Radioactive Decay

1. Background

It is well known that many nuclei are unstable and are transformed into other nuclear species by means of either alpha decay or beta decay. The rate at which those radioactive nuclei decay in a sample of material is called the *activity* of the sample. The greater the activity, the more nuclei decay per second. The decay probability per nucleus per second is call the *decay constant* λ. One assumes that λ is a small number, and that it is constant in time for any particular material, i.e. the probability of any one nucleus decaying does not depend on the age of the sample. The activity $\mathcal{A}$ depends on the number N of radioactive nuclei in the sample and also on the probability λ for each nucleus to decay, $\mathcal{A} = \lambda N$. As the radioactive nuclei in the sample decay, there are fewer of them left. If N decreases and λ is a constant, then the number of radioactive nuclei in the sample is governed by the differential equation

$$\frac{dN}{dt} = -\lambda N. \tag{1}$$

This equation can be directly integrated to yield

$$N = N_0 e^{-\lambda t}, \tag{2}$$

where N_0 represent the number of radioactive nuclei originally present at $t = 0$. Eq. (2) is the *exponential law of radioactive decay*, which tells us how the number of radioactive nuclei in a sample decreases with time. Similarly, for the activity holds

$$\mathcal{A} = \mathcal{A}_0 e^{-\lambda t}. \tag{3}$$

The *half-life* $t_{1/2}$ of the decay is the time that it takes for the activity to be reduced by half, i.e. $\mathcal{A} = \mathcal{A}_0/2$ when $t = t_{1/2}$. From that follows a simple relation of $t_{1/2}$ to the decay constant

$$t_{1/2} = \frac{1}{\lambda} \ln 2 = \frac{0.693}{\lambda}. \tag{4}$$

Another useful parameter is the *mean lifetime* τ,

$$\tau = \frac{1}{\lambda}. \tag{5}$$

2. Numerical Approach

While the differential equation Eq. (2) can be solved without resorting to a numerical approach, this problem is useful for introducing a computational method for differential equations. Starting from the simplest method for obtaining a numerical derivative, the forward different method, we now consider a simple method for solving this problem numerically. Given a value N_0 at one particular value of t (usually $t = 0$), we want to estimate its value at later times. One line of attack can be derived by examining the Taylor expansion for N_0

$$N(\Delta t) = N_0(0) + \frac{dN}{dt}\Delta t + \frac{1}{2}\frac{d^2N}{dt^2}(\Delta t)^2 + \cdots, \tag{6}$$

where $N(\Delta t)$ is the value of N at $t = \Delta t$, and the derivatives are evaluated at $t = 0$. If we take Δt to be small, then it is usually a good approximation to simply ignore the terms that involve second and higher powers of Δt, leaving

$$N(\Delta t) \approx N_0(0) + \frac{dN}{dt}\Delta t. \tag{7}$$

The same result can be obtained from the definition of the derivative

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} = \frac{N(t + \Delta t) - N(t)}{\Delta t} \approx \frac{N(t + \Delta t) - N(t)}{\Delta t}, \tag{8}$$

where in the last approximation we have assumed that Δt is small, but nonzero. Rearranging the terms gives

$$N(t + \Delta t) \approx N(t) + \frac{dN}{dt}\Delta t, \tag{9}$$

which is equivalent to (7). It is important to recognize that this is an *approximation*. The error terms that were dropped in deriving this result are of order $(\Delta t)^2$, which makes them at least one factor of Δt smaller than any of the other terms in (9). Hence, by making Δt small, one would expect that the error terms can be made negligible. From our study of differentiation, we already know that one has to pay attention. The above simple approximation to solve a first order differential equation is called **Euler method**.

For the physics problem we want to solve we know the functional form of the derivative. Inserting this into Eq. (9) and rearranging the terms leads to

$$N(t + \Delta t) \approx N(t) - \lambda N \Delta t, \tag{10}$$

which forms the basis for a numerical solution of our radioactive decay problem. Given that we know the value of N at some value of T, we can use (10) to *estimate* its value at time Δt later. In our case the number of radioactive nuclei at $t = 0$ is given.

The overall structure of a program to simulate the radioactive decay consists of three basic tasks:

1. Initialize all variables and parameters
2. Carrying out the calculation according to (10) for a finite number of time steps.
3. Display the results.
4. Estimate the numerical error.