Project I

Due 10/10/2016

Please put your final tar-file at the due date by 8:15 am in the same directory as your homework. Please name your tar-file **Project1.tar**.

1. The function

$$f(r) = \frac{1}{1 + e^{\frac{r-c}{a}}} \tag{1}$$

is known by the name of "Wood-Saxon" in nuclear physics. The quantity c is related to the nuclear radius and a is the surface thickness of the varying density. The idea is that a nucleus does not behave like a hard sphere, but rather like a 'fuzzy' ball. The 'volume' of $f(r)$ can be written as a rapidly converging series:

$$V \equiv 4\pi \int_0^\infty \frac{dr\, r^2}{1 + e^{\frac{r-c}{a}}} = 4\pi a^3 \left[\frac{1}{3}\left(\frac{c}{a}\right)^3 + \frac{\pi^2}{3}\frac{c}{a} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} e^{-\frac{nc}{a}} \right]. \tag{2}$$

Use $c = 2.0$ fm and $a = 0.5$ fm. In these units r has the units fm.

(a) Evaluate the integral for V by Simpson and by Gaussian integration. When employing the Gaussian integration method you need to map the interval in which the gauss points are given to the integration interval, i.e. you need to map $(0, 1) \to (0, \infty)$. You may use the map (4.43) and (4.44) of the textbook or use a map

$$x_i = A \tan\left(y_i \frac{\pi}{2}\right) \tag{3}$$

$$w_i = A \frac{\pi}{2} \frac{1}{\cos^2\left(y_i \frac{\pi}{2}\right)} w_i' \tag{4}$$

Here A is a mapping factor. Convince yourself (and the instructor) that in this map half of the points are located in the interval $(0, A)$ and half of them in the interval A, ∞. Thus A is the midpoint for the integration.

(b) Make an error analysis of your Simpson integration. You will need to check how far out you have to integrate in addition to varying your step size.

(c) Make an error analysis of your Gaussian integration. Since you map your points to the interval $(0, \infty)$, the upper integration limit is taken into account exactly. You will need to check in your error analysis your choice of the midpoint and determine how the number of points needed for a converged result depend on the choice of your midpoint. [Hint: plot the function $f(r)$ to locate its region of importance.]

(d) Compare your result to the result obtained by summing the series. When summing your series, remember the numerical issues related to summing a series.

(e) Carry out the a Fourier-Bessel transform numerically:

$$\hat{f}(q) = \frac{1}{4\pi} \int_0^\infty dr\, r^2\, j_o(qr)\, f(r). \tag{5}$$

Normalize $\hat{f}(q)$ such that $\hat{f}(0) = 1$. Plot $\hat{f}(q)$.

(f) Calculate the integral

$$\hat{V} = 4\pi \int_0^\infty dq\, q^2 \hat{f}(q) \tag{6}$$

and attempt an accuracy of at least 3 digits. Have in mind that the accuracy of your result is determined by the accuracy of your integration as well as your Fourier transform.

2. Radioactive Decay

(a) Consider a radioactive decay problem involving two types of nuclei, A and B, which populations $N_A(t)$ and $N_B(t)$. Suppose that type A nuclei decay to form type B nuclei, which then also decay according to the differential equations

$$\begin{aligned} \frac{dN_A}{dt} &= -\frac{N_A}{\tau_A} \\ \frac{dN_B}{dt} &= \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B}, \end{aligned} \tag{7}$$

where τ_A and τ_B are the decay time constants for each type of nucleus. Use the Euler method to solve these coupled equations for N_A and N_B as functions of time. Explore the behavior found for values of $\tau_A/\tau_B = 3$, 5, and 1/5. you may choose $\tau_A = 1$ s. Check different time steps to make sure your calculations do not depend on the choice of your time step. Plot your final result

(b) Consider again the decay problem with two nuclei A and B, but now suppose that nuclei of type A decay into ones of type B, while nuclei of type B decay into ones of type A. The corresponding equations are then

$$\begin{aligned} \frac{dN_A}{dt} &= \frac{N_B}{\tau} - \frac{N_A}{\tau} \\ \frac{dN_B}{dt} &= \frac{N_A}{\tau} - \frac{N_B}{\tau}, \end{aligned} \tag{8}$$

where for simplicity it is assumed that the two types of decay are characterized by the same time constant τ. Solve this system of equations for the number of nuclei,

N_A and N_B, as function of time. Consider as initial conditions e.g. $N_A = 100$, $N_B = 0$, and take $\tau = 1$ s. Show that your numerical results are consistent with the idea that the system reaches a steady state in which N_A and N_B are constant. In such a steady state, the time derivatives dN_A/dt and dN_B/dt should vanish. Check different time steps to make sure your calculations do not depend on the choice of your time step. Plot your final result