{2.18}$$

Differentiating Eq. (2.18) yields

$$\frac{\partial S}{\partial x} = p, \quad \frac{\partial S}{\partial t} = -E \tag{2.19}$$

so that from Eq. (2.12) we recover the Hamilton–Jacobi equation.

III. HAMILTON–JACOBI THEORY AND GEOMETRICAL OPTICS

The Hamilton–Jacobi equation also appears in the classical theory of optics which is concerned with wave motion rather than particle motion. Specifically, the wave equation in optics for quantities such as a scalar electromagnetic potential ϕ is

$$\nabla^2\phi - \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{3.1}$$

where c is the velocity of light *in vacuo* and n is the refractive index of the medium in which the light is propagating. When n has no space dependence the solution of Eq. (3.1) is a plane wave

$$\phi = e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{3.2}$$

where

$$|\mathbf{k}| = n\omega/c. \tag{3.3}$$

The frequency in the medium is thus given by

$$\omega = k_0 c, \tag{3.4}$$

where k_0 is the wavenumber *in vacuo*. In the more general case the refractive index will have some space dependence and so we replace the plane wave by

$$\phi = e^{A(\mathbf{r}) + ik_0[B(\mathbf{r}) - ct]}, \tag{3.5}$$

where $A(\mathbf{r})$ and $B(\mathbf{r})$ are arbitrary real functions of position. This general form will satisfy the wave equation if

$$\nabla^2 A + |\overline{\nabla} A|^2 + k_0^2(n^2 - |\overline{\nabla} B|^2) = 0,$$
$$\nabla^2 B + 2\overline{\nabla} A\cdot\overline{\nabla} B = 0. \tag{3.6}$$

In the geometrical optics limit we consider light with wavelength small compared to the characteristic distances over which n varies. In this limit $A(\mathbf{r})$ is also slowly varying so that for large k_0 Eqs. (3.6) reduce to

$$|\overline{\nabla} B|^2 = n^2, \tag{3.7}$$

the eikonal equation of optics. We immediately recognize the form equivalence of Eqs. (3.7) and (2.16), so that the eikonal function $B(\mathbf{r})$ and Hamilton's characteristic function $W(q,\alpha)$ have analogous propagation properties. In classical physics this simply means that by making the wavelength of the light so short we have eliminated its diffractive and interference aspects completely (i.e., all of its wave nature) so that all that remains is reflection and refraction, which of course can also be described entirely by particle mechanics.

Thus in classical mechanics we see that a short wavelengths the wave equation becomes the particle Hamilton–Jacobi equation. The great leap of faith of quantum mechanics is to make the complementary assertion, i.e., that Hamilton's characteristic function satisfies a wave equation. Specifically we can introduce wave–particle duality by requiring that the stationary classical action for particle motion be identified with the phase of an associated wave motion. Thus we set

$$S = W - Et = \hbar k_0 [B - ct\,], \tag{3.8}$$

where $\hbar$ is some constant of proportionality. From Eq. (3.8) we obtain the Einstein and de Broglie energy and momentum relations for the particle associated with a quantized wave, viz.

$$E = \hbar\omega \tag{3.9}$$

and

$$|\mathbf{p}| = |\overline{\nabla} W| = \hbar k_0 |\overline{\nabla} B| = \hbar k_0 n = \hbar |\mathbf{k}|. \tag{3.10}$$

Further, as for the wave equation itself, Eq. (3.1) gives

$$\nabla^2\phi + n^2 k_0^2 \phi = 0. \tag{3.11}$$

Thus using Eqs. (3.10) and (2.16) we obtain, finally,

$$\nabla^2\phi + [2m(E - V)/\hbar^2]\phi = 0, \tag{3.12}$$

which we recognize as the Schrödinger equation. Thus quantizing the action yields the Schrödinger equation.

While all of this is of course familiar ground[4] our interest here is in noting that according to the quantization postulate of Eq. (3.8) the phase of the quantum-mechanical wave function is in fact none other than $iS_{CL}/\hbar$. Thus we see why this particular phase has something to do with quantum-mechanical propagation.

It is also instructive to derive this result in slightly different manner using a semiclassical approach.[6] We consider an arbitrary wave packet (taken to be one dimensional for simplicity)

$$\psi(x,t) = \int dE'\, f(E') e^{(i/\hbar)[\phi(x,E') - E't]}, \tag{3.13}$$

which is built out of energy eigenstates each with a so far arbitrary space-dependent phase $\phi(x,E')$. The wave packet is semiclassical in the sense that $f(E')$ is peaked at the classical energy E of Eqs. (2.18) and (2.19), i.e., at the energy of an associated classical particle, while the integration of Eq. (3.13) is dominated by a stationary phase at E. At this stationary phase

$$\frac{\partial\phi(x,E)}{\partial E} = t. \tag{3.14}$$

Hence according to Eq. (2.17) we can identify

$$\phi(x,E) = \int_0^x dx'\, p(x',E) \tag{3.15}$$

and can thus approximate the wave packet by

$$\psi(x,t) = f(E) e^{(i/\hbar)\left[\int_0^x dx'\, p(x',E) - Et\right]}. \tag{3.16}$$

According to Eq. (2.18) we thus see that the phase of the wave function is precisely $iS_{CL}/\hbar$, i.e., the phase is proportional to the stationary classical action of the associated classical particle motion. Finally, differentiating Eq. (3.16) yields

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - V(x)\psi = \frac{i\hbar}{2m}\frac{\partial p}{\partial x}\psi. \tag{3.17}$$

The right-hand side of this equation is negligible when

$$\hbar\frac{\partial p}{\partial x} \ll p^2, \tag{3.18}$$

i.e., when

$$\frac{\partial k}{\partial x} \ll k^2 \tag{3.19}$$

according to Eq. (3.10). Thus we again recover the Schrödinger equation in the high-energy limit where the wavelength is small compared to the characteristic distances over which $V(x)$ varies.

Though we have now indeed found a connection between quantum-mechanical propagation and the classical action we note that for the moment we have only found a connection in the high-energy limit. Moreover, there is as of yet no sign of any sum over paths since the Hamilton–Jacobi equation only refers to the path which minimizes the classical action. We shall therefore now seek to find a role for the classical action in a different limit.

IV. QUANTUM MECHANICS IN THE CLASSICAL LIMIT

Having now quantized classical mechanics by studying its short-wavelength limit we next make a complementary analysis by studying quantum mechanics in the small $\hbar$ classical limit. We consider a general quantum-mechanical wave function written in a convenient form [$C(\mathbf{r},t)$ and $D(\mathbf{r},t)$ are arbitrary real functions]

$$\psi(\mathbf{r},t) = C(\mathbf{r},t) e^{(i/\hbar)D(\mathbf{r},t)}. \tag{4.1}$$

Substituting into the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{4.2}$$

yields

$$\frac{\partial D}{\partial t} + \frac{|\overline{\nabla} D|^2}{2m} + V = \frac{\hbar^2}{2m}\frac{\nabla^2 C}{C} \tag{4.3}$$

and

$$\frac{\partial}{\partial t}C^2 + \frac{1}{m}\overline{\nabla}\cdot(C^2\overline{\nabla} D) = 0. \tag{4.4}$$

We recognize Eq. (4.4) as the continuity equation for a particle with probability density C^2 and momentum

$$m\mathbf{v} = \overline{\nabla} D. \tag{4.5}$$

In the small $\hbar$ limit we therefore recognize Eqs. (4.3) and (4.5) as the Hamilton–Jacobi equations, so that we can identify $D(\mathbf{r},t)$ with the stationary classical action S_{CL}. Thus in the small $\hbar$ limit the phase of the quantum-mechanical wave function is given by $iS_{CL}/\hbar$. Further, in a stationary state with energy E

$$\frac{\partial C}{\partial t} = 0, \quad \frac{\partial D}{\partial t} = -E, \tag{4.6}$$

so that Eq. (4.3) becomes

$$|\overline{\nabla} D|^2 = 2m(E - V) + \hbar^2(\nabla^2 C)/C. \tag{4.7}$$

Hence, we recover Eq. (2.15) in two limits, namely small $\hbar$, or E large as in Sec. III.

While we have now established a connection between quantum-mechanical propagation and the classical action in two distinct limits, in practice it is difficult to extend the range of validity of either of these limits to longer wave-

lengths or higher orders in $\hbar$. Moreover, an expansion in powers of $\hbar$ would suggest that the quantum action should appear rather many classical actions. Thus in order to reach the path integral formula we shall consider instead yet another limit, namely quantum mechanics at small times.

V. QUANTUM MECHANICS FOR SMALL TIMES

Our approach now is that of standard quantum mechanics. In order to obtain the required limit we evaluate the propagator of conventional quantum mechanics given in Eq. (1.4) for $t' = t + \epsilon$, where ϵ is a small time interval. To simplify the discussion we shall work in one space dimension. Expanding Eq. (1.4) for small ϵ we obtain

$$G(q',t+\epsilon;q,t) = \langle q|q'\rangle - \frac{i\epsilon}{\hbar}\left\langle q\left|\left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)\right|q'\right\rangle + O(\epsilon^2), \tag{5.1}$$

where $\hat{p}$ and $\hat{q}$ are, respectively, momentum and position operators. We evaluate Eq. (5.1) by inserting complete sets of momentum and position eigenstates with eigenvalues p and q, respectively. (We note in passing that this in no way requires that the eigenstates of $\hat{H}$ be plane waves. Our use of momentum eigenstates is just a convenient way of representing the eigenspectrum of the operator $\partial^2/\partial\hat{q}^2$.) Thus we obtain

$$G(q',t+\epsilon;q,t) = \delta(q'-q)\left[1 - \frac{i\epsilon}{\hbar}\overline{V}(q,q')\right] - \frac{i\epsilon}{\hbar}\int\frac{dp}{2\pi\hbar}e^{(i/\hbar)p(q'-q)}\frac{p^2}{2m}, \tag{5.2}$$

where $\overline{V}(q,q')$ is some appropriate average value of the classical potential. The particular averaging chosen is immaterial as long as $\overline{V}(q,q')$ is multiplied by $\delta(q'-q)$. Below, however, we shall undo the delta function and so we would appear to have an ambiguity in how to choose the appropriate $\overline{V}(q,q')$. We shall see below however that the actual path integration will in fact resolve this ambiguity. The utility of Eq. (5.2) is that of the propagator is now expressed entirely in terms of c numbers.

While the above analysis is completely straightforward we note that it is only in $O(\epsilon^2)$ that the expansion of the exponential of the operator $i\epsilon\hat{H}/\hbar$ would first be sensitive to the fact that $\hat{p}$ and $\hat{q}$ do not commute. Thus in order ϵ the calculation proceeds as though $\hat{p}$ and $\hat{q}$ are c numbers. Hence we have lost some of the information contained in the uncertainty principle, and may have thrown too much information away. However, this limit is in fact useful and we will return to this point below in Secs. VI and VII where we show how the existence of many paths restores this information at finite times.

To complete the evaluation of the propagator in order ϵ we rewrite Eq. (5.2) as

$$\begin{aligned} G(q',t+\epsilon;q,t) &= \int\frac{dp}{2\pi\hbar}e^{(i/\hbar)p(q'-q)}\left[1 - \frac{i\epsilon}{\hbar}\left(\frac{p^2}{2m} + \overline{V}(q,q')\right)\right] \\ &= \int\frac{dp}{2\pi\hbar}e^{(i\epsilon/\hbar)[\,p(q'-q)/\epsilon - p^2/2m - \overline{V}(q,q')]}. \end{aligned} \tag{5.3}$$

Performing the Gaussian integration gives

$$G(q',t+\epsilon;q,t) = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}e^{(i\epsilon/\hbar)(m/2)[(q'-q)/\epsilon]^2 - \overline{V}(q,q')}. \tag{5.4}$$

We recognize the factor in the exponential as the classical Lagrangian (i.e., the Lagrangian is the Gaussian transform of the Hamiltonian), and it is tempting to identify its product with ϵ as the classical action that we are seeking. However, the phase that appears in Eq. (5.4) is only equal to the small ϵ classical action when q and q' are close, while our derivation of Eq. (5.4) required no such restriction. However, we now note that our only interest in Eq. (5.4) is to use it in Eq. (1.3) to evaluate $\psi(q',t+\epsilon)$, or in the path integral of Eq. (1.8). This then requires an integration over all q. In this integration the factor $im(q'-q)^2/2\hbar\epsilon$ in the exponential causes rapid mutually cancelling oscillations except in the region where $(q'-q)^2$ is of order ϵ. Hence, unless the potential has a strong velocity dependence, a case we are not considering, the integration on q will precisely only receive appreciable contributions from the region where q and q' are close to each other. This is just the region where the choice of $\overline{V}(q,q')$ is not ambiguous, as it can be evaluated by a simple mean value. Further, in this region we can indeed identify the phase as the stationary classical action between q and q' so that we can replace Eq. (5.4) by

$$G(q',t+\epsilon;q,t) = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}e^{(i/\hbar)S_{CL}(q',q)}. \tag{5.5}$$

As long as we subsequently integrate over coordinates (as we expressly do in the path integral in Sec. VI below) the difference in contributions between Eqs. (5.4) and (5.5) will be negligibly small for velocity independent potentials. [In his original paper Feynman inserted Eq. (5.5) into Eq. (1.3) and showed that it immediately led to the Schrödinger equation for $\psi(x,t)$ for small ϵ provided the potential was velocity independent. Thus our restriction to velocity independent potentials—or strictly speaking to potentials whose velocity dependence does not destroy the stationary phase dominance of the region in which $(q-q')^2$ is of order ϵ—is just the standard one for which Feynman was able to show the equivalence of the path integral formulation to ordinary Schrödinger theory in the first place.]

It is instructive to illustrate these above remarks in an explicit example, and so we consider a simple harmonic oscillator. The classical Lagrangian is

$$L = \tfrac{1}{2}m(\dot{x}^2 - \omega^2x^2) \tag{5.6}$$

and the classical trajectory is given by

$$x = \left(\frac{2E}{m}\right)^{1/2}\frac{\sin\omega t}{\omega}, \tag{5.7}$$

where E is the classical energy. The stationary classical action is

$$S(x,t) = \frac{mx^2\omega\cos\omega t}{2\sin\omega t} = \frac{E\sin 2\omega t}{2\omega} \tag{5.8}$$

after integrating exactly along the path of Eq. (5.7). To use $S(x,t)$ in the path integral or in Eq. (1.3) we must now consider the variation of $S(x,t)$ with respect to its end points. This is a variation over points x which do not satisfy Eq. (5.7)—Eq. (5.7) only refers to the minimum classical path between the origin and (x,t). Consequently, there is an apparent ambiguity in choosing the right $S(x,t)$ since the stationary classical action of Eq. (5.8) can be written in many equivalent forms by the use of Eq. (5.7), and can even be written in the form given in the second part of Eq. (5.8) which is independent of x altogether. As we shall see momentarily, however, we should in fact use the first form of $S(x,t)$ given in Eq. (5.8), i.e., the unique function of x and t,

which is independent of E. Thus classical mechanics supplies us with a function $S(x,t)$. To use it in the path integral or in Eq. (1.3) for small times we hold x fixed and let t become small in $S(x,t)$ to obtain

$$S(x,\epsilon) = mx^2/2\epsilon - mx^2\omega^2\epsilon/4. \tag{5.9}$$

For small x and small ϵ we can identify this quantity as

$$S(x,\epsilon) = \epsilon\{(m/2)(x/\epsilon)^2 - \tfrac{1}{2}[V(0) + V(x)]\}. \tag{5.10}$$

This is precisely the factor which appears in the exponent in Eq. (5.4) as required. Thus it is Eq. (5.4) which tells us how to resolve the ambiguity inherent in the various choices for the classical action, as it requires us to choose a function of x and ϵ only with no reference to E. Finally, we also note in passing that to this order in ϵ the potential dependent term in Eq. (5.9) is negligible while the dominant kinetic-energy-dependent term is just the classical action for a free particle. Hence for small ϵ we can even evaluate the phase factor in Eq. (5.5) along the stationary classical trajectory for a free particle, i.e., along a straight line. Thus we can simplify our stepwise definition of the multiple path integral limit given in Eq. (1.8) since we can even connect the various points q_i and q_{i+1} by straight line paths. In the subsequent integrations over all space the difference between using the exact classical orbit between q_i and q_{i+1} or a straight line path will be negligible for all but the most badly behaved potentials.

Thus to conclude, we see that for small times the propagator is given exactly by the phase $iS_{CL}/\hbar$ in Eq. (5.5). Moreover, and this will be crucial below, the classical action required for Eq. (5.5) is in fact the stationary one which minimizes the classical action for motions between q,t and $q',t+\epsilon$. Thus despite the fact that Eq. (5.5) is fully quantum mechanical, for small enough times the propagator still follows the classical trajectory. Thus for the moment there is still only one path.

VI. QUANTUM MECHANICS FOR FINITE TIMES

Having now constructed the propagator for small times we have a limit which is readily generalizable to arbitrary finite times. Thus we can now build up the finite time propagator by iteration. According to the superposition principle the complete wave function arriving at q'' at time $t+2\epsilon$ is given in terms of states $|q',t+\epsilon\rangle$ as

$$|q'',t+2\epsilon\rangle = \int dq'\, G(q'',t+2\epsilon;q',t+\epsilon)|q',t+\epsilon\rangle \tag{6.1}$$

to yield a quantum-mechanical Huygens's principle.[1] Thus the full propagator from $|q,t\rangle$ to $|q'',t+2\epsilon\rangle$ is given as

$$G(q'',t+2\epsilon;q,t) = \int dq'\, G(q'',t+2\epsilon;q',t+\epsilon)G(q',t+\epsilon;q,t). \tag{6.2}$$

Using Eq. (5.5) we obtain

$$G(q'',t+2\epsilon;q,t) = \left(\frac{m}{2\pi i\hbar\epsilon}\right)\int dq'\, e^{(i/\hbar)S_{CL}(q'',q') + (i/\hbar)S_{CL}(q',q)} \tag{6.3}$$

and are thus able to express the quantum-mechanical propagator entirely in terms of classical actions between various points.

We give a pictorial representation of Eq. (6.3) in Fig. 1 and see immediately that the construction is that of Huygens's principle. We note that unlike the $\Delta t = \epsilon$ case, for

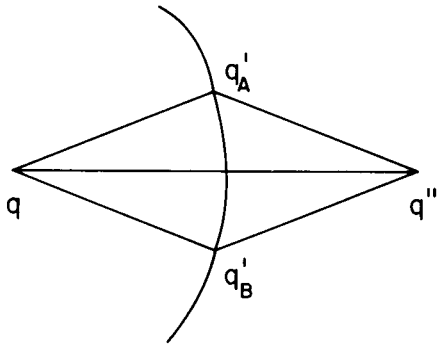

Fig. 1. Quantum-mechanical propagator $G(q'',t+2\epsilon;q,t)$ built up by intermediate propagations to points q' at time $t+\epsilon$. The paths via points q'_A and q'_B are typical intermediate paths.

$\Delta t = 2\epsilon$ there is more than one path involved. For instance we can propagate via points such as q'_A or q'_B of Fig. 1. Now the propagation from q to q'_A follows the minimizing classical path between q and q'_A (while the propagation from q to q'_B likewise follows the minimizing classical path between q and q'_B); and the propagation from q'_A on to q'' follows the minimizing classical path between q'_A and q''. However, and this is the key point, the complete propagation from q to q'' via q'_A does *not* follow the minimizing classical path between q and q''. In other words, *the sum of stationary paths for two subintervals is not necessarily a stationary path for the full interval.*[7] Thus as soon as we go beyond $\Delta t = \epsilon$ we have to start including nonstationary classical paths. Nonetheless, the phase for each of these nonstationary paths is still the classical action for that path, since for the path via q'_A, for instance,

$$S_{CL}(q'',q) = S_{CL}(q'',q'_A) + S_{CL}(q'_A,q). \tag{6.4}$$

The path is still classical, it is just no longer stationary between its end points. The same is true for all but one of the points q'. Hence there are many paths.

Proceeding in this way we can then iterate to longer times to obtain finally the standard finite time path integral formula of Eq. (1.8). The advantage of our approach here is that we explicitly see where the paths come from. As a bonus we are also able to identify the complete set of paths required for the path integral by applying Huygens's principle iteratively. Thus we obtain a constructive method for determining the actual measure required for the path integration, and also even obtain the appropriate normalizing factors such as the factor $(m/2\pi i\hbar\epsilon)^{1/2}$ given in Eq. (5.5). Starting from Eq. (1.6) as a postulate it is not immediately clear what measure to use.

As we have just seen it is the extension to finite time propagation which obliges us to include many classical paths. To complete our study we now show that the existence of all of these paths is a consequence of the uncertainty principle.

VII. PATH INTEGRALS AND THE SPREADING OF THE WAVE FUNCTION

To complete our analysis we discuss some intuitive aspects of the structure we have just found for the path integral. As we noted in Eq. (5.5) for small times there is only one path, namely the classically stationary one. Despite this, and despite the fact that the derivation of Eq.

(5.5) did not depend on the nonvanishing of the $\hat{p},\hat{q}$ commutator, Eq. (5.5) nontheless still contains some quantum mechanics. For instance, in order to prepare the $|q,t\rangle$ state in the first place we must still use a wave packet with a position uncertainty $\Delta q(t)$; and we must then subsequently let the packet propagate for a time ϵ. Now we recall that in quantum mechanics the spreading of the wave function is given at a later time $t+\epsilon$ by

$$\left(\frac{\Delta q(t+\epsilon)}{\Delta q(t)}\right)^2 = 1 + \frac{1}{4}\left(\frac{\hbar\epsilon}{m[\Delta q(t)]^2}\right)^2, \tag{7.1}$$

which is $O(\epsilon^2)$. Thus in $O(\epsilon)$ the wave packet has not yet had a chance to spread and hence follows the classical trajectory in Eq. (5.5). Now while it is true that for small or large times the center of the wave packet always follows the stationary classical path, our point here is that for small times of $O(\epsilon)$ the *wings* of the initial wave packet also follow this trajectory. Hence there is only one path in Eq. (5.5). As soon as we go to $O(\epsilon^2)$, [or $O(\hbar^2)$], the wings of the wave packet start to spread, so that we cannot specify position and momentum simultaneously as we become sensitive to the noncommutativity of $\hat{p}$ and $\hat{q}$. Hence we have to include many classical momentum and position configurations, and thus need many classical paths to specify the quantum-mechanical motion. Thus the existence of all of the paths is a consequence of the spreading of the quantum-mechanical wave function.

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[1]R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).

[2]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[3]In the literature the term "classical action" is often used only to denote the value of $\int L\,dt$ for that particular classical path which minimizes the classical equations of motion. Here we use it in the broader sense of also including paths which are not stationary classically. In this paper when we do have in mind the stationary classical action we shall state it explicitly.

[4]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950).

[5]We note that it is only the stationary classical action which satisfies the Hamilton–Jacobi equation. Consequently in classical mechanics Eq. (2.9) is not of much practical value for calculational purposes since it can only be used after the problem has already been solved.

[6]E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961).

[7]The converse of course is true, i.e., any subinterval of a given stationary path is itself stationary.

Testing laboratory performance

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A practical test seems to be a more adequate test type for measuring laboratory achievement than written or oral tests, for it is constructed specifically to examine outcomes in laboratory objectives, which can roughly be described as experimental skills. Several suggestions from various authors are reviewed. Their methods are discussed, and an improved way of testing experimental abilities is proposed. It is shown that under certain conditions performance tests may be highly reliable instruments.

I. INTRODUCTION

Grading laboratory performance of students often is a source of trouble for teachers as well as for students. For the most part, one of the following three methods is applied to measure achievement:

–a written examination,

–a judgment on the quality of the students' written reports,

–and, finally, a general estimation of the students' abilities by their tutors.

But all three methods have disadvantages which demand we consider other ways of testing laboratory achievement. Written tests, for example, contradict the purpose of testing practical abilities. They do not ask if a student is able to handle measuring instruments correctly, if he has experimental skill in building up experimental apparatus, in finding and repairing errors, and so forth.

In a similar sense the evaluation of the students' written reports does not allow a precise judgment on the very manner how they came about, and how logically and skillfully the problems were solved.

The third method, estimation by tutors, though it seems to be most adequate to a laboratory course, has the shortcoming that it disturbs good relations between tutors and students. The students will avoid confessing difficulties and hesitate to ask questions, and the cooperation between students and tutors is hindered.

In view of these drawbacks we tried another method of laboratory achievement control,[1,2] which, though obvious, has hardly been applied: the so-called performance test. This kind of examination does not intend to ask for theoretical knowledge, but should test real experimental skills.

Unfortunately, only a few experiences with performance tests or ideas on their structure have been reported; performance tests cannot reliably be consulted as routine techni-