Chapter 1

The Nucleon-Nucleon System

1.1 The Lippmann-Schwinger Equation for the Scattering Process

Let us consider two-nucleon scattering and define $\vec{k}_1$ and $\vec{k}_2$ to be the individual nucleon momenta. The relative momentum is then given as

$$\vec{p} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2) \tag{1.1}$$

The momentum eigenstates in the nucleon-nucleon (NN) c.m. system are then

$$\mid \vec{p}\rangle \tag{1.2}$$

and are chosen to be normalized as

$$\langle \vec{p} \mid \vec{p}\,'\rangle = \delta^3(\vec{p} - \vec{p}\,') \ . \tag{1.3}$$

They are eigenstates to the free Hamiltonian for the relative motion

$$H_0 = \frac{\vec{p}^{\,2}}{m} \tag{1.4}$$

where m is the nucleon mass.
Let V be the NN potential, which is assumed to be energy independent.
The Schrödinger equation for a scattering state $\Psi^{(+)}_{\vec{p}}$

$$(H_0 + V)\ \Psi^{(+)}_{\vec{p}} \ = \ E\ \Psi^{(+)}_{\vec{p}} \tag{1.5}$$

can be cast into an integral equation, the Lippmann-Schwinger equation (LSE):

$$(H_0 - E)\ \Psi_{\vec{p}}^{(+)} \ = \ -V\Psi_{\vec{p}}^{(+)} \tag{1.6}$$

$$\mid \Psi_{\vec{p}}^{(+)}\rangle \ = \mid \vec{p}\rangle + \frac{1}{E + i\epsilon - H_0} V \mid \Psi_{\vec{p}}^{(+)}\rangle \tag{1.7}$$

Let us consider the configuration space representation. The conjugate variable to $\vec{p}$ is

$$\vec{x} = \vec{r}_1 - \vec{r}_2 \ , \tag{1.8}$$

and we choose $\mid \vec{x}\rangle$ to be normalized as

$$\langle \vec{x} \mid \vec{x}'\rangle = \delta^3(\vec{x} - \vec{x}'). \tag{1.9}$$

Then the Fourier transform is given by

$$\langle \vec{x} \mid \vec{p}\rangle = \frac{1}{(2\pi)^{3/2}}\ e^{i\vec{p}\cdot\vec{x}} \ . \tag{1.10}$$

The configuration space representation of the free propagator

$$G_0 \equiv \frac{1}{E + i\epsilon - H_0} \tag{1.11}$$

is given as

$$\begin{aligned}
\langle \vec{x} \mid \frac{1}{E + i\epsilon - H_0} \ \mid \vec{x}\,'\rangle \ &= \ \int \ d\vec{p}\, \langle \vec{x} \mid \vec{p}\rangle \ \frac{1}{E + i\epsilon - p^2/m}\ \langle \vec{p} \mid \vec{x}\,'\rangle \\
&= \ \frac{1}{(2\pi)^{3/2}}\ \int d^3p\ e^{i\vec{p}\cdot\vec{x}} \frac{1}{E + i\epsilon - p^2/m} e^{-i\vec{p}\cdot\vec{x}\,'} \\
&= \ \frac{1}{(2\pi)^3}\ \int d^3p\ e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \frac{1}{E + i\epsilon - p^2/m} \\
&= \ \frac{1}{(2\pi)^3}\ 4\pi\ \int_0^\infty dp\ p^2 j_0(p\rho)\ \frac{1}{E + i\epsilon - p^2/m}
\end{aligned} \tag{1.12}$$

with $\rho \equiv \mid \vec{x} - \vec{x}' \mid$
Standard residue techniques lead to

$$\langle \vec{x} \mid G_0 \mid \vec{x}\,'\rangle = -\frac{m}{4\pi}\ \frac{e^{i\sqrt{mE}|\vec{x}-\vec{x}'|}}{\mid \vec{x} - \vec{x}' \mid} \ , \tag{1.13}$$

which exhibits the outgoing wave behavior from the source point $\vec{x}'$ to $\vec{x}$. It results the configuration space representation of the LSE

$$\begin{aligned} & \langle \vec{x} \mid \ \Psi_{\vec{p}}^{(+)} \rangle \equiv \ \Psi_{\vec{p}}^{(+)} \ (\vec{x}) \\ = \ & \frac{1}{(2\pi)^{3/2}} \ e^{i\vec{p}\cdot\vec{x}} \ - \frac{m}{4\pi} \ \int d^3x' \ \frac{e^{i\sqrt{mE}|\vec{x}-\vec{x}'|}}{\mid \vec{x} - \vec{x}' \mid} \ V(x') \ \Psi_{\vec{p}}^{(+)}(\vec{x}') \ . \end{aligned} \tag{1.14}$$

Thereby we assumed a local potential

$$\langle \vec{x} \mid V \mid \vec{x}' \rangle = \ \delta^3(\vec{x} - \vec{x}\,') \ V(x') \tag{1.15}$$

In a well known manner one reads off the asymptotic form for $\mid \vec{x} \mid \to \infty$:

$$\Psi_{\vec{p}}^{(+)}(\vec{x}) \ \to \ \frac{1}{(2\pi)^{\frac{3}{2}}} \ (\ e^{i\vec{p}\vec{x}} \ + \ \frac{e^{ipx}}{x} \ f(\hat{x}) \ ), \tag{1.16}$$

with the scattering amplitude $f(\hat{x})$ depending on the direction $\hat{x}$ of observation

$$f(\hat{x}) \ = \ -m \ \sqrt{\frac{\pi}{2}} \ \int \ d^3x' \ e^{-ip\hat{x}\cdot\vec{x}'} \ V(x') \ \Psi_{\vec{p}}^{(+)}(\vec{x}') \ . \tag{1.17}$$

This can be interpreted in terms of a scattered momentum

$$\vec{p}\,' \equiv \hat{x}p \ , \tag{1.18}$$

and one introduces a transition amplitude

$$\begin{aligned} \langle \vec{p}\,' \ \mid \ t \ \mid \vec{p} \rangle \ & \equiv \ \frac{1}{(2\pi)^{\frac{3}{2}}} \ \int \ d^3x' \ e^{-i\vec{p}\,'\vec{x}\,'} \ V(x') \ \Psi_{\vec{p}}^{(+)}(\vec{x}\,') \\ & = \ \langle \vec{p}\,' \ \mid V \mid \ \Psi_{\vec{p}}^{(+)} \rangle \ . \end{aligned} \tag{1.19}$$

Apparently t is the result of the scattering process and determines all scattering observables.
Is there an integral equation directly for t? From (1.24) we read off

$$t \mid \vec{p} \rangle \equiv V \ \mid \Psi_{\vec{p}}^{(+)} \rangle \ , \tag{1.20}$$

and using (1.7) we find

$$\begin{aligned} t \mid \vec{p} \rangle \ & = \ V \mid \vec{p} \rangle + V \ G_0 \ V \mid \Psi_{\vec{p}}^{(+)} \rangle \\ & = \ V \mid \vec{p} \rangle + V \ G_0 \ t \mid \vec{p} \rangle \end{aligned} \tag{1.21}$$

We can strip off the initial state $\mid \vec{p}\rangle$ and get the operator relation

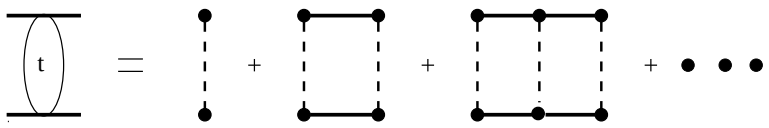
$$t = V + V\ G_0\ t \tag{1.22}$$

which is the LSE for the transition operator. Its simple physical interpretation results by iterating that equation:

$$\begin{aligned} t &= V + VG_0\ (\ V + VG_0 t\ ) \\ &= V + VG_0V + VG_0VG_0V + VG_0VG_0VG_0V + \ldots \end{aligned} \tag{1.23}$$

This is the Born series for scattering on V, a sum of terms of increasing order in V. Each term consists of a sequence of $V's$ with free propagations in between. This is a general structure valid for any number of particles.

It is useful to visualize that multiple scattering process in the form



where the dashed lines stand for the action of V and two horizontal lines for the free propagation G_0 between two interactions. Intuitively one can start from that sum of terms (1.23)

$$t = V + V\ G_0\ V + V\ G_0\ VG_0\ V + \ldots \tag{1.24}$$

and ask the question: can this series be summed up into an integral equation for t ? Obviously it can:

$$t\ =\ V\ +\ VG_0\ (\ V\ +\ VG_0V\ +\ VG_0VG_0V\ +\ \ldots)\ , \tag{1.25}$$

and we recover t again on the right hand side and thus get

$$t = V + V\ G_0\ t \tag{1.26}$$

which is the LSE.
Of course if one starts from Eq. (1.23) in an ad hoc manner one has to know the form of the free propagator G_0 and therefore one has to make contact to the underlying dynamical equation, in our case the Schrödinger equation. Formally however this multiple scattering series is quite general and also valid for the Bethe-Salpeter equation, where G_0 is different from the G_0 used in our nonrelativistic context.

1.2 Alternative Derivation of the Lippmann-Schwinger Equation

A free momentum eigenstate obeys

$$H_0 \mid \vec{p} \rangle \;=\; E_p \mid \vec{p} \rangle \tag{1.27}$$

and a scattering state obeys

$$H \mid \vec{p} \rangle^{(+)} \;=\; E_p \mid \vec{p} \rangle^{(+)} \; . \tag{1.28}$$

Here a scattering state is defined via

$$\mid \vec{p} \rangle^{(+)} \;=\; \Omega^{(+)} \mid \vec{p} \rangle \;=\; lim_{\epsilon \to 0} \; i\epsilon \; G(E + i\epsilon) \mid \vec{p} \rangle \; , \tag{1.29}$$

where $\Omega^{(+)}$ is the Møller operator, which maps a free state $\mid \vec{p} \rangle$ onto a scattering state $\mid \vec{p} \rangle^{(+)}$.

The propagators, or Resolvents, are given by

$$G_0(z) \;=\; \frac{1}{z - H_0} \tag{1.30}$$

and

$$G(z) \;=\; \frac{1}{z - H} \; . \tag{1.31}$$

Here $G_0(z)$ is the free propagator (Resolvent) and $G(z)$ the full propagator (Resolvent).

Let us consider

$$G_0^{-1} - G^{-1} \;=\; (z - H_0) - (z - H) \;=\; -H_0 + H \;=\; V \; , \tag{1.32}$$

where we use that $H \;=\; H_0 + V$.

Multiplying Eq. (1.32) from the left with G_0 and from the right with G yields

$$G_0(G_0^{-1} - G^{-1})G \;=\; G - G_0 \;=\; G_0 V \; G \tag{1.33}$$

or

$$G \;=\; G_0 + G_0 V \; G \; . \tag{1.34}$$

The above relation, Eq. (1.34), is called first Resolvent Identity.

Applying the first Resolvent Identity on a momentum eigenstate and multiplying both sides of the resulting equation with $i\epsilon$ yields

$$\begin{aligned} i\epsilon\, G(E+i\epsilon)\mid \vec{p}\,\rangle &= i\epsilon\, G_0(E+i\epsilon)\mid \vec{p}\,\rangle \;+\; i\epsilon\, G_0(E+i\epsilon)V\; G(E+i\epsilon)\mid \vec{p}\,\rangle \\ &= \frac{i\epsilon}{E+i\epsilon-H_0}\;\mid \vec{p}\,\rangle \;+\; i\epsilon\, G_0(E+i\epsilon)V\; G(E+i\epsilon)\mid \vec{p}\,\rangle \\ &= \mid \vec{p}\,\rangle \;+\; i\epsilon\, G_0(E+i\epsilon)V\; G(E+i\epsilon)\mid \vec{p}\,\rangle\;. \end{aligned} \tag{1.35}$$

Taking the limit $\epsilon \to 0$ gives, together with the definition Eq. (1.29), the Lippmann-Schwinger equation for states

$$\mid \vec{p}\,\rangle^{(+)} \;= \mid \vec{p}\,\rangle \;+\; G_0(E+i\epsilon)V \mid \vec{p}\,\rangle^{(+)} \tag{1.36}$$

If we multiply Eq. (1.36) by V and define

$$V \mid \vec{p}\,\rangle^{(+)} \;=\; t \mid \vec{p}\,\rangle\;, \tag{1.37}$$

we obtain

$$V \mid \vec{p}\,\rangle^{(+)} = V \mid \vec{p}\,\rangle \;+\; V\; G_0(E+i\epsilon)V \mid \vec{p}\,\rangle^{(+)} \tag{1.38}$$

or

$$t \mid \vec{p}\,\rangle \;=\; V \mid \vec{p}\,\rangle \;+\; V\; G_0(E+i\epsilon)t \mid \vec{p}\,\rangle\;. \tag{1.39}$$

Since the operators in Eq. (1.39) are applied on a general state $\mid \vec{p}\,\rangle$, we can consider this equation as operator equation:

$$t \;=\; V + V\; G_0(E+i\epsilon)t\;. \tag{1.40}$$

This equation is also called operator Lippmann-Schwinger equation.

A next task is to derive from Eq. (1.36) a relation to the scattering wave function $\psi^+(\vec{r})$. Let us consider

$$\langle \vec{r}\mid \vec{p}\rangle^{(+)} \;=\; \langle \vec{r}\mid \vec{p}\,\rangle \;+\; \langle \vec{r}\mid G_0\; V \mid \vec{p}\rangle^{(+)} \tag{1.41}$$

which leads to

$$\begin{aligned} \psi^{(+)}(\vec{r}) \;\equiv\; \langle \vec{r}\mid \vec{p}\,\rangle^{(+)} &= \langle \vec{r}\mid \vec{p}\,\rangle \;+\; \langle \vec{r}\mid G_0\; t \mid \vec{p}\rangle & (1.42)\\ &= \langle \vec{r}\mid \vec{p}\rangle \;+\; \int d^3p'\; \langle \vec{r}\mid \vec{p}'\rangle\; \langle \vec{p}'\mid G_0\; t \mid \vec{p}\rangle\;. & (1.43) \end{aligned}$$

Applying the definition of G_0 leads to

$$\langle \vec{r}\mid \vec{p}\,\rangle^{(+)} \;=\; \langle \vec{r}\mid \vec{p}\,\rangle \;+\; \int d^3p'\; \langle \vec{r}\mid \vec{p}'\,\rangle\; \frac{1}{E+i\epsilon-\frac{p'^2}{m}}\; \langle \vec{p}'\mid t \mid \vec{p}\,\rangle\;, \tag{1.44}$$

which is the desired equation for the scattering wave function $\psi^{(+)}(r)$.

Energy conservation leads to an additional constraint for the t-operator. If a momentum before the scattering event is denoted with $\vec{p}$, and after the scattering event with $\vec{p}\,'$, then energy conservation requires

$$\frac{\vec{p}\,'^2}{m} \;=\; \frac{\vec{p}^{\,2}}{m} \;\Rightarrow\; \vec{p}\,'^2 \;=\; \vec{p}^{\,2}\;. \tag{1.45}$$

This means that we can extract an energy conserving δ-function from the matrix element

$$\langle \vec{p}\,' \mid T(E) \mid \vec{p}\,\rangle \;=\; \delta(E_{p'} - E_p)\;\langle \hat{p}' \mid t(E) \mid \hat{p}\rangle\;. \tag{1.46}$$

The latter relation is sometimes called on-shell condition. The physical meaning is that the observables of NN scattering only determine the matrix elements consistent with the relation (1.46).

1.3 The Lippmann-Schwinger Equation for the Bound State

Let us assume, that V supports a bound state $\mid \Psi_b\rangle$ at $E = E_b\langle 0$. Then

$$(H_0 + V) \mid \Psi_b\rangle = E_b \mid \Psi_b\rangle \tag{1.47}$$

or

$$(H_0 - E_b) \mid \Psi_b\rangle = -V \mid \Psi_b\rangle \tag{1.48}$$

Since $E_b\langle 0$ there is no regular and square integrable solution to the left hand side alone and $\mid \Psi_b\rangle$ obeys the homogeneous LSE

$$\mid \Psi_b\rangle \;=\; \frac{1}{E_b - H_0}\; V \mid \Psi_b\rangle \tag{1.49}$$

Using the configuration space representation Eq. (1.12) for $E = E_b\langle 0$ we see that (1.21) guarantees the correct exponential fall-off behavior of

$$\langle \vec{x}|\Psi_b\rangle \equiv \Psi_b(\vec{x}) = -m\sqrt{\frac{\pi}{2}} \int d^3x' \frac{e^{-\sqrt{m|E_b|}|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} V(x')\Psi_b(\vec{x}') \tag{1.50}$$

1.4 Connection Between Homogeneous and Inhomogeneous LSE's

It is of interest and importance to relate the homogeneous equation, valid at the discrete energy $E = E_b$

$$\mid \Psi_b\rangle \;=\; G_0(E_b)V \mid \Psi_b\rangle \tag{1.51}$$

and the inhomogeneous equation, derived for $E\rangle 0$

$$t(E) = V \;+\; VG_0(E)t(E)\;. \tag{1.52}$$

The transition operator $t(E)$ can be evaluated also for $E\langle 0$. What happens for $E \to E_b$? We rewrite (1.52)

$$(\;1 \;-\; VG_0(E)\;)\;t(E) \;=\; V \tag{1.53}$$

$$t(E) \;=\; (\;1 \;-\; VG_0(E)\;)^{-1}\;V \tag{1.54}$$

Let us expand

$$\begin{aligned} t(E) \;&=\; (\;1 + VG_0 \;+\; VG_0VG_0 \;+\; \ldots\;)\;V \\ &=\; V\;(\;1 + G_0V \;+\; G_0VG_0V \;+\; \ldots\;)\;. \end{aligned} \tag{1.55}$$

If we apply $t(E)$ onto $\mid \Psi_b\rangle$ and choose $E = E_b$, then we find, using Eq. (1.31)

$$t(E_b) \mid \Psi_b\rangle \;=\; V\;(\;1 \;+\; 1 \;+\; 1 \;+\; \ldots)\;\mid \Psi_b\rangle\;, \tag{1.56}$$

which is clearly diverging.

More precisely

$$\begin{aligned} t(E) \;&=\; [\;(\;G_0^{-1} \;-\; V\;)\;G_0\;]^{-1}V \\ &=\; G_0^{-1}\frac{1}{E - H_0 - V}\;V \\ &=\; G_0^{-1}\;\frac{1}{E - H}\;V \\ &=\; G_0^{-1}\;G^{-1}\;V \end{aligned} \tag{1.57}$$

Inserting the completeness relation

$$\mid \Psi_b\rangle\langle\Psi_b \mid \;+\; \int d^3p \;\mid \Psi_{\vec{p}}^{(+)}\rangle\langle\Psi_{\vec{p}}^{(+)} \mid = \mathbf{1} \tag{1.58}$$

to the left of V gives

$$\begin{aligned} t(E) &= (E - H_0) \mid \Psi_b\rangle \frac{1}{E - E_b} \langle \Psi_b \mid V \\ &+ \int d\vec{p}\, (E - H_0) \mid \Psi_{\vec{p}}^{(+)}\rangle \frac{1}{E - \vec{p}^{\,2}/m} \langle \Psi_{\vec{p}}^{(+)} \mid V \\ &= V \mid \Psi_b\rangle \frac{1}{E - E_b} \langle \Psi_b \mid V \\ &+ \int d\vec{p}\, V \mid \Psi_{\vec{p}}^{(+)}\rangle \frac{1}{E - \vec{p}^{\,2}/m} \langle \Psi_{\vec{p}}^{(+)} \mid V \ . \end{aligned} \tag{1.59}$$

We see explicitly that $t(E)$ has a pole at $E = E_b$

$$t(E) \ \rightarrow \ V \mid \Psi_b\rangle \frac{1}{E - E_b} \langle \Psi_b \mid V \qquad \text{for} \ \ E \rightarrow E_b \tag{1.60}$$

Thus $t(E)$ has a pole at the energy where the homogeneous LSE has a solution, which is the same as requiring that the homogeneous part of the inhomogeneous LSE has a solution:

$$\Theta(E) \ = \ V \, G_0(E) \, \Theta(E) \tag{1.61}$$

Put

$$\Theta(E) \ \equiv \ V \, \chi(E) \tag{1.62}$$

then

$$\chi(E) \ = \ G_0(E) \, V \, \chi(E) \tag{1.63}$$

This is identical to (1.38) and thus

$$\chi(E) \ = \ \Psi_b \qquad \text{at } E = E_b \tag{1.64}$$

This pole in $t(E)$ at the NN bound state will be of decisive importance for describing an interacting system of 3 or more nucleons.

1.5 Realization in a Partial Wave Representation in Momentum Space

We introduce the momentum space basis to a fixed orbital angular momentum l and magnetic quantum number m

$$\mid plm\rangle \tag{1.65}$$

These states are defined via

$$\langle\ \vec{p}\,'\ \ |\ p\ l\ m\rangle\ \equiv\ \frac{\delta(p\,'-p)}{p\ p'}\ Y_{lm}(\hat{p}\,')\ . \tag{1.66}$$

They are complete and orthonormal

$$\sum_{lm}\ \int dp\ p^2\ \ |\ p\ l\ m\rangle\langle p\ l\ m\ |\ =\ 1 \tag{1.67}$$

$$\langle p\ l\ m\ |\ p'\ l'\ m'\rangle\ =\ \frac{\delta(p\,'-p)}{p\ p'}\ \delta_{ll'}\ \delta_{mm'} \tag{1.68}$$

Let us consider the LSE for $t(E)$ in this basis

$$\begin{aligned}\langle p'l'm'|t(E)|plm\rangle\ \ =\ \ & \\ & \langle p'l'm'|V|plm\rangle + \sum_{l''m''}\int_0^\infty dp''\ p''^2\langle p'l'm'|V|p''l''m''\rangle \\ & \times\frac{1}{E+i\epsilon-p''^2/m}\langle p''l''m''|t(E)|plm\rangle\end{aligned} \tag{1.69}$$

We take V to be rotationally invariant:

$$\langle p'\ l'\ m'\ |\ V\ |\ p\ l\ m\rangle\ =\ \delta_{ll'}\ \delta_{mm'}\ V_l(p',p) \tag{1.70}$$

which leads to an integral equation in one variable:

$$t_l(p'p)\ =\ V_l(p'p)\ +\ \int_0^\infty dp''\ p''^{\,2}\ V_l(p'p'')\ \frac{1}{E\ +\ i\epsilon\ -\ p''^{\,2}/m}\ t_l(p''p) \tag{1.71}$$

What is V_l, assuming $V(r)$ to be given? Introduce states

$$|\ rlm\rangle \tag{1.72}$$

defined analogously to (1.59) via

$$\langle\vec{x}\ |\ r\ l\ m\rangle\ \equiv\ \frac{\delta(x-r)}{xr}\ Y_{lm}(\hat{x}) \tag{1.73}$$

Then

$$\begin{aligned}\langle\ \ & p\ l\ m\ |\ r\ l\ m\rangle = \int d\vec{p}\,'\ \int\ d\vec{x}\ \langle p\ l\ m\ |\ \vec{p}\,'\rangle\ \langle\vec{p}\,'\ |\ \vec{x}\rangle\langle\vec{x}\ |\ r\ l\ m\rangle \\ =\ & \int\ d^3p'\ \int d^3x\ \frac{\delta(p'-p)}{p'p}\ Y^*_{lm}(\hat{p}\,')\ \frac{1}{(2\pi)^{(3/2)}}\ e^{-i\vec{p}'\cdot\vec{x}}\ \frac{\delta(x-r)}{xr}Y_{lm}(\hat{x}) \\ =\ & \sqrt{\frac{2}{\pi}}\ j_l(pr)\ i^l\end{aligned} \tag{1.74}$$

Therefore, assuming a local potential:

$$\begin{aligned} V_l(p'p) &= \langle p'lm \mid V \mid plm\rangle \\ &= \int_0^\infty dr\ r^2 \int_0^\infty dr' r'^{\,2}\ \langle p'\ l\ m \mid r'\ l\ m\rangle \\ &\times \langle r'\ l\ m \mid\ V\ \mid r\ l\ m\rangle\langle\langle r\ l\ m \mid p\ l\ m\rangle \\ &= \frac{2}{\pi}\int_0^\infty dr\ r^2 \int_0^\infty dr'\ r'^{\,2}\ j_l(p'r')\ \frac{\delta(r-r')}{rr'}\ V(r)\ j_l(pr) \\ &= \frac{2}{\pi}\int_0^\infty dr\ r^2\ j_l(p'r)\ V(r)\ j_l(pr) \end{aligned} \tag{1.75}$$

This is one way to determine the momentum space representation of a local potential. The LSE for t_l can easily be solved by standard methods.
Let us now consider the full space for two nucleons including spin and isospin:

$$\begin{aligned} |p(ls)jm(\frac{1}{2}\frac{1}{2})tm_t\rangle \ \equiv\ & \\ & \sum_{m_l} C(lsj, m_l m-m_l)|plm_l\rangle|sm-m_l\rangle \\ & \times\sum_{\nu} C(\frac{1}{2}\frac{1}{2}t, \nu m_t-\nu)|\frac{1}{2}\nu\rangle|\frac{1}{2}|m_t-m_\nu\rangle \end{aligned} \tag{1.76}$$

Clearly one has $s=0,1$ and $t=0,1$. The antisymmetry (working in isospin formalism) leads to the well known restriction

$$(-)^{l+s+t} = -1 \tag{1.77}$$

for the allowed quantum numbers. Thus $t=1$ states are

$$^1S_0\ ,\ ^3P_0\ ,\ ^3P_1\ ,\ ^1D_2\ ,\ ^3P_2 -^3 F_2, \ldots \tag{1.78}$$

and $t=0$ states are

$$^1P_1\ ,\ ^3S_1 -^3 D_1\ ,\ ^3D_2\ , \ldots \tag{1.79}$$

The hyphen denotes coupled states, where l is not conserved. A well known mechanism for that is the tensor force.
For a general NN force, which conserves spin and parity one has

$$\langle p'(l's')j'm't'm'_t|V|p(ls)jmtm_t\rangle = \delta_{jj'}\delta_{mm'}\delta_{tt'}\delta_{m_t m_{t'}}\delta_{ss'}V_{ll'}^{sjtm_t}(p',p) \tag{1.80}$$

Because of (1.71) conservation of isospin follows and the indicated t-dependencies for V is redundant.
There is a dependence on m_t, the charge state of the two nucleons, in case of charge-independence breaking (CIB) or charge-symmetry breaking (CSB):

CIB means: $np \neq pp/_{strong}$ forces CSB means: $nn \neq pp/_{strong}$ forces

It is well established that in the state $^1S_0$ the np force is different from the nn or pp force. This is evident in the different scattering lengths:

$$a_{np} = -23.48 \pm 0.009 fm$$

$$a_{pp}/_{strong} = -17.36 \pm 0.4 fm \text{ (recommended value)}$$

(G.A. Miller et al, Phys. Rep. 194 (1990) 1)

$$a_{nn} = -18.6 \pm 0.3 fm$$

(extracted from $\pi^- + d \to n + n + \gamma$; B. Gabioud et al, Phys. Rev. Lett. 42 (1979) 1508; O. Schori et al, Phys. Rev. C35 (1987) 2252). That $\pi-$ absorption experiment has been redone at Los Alamos and is presently being analyzed.

In addition, nd breakup experiments are being presently performed (W. Tornow, TUNL), in order to extract a_{nn} using modern Faddeev calculations.

In $t = 1$ states different from $^1S_0$ CIB or CSB is not yet convincingly established, though small effects at least have to be there, simply because of the different pion masses.
We shall drop in the following the possible m_t-dependence in the notation.

In this most general basis, the LSE for t is represented as

$$\begin{aligned} \langle p'(l's')jt|t(E)|p(ls)jt\rangle &= \langle p'(l's)jt|V|p(ls)jt\rangle \\ &+ \sum_{l''} \int_0^\infty dp''^2 \; p''^2 \langle p' \; (l's)jt|V|p''(l''s)jt\rangle \\ &\times \frac{1}{E + i\epsilon - p''^2/m} \langle p''(l''s)jt|t(E)|p(ls)jt\rangle \end{aligned} \tag{1.81}$$

or

$$t^{sj}_{l'l}(p', p) = V^{sj}_{l'l}(p'p) + \sum_{l''} \int_0^\infty dp'' \; p''^2 V^{sj}_{l'l''}(pp'') \frac{1}{E + i\epsilon - p''^2/m} \; t^{sj}_{l''l}(p''p) \tag{1.82}$$

Since s is at most 1 and parity is conserved

$$l = l' \text{ or } l = l' \pm 2 \tag{1.83}$$

Thus one has either a single equation or two coupled equations. A prominent example for the coupled case is

$$^3S_1 -^3 D_1 \tag{1.84}$$

acting in the deuteron.

1.6 NN Phase-Shifts

The t-matrix generated by the coupled or uncoupled LSE is unitary. Let us choose a matrix notation

$$t \equiv t^{sj}_{l'l}(p'p) \qquad \text{etc.} \tag{1.85}$$

Then

$$t = V + V\,G_0\,t = V + t\,G_0\,V \tag{1.86}$$

The adjoint of that is

$$t^\dagger = V + V\,G_0{}^*\,t^\dagger \tag{1.87}$$

since $V^\dagger = V$. This is valid on physical grounds. Subtraction yields

$$\begin{aligned} t - t^\dagger &= VG_0t - VG_0^*t^\dagger \\ &= VG_0(t - t^\dagger) + V(G_0 - G_0^*)t^\dagger \\ &\quad (1 - VG_0)(t - t^\dagger) = V(G_0 - G_0^*)\,t^\dagger \\ t - t^\dagger &= (1 - VG_0)^{-1}V(G_0 - G_0^*\,)\,t^\dagger \\ &= t\,(\,G_0 - G_0^*\,)\,t^\dagger \end{aligned} \tag{1.88}$$

Now

$$G_0 = \frac{1}{E + i\epsilon - H_0}\,\mathbf{1} \tag{1.89}$$

thus

$$G_0 - G_0^* = -2\pi i\,\delta(E - H_0)\,\mathbf{1} \tag{1.90}$$

and we get, back in explicit notation

$$\begin{aligned} t_{l'l}(p'p) - t^*_{ll'}(pp') &= \int_0^\infty dp''\,p''^{\,2}\,\sum_{l''}\,t_{l'l''}(p'p'')(-2\pi i)\,\delta(E - \frac{p''^{\,2}}{m})\,t^*_{ll''}(pp'') \\ &= -2\pi i\,m\frac{\sqrt{mE}}{2}\,\sum_{l''}\,t_{l'l''}(p'\sqrt{mE})\,t^*_{ll''}(p\sqrt{mE}) \end{aligned} \tag{1.91}$$

Let us choose the on-the-energy shell values $p = p' = \sqrt{mE}$:

$$t_{l'l}(pp) - t^*_{ll'}(pp) = -\pi imp\,\sum_{l''}\,t_{l'l''}(pp)t^*_{ll''}(pp) \tag{1.92}$$

Back in matrix notation this is

$$t - t^\dagger = -\pi imp\;t\;t^\dagger \tag{1.93}$$

Now we introduce a S-matrix

$$S \;=\; 1 \;-\; i\pi mp\; t \tag{1.94}$$

and find

$$\begin{aligned} S\; S^\dagger \;&=\; (1 \;-\; i\pi mp\; t)\; (1 \;+\; i\pi mp\; t^\dagger) \\ &=\; 1 \;-\; i\pi mp\; (t \;-\; t^\dagger \;-\; i\pi mp\; t\; t^\dagger) \;=\; 1 \end{aligned} \tag{1.95}$$

Thus S is unitary and can be parameterized in the coupled case by 3 parameters:

$$S \;=\; \begin{pmatrix} cos2\overline{\epsilon}\; e^{2i\overline{\delta_1}} & isin2\overline{\epsilon}\; e^{i(\overline{\delta_1}+\overline{\delta_2})} \\ isin2\overline{\epsilon}\; e^{i(\overline{\delta_1}+\overline{\delta_2})} & cos2\overline{\epsilon}\; e^{2i\overline{\delta_2}} \end{pmatrix} \tag{1.96}$$

which is the "Stapp" or "bar"-phase shift parameterization
(H.P. Stapp et al, Phys. Rev. 105 (1957) 302).
In the uncoupled case S is simply

$$S \;=\; e^{2i\delta} \tag{1.97}$$

with δ real.

The most recent NN phase-shift parameters by the Nijmegen group
(V.G.J. Stoks et al, Phys. Rev C48 (1993) 792)
can be viewed on-line at

http://nn-online.sci.kun.nl/

and by the Arndt group
(R. A. Arndt et al, Phys. Rev. D45 (1992) 3995)
Their on-line facility is called *SAID* and can be accessed via

telnet *said.phys.vt.edu*

using the login *said* . You need an xterm if you want to do the graphics.

1.7 Deuteron Properties

The homogeneous LSE Eq. (1.51) is now projected onto the basis given in Eq. (1.73). Thus for

$$\Psi_l(p) \;\equiv\; \langle p\; (ls)\; j\; t \mid \Psi_b \rangle \tag{1.98}$$

with $l = 0, 2$, $s = j = 1$, $t = 0$ one gets the set of two coupled equations

$$\Psi_l(p) \;=\; \frac{1}{E_b - \frac{p^2}{m}} \sum_{l'=0,2} \int_0^\infty dp' \; p'^{\,2} \; V_{ll'}(pp') \; \Psi_{l'}(p') \tag{1.99}$$

This can be solved numerically by standard techniques. Realistic forces are adjusted to reproduce various measurable quantities:

- $E_b = -2.2246$ MeV
- $Q = 0.2859 \; fm^2$ (there are theoretical uncertainties in the description of that experimental value caused by MEC)
- $A_s = 0.8883$ fm$^{-1/2}$ (asymptotic normalization constant for the s-wave component)
- $\eta = A_D/A_s = 0.02564$ (asymptotic d/s ratio)

The deuteron d-state probability

$$p_d \;\equiv\; \frac{\int_0^\infty \; \Psi_2^2(p) \; p^2 dp}{\int_0^\infty \; \Psi_0^2(p) \; p^2 \; dp \; + \; \int_0^\infty \; \Psi_2^2(p) \; p^2 \; dp} \tag{1.100}$$

is not a measurable quantity, but strongly correlated to nuclear binding energies, as we shall see later. In general, the smaller p_d the larger the triton and α-particle binding energies.
Let us now consider the single nucleon momentum distribution

$$n(k) \;\equiv\; \frac{1}{2} \, \frac{1}{3} \sum_m \langle \Psi_b \; m \mid \sum_{i=1}^{2} \; \delta(\vec{k} - \vec{k}_i^{cm}) \mid \Psi_b \; m \rangle \tag{1.101}$$

$$= \; \frac{1}{3} \sum_m \langle \Psi_b \; m \mid \delta(\vec{k} - \vec{k}_1^{cm}) \mid \Psi_b \; m \rangle \tag{1.102}$$

We have

$$\vec{p} \;=\; \frac{1}{2} \; (\vec{k}_1^{cm} \; - \; \vec{k}_2^{cm}) \;=\; \vec{k}_1^{cm} \tag{1.103}$$

and thus

$$n(k) = \frac{1}{3} \sum_m \int \; d^3p \; \langle \Psi_b \; m \mid \vec{p} \rangle \; \delta(\vec{k} - \vec{p}) \; \langle \vec{p} \mid \Psi_b m \rangle. \tag{1.104}$$

One has

$$\begin{aligned} \langle \vec{k} | \Psi_b m \rangle \;&=\; \sum_l \int_0^\infty dp \; p^2 \langle \vec{k} | p(ls) j m \rangle \Psi_l(p) \\ &=\; \sum_l \sum_{m_l} C(lsj, m_l, m - m_l) Y_{lm_l}(\hat{k}) | sm - m_l \rangle \Psi_l(k) \end{aligned} \tag{1.105}$$

and therefore

$$\begin{aligned} n(k) &= \frac{1}{3} \sum_{m} \sum_{ll'} \sum_{m_l} C(l' \ s \ j, m_l \ m - m_l) \\ &\times \ C(l \ s \ j, m_l \ m - m_l) \ Y^*_{l'm_l}(\hat{k}) \ Y_{lm_l}(\hat{k}) \\ &\times \ \Psi_{l'}(k) \ \Psi_l(k) \end{aligned} \tag{1.106}$$

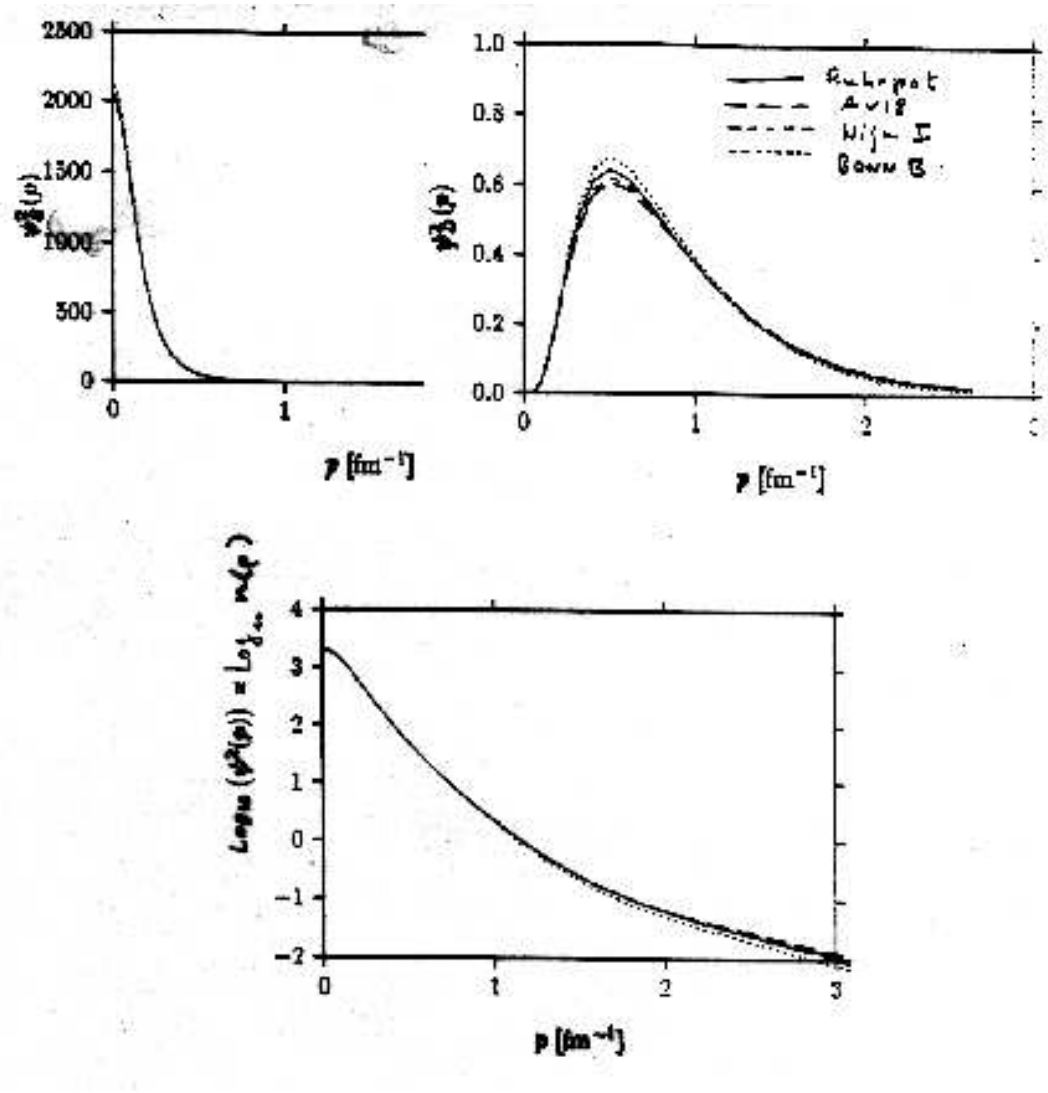
Now, with $\hat{a}$ being defined as $\hat{a} \equiv 2a + 1$ we have

$$C(l \ s \ j, m_l \ m - m_l) \ = \ (-)^{s+m-m_l} \ \sqrt{\frac{\hat{j}}{\hat{l}}} \ C(j \ s \ l \ , -m, m - m_l) \tag{1.107}$$

Using the above relation we find

$$\begin{aligned} n(k) &= \frac{1}{3} \sum_{m_l} \sum_{ll'} Y^*_{l'm_l}(\hat{k}) \ Y_{lm_l}(\hat{k}) \qquad (1.108) \\ &\times \ \Psi_{l'}(k) \ \Psi_l(k) \ \sum_{m} \ \sqrt{\frac{\hat{j}}{\hat{l}}} \ \sqrt{\frac{\hat{j}}{\hat{l}'}} \\ &\times \ C(j \ s \ l, -m, m - m_l) \ C(j \ s \ l', -m, m - m_l) \\ &= \frac{1}{3} \sum_{m_l} \sum_{l} Y^*_{lm_l}(\hat{k}) \ Y_{lm_l}(\hat{k}) \ \frac{\hat{j}}{\hat{l}} \ \Psi^2_l(k) \qquad (1.109) \\ &= \frac{\hat{j}}{3} \ \frac{1}{4\pi} \ \sum_{l} \ \Psi^2_l(k) \ = \ \frac{1}{4\pi} \sum_{l=0,2} \Psi^2_l(k) \qquad (1.110) \end{aligned}$$

This is displayed for several realistic NN forces in the next figure, where different short range behavior of NN forces is reflected for $k \gtrsim 1 fm^{-1}$.



There is hope to measure these quantities in electron scattering on deuterons.

Of interest is also the NN correlation function, the probability to find 2 nucleons at a distance r:

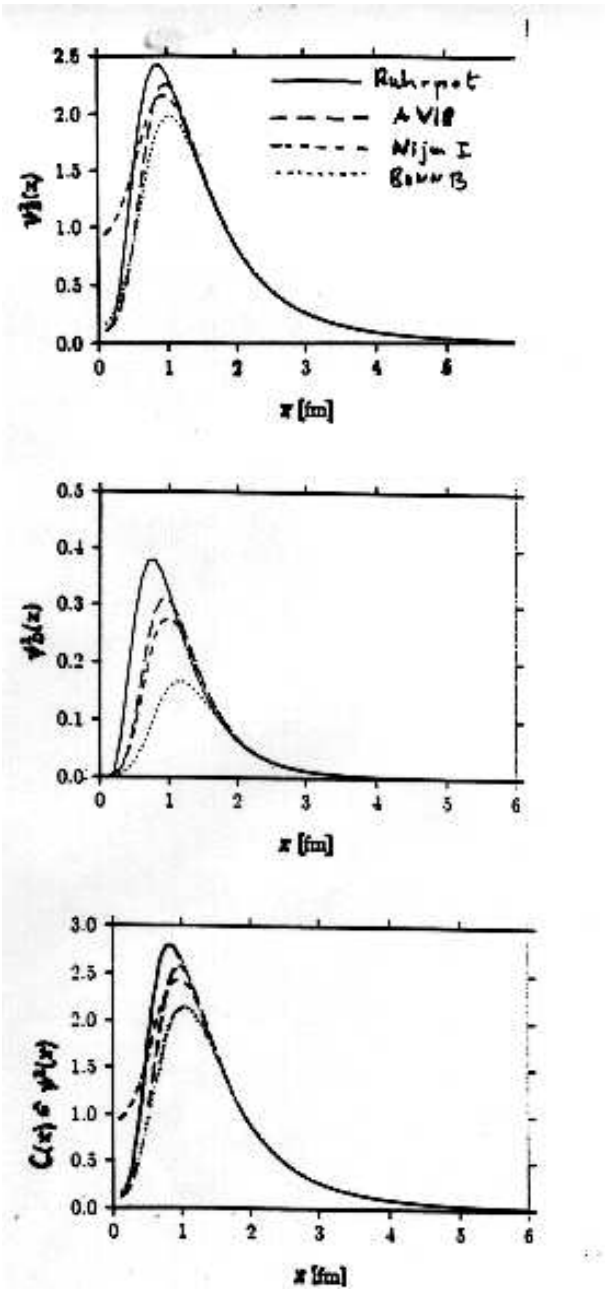
$$\begin{aligned} C(r) &\equiv \frac{1}{3}\sum_m \langle \Psi_b m|\delta(\vec{r}-\vec{x})|\Psi_b\ m\rangle \\ &= \frac{1}{3}\sum_m \langle \Psi_b m|\vec{r}\rangle\langle\vec{r}|\Psi_b\ m\rangle \end{aligned}$$

$$= \frac{1}{4\pi} \sum_{l=0,2} \Psi_l^2(r) \tag{1.111}$$

The connection between configuration and momentum space is given by

$$\Psi_l(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty dp\ p^2 j_l(pr) \Psi_l(p) \tag{1.112}$$

The $l = 0$ and 2 parts of $C(r)$ together with their sum are displayed below. We see differences at short distances, depending on the strengths of the short range repulsions, as shown in the next figure


Ruhrpot
AV18
Nijm I
Bonn B
x [fm]
x [fm]
x [fm]

1.8 Nuclear Forces

The determination of the nuclear force is a longstanding and still unsolved basic problem. The whole issue on how to set up a framework for deriving a nuclear force is not touched here. Simply a list of so called "realistic" NN forces is given:

- Paris potential (dispersion theoretical background) by M. Lacombe et al, Phys. Rev. C21 (1980) 861
- Nijmegen 78 potential (one-boson-exchange background) by M.M. Nagels et al, Phys. Rev. D17 (1978) 768
- AV14 potential (one-pion tail, otherwise phenomenological) by R. B. Wiringa et al, Phys. Rev. C29 (1984) 1207
- Bonn potential (meson exchange potential (multiple meson), based on time-ordered perturbation theory by R. Machleidt, K. Holinde, Ch. Elster, Phys. Rep. 149 (1987) 1 and R. Machleidt, Adv. Nucl. Phys. 19 (1989) 189

All those potential have $\chi^2/N_{data} \geq 2$ with respect to the Nijmegen data base.

Most recent NN potential, however **all** phenomenological with about 30-50 parameters fit the Nijmegen data base with a $\chi^2/N_{data} \sim 1$ and are

- Nijmegen I (includes ∇^2-term)
- Nijmegen II (local)
- Reid 93 (local) by V.G.J. Stoks et al, Phys. Rev. C49 (1994) 2950
- AV18 (updated AV14, local, as operators defined) by R.B. Wiringa et al, Phys. Rev. C51 (1995) 38
- CD-Bonn (nonlocal) by R. Machleidt, F. Sammarruca, Y. Song, Phys. Rev. C53, (1996), R1483.

They come in charge-dependent versions and describe the NN data up to 350 MeV perfectly well with $\chi^2/N_{data} \sim 1$

This is the first time that one has a set of "realistic" nearly phase-equivalent NN forces. They cover a certain range of properties, a NN force can have:

- local versus nonlocal
- soft or hard core

What is still missing in that family are potentials with dynamical nonlocalities at very short distances $r \leq 0.8fm$, say, resulting from the overlap regions of extended nucleons. They might be good for surprises.

1.9 Construction of the NN Potential From Invariance Requirements

We want to investigate to what extent the form of the potential $V_{NN}(1,2)$ acting between two nucleons is determined by the requirement that the Hamiltonian describing the system be invariant under various symmetry transformations. This analysis will be made considering the two nucleons as identical particles, i.e., disregarding the difference of the mass and charge between the neutron and proton. The Hamiltonian has then the form

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + V_{NN}(1,2) , \tag{1.113}$$

m being the nucleon mass.

Regarding the symmetry properties of H, we shall assume first of all invariance under the restricted Galilei group. Then we shall assume invariance under the discrete transformations of space reflection, time invariance and permutation of the two nucleons. Finally, we shall assume invariance under the isospin transformations of the group $SM(2)$.

The operators we have at our disposal to build up the potential are the coordinates $\vec{r}_1, \vec{r}_2$, the momenta $\vec{p}_1, \vec{p}_2$, the spin vector operators $\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}$ and the isospin vector operators $\vec{\tau}^{(1)}, \vec{\tau}^{(2)}$. Going to the two nucleon c.m. frame gives

$$\begin{aligned} \vec{r} &= \vec{r}_2 - \vec{r}_1 \\ \vec{R} &= \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \\ \vec{p} &= \vec{p}_2 - \vec{p}_1 \\ \vec{P} &= \vec{p}_2 + \vec{p}_1 . \end{aligned} \tag{1.114}$$

1. Assume that the potential operator is hermitian

$$V_{NN} = V_{NN}^{\dagger} . \tag{1.115}$$

2. Using time-translation invariance, which makes V_{NN} not explicitly dependent on the time t, gives

$$V_{NN} \equiv V_{NN}\ (\vec{r}, \vec{R}, \vec{p}, \vec{P}, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}, \vec{\tau}^{(1)}, \vec{\tau}^{(2)})\ . \tag{1.116}$$

Let us now discuss the implications of the assumed invariance on the dependencies of V_{NN} on the indicated variables.

3. Consider the invariance for rotations in charge space, which determines the dependence of V_{NN} on the isospin vectors $\vec{\tau}^{(1)}$ and $\vec{\tau}^{(2)}$. The unitary operator representing a rotation in charge space is given by

$$U_I(w) = e^{i\vec{I}\cdot\vec{w}}\ , \tag{1.117}$$

where $\vec{I}$ is the total isospin and $\vec{w} = \vec{n}\ w$. The required invariance is expressed by

$$U_I^\dagger\ V_{NN}\ U_I = V_{NN} \tag{1.118}$$

with arbitrary $\vec{n}$ and w. (1.118) will be satisfied if V_{NN} is a scalar in isospin space. In order to construct all possible scalars from $\vec{\tau}^{(1)}$ and $\vec{\tau}^{(2)}$, it is remarked that any polynomial expression in $\vec{\tau}^{(i)}$ can be reduced to a linear expression by using

$$\begin{aligned} [\tau_j^{(i)}, \tau_k^{(i)}] &= i\varepsilon_{jk\ell}\ \tau_\ell^{(i)} \\ (\tau_j^{(i)})^2 &= 1\ , \end{aligned} \tag{1.119}$$

so that, e.g.,

$$(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})^2 = 3 - 2(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})\ . \tag{1.120}$$

Hence, the most general expression that has to be considered is linear, both in $\vec{\tau}^{(1)}$ and $\vec{\tau}^{(2)}$. The only scalar quantity obtained in this way is $\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}$. It follows that V_{NN} is a function only of this quantity as regards its dependence on the isospin variables of the two particles:

$$V_{NN} \equiv V_{NN}(\vec{r}, \vec{R}, \vec{p}, \vec{P}, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}, [\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}])\ . \tag{1.121}$$

Expanding V_{NN} in a power series of $\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}$ and expressing $(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})^n$ with (1.120) in terms of $\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}$ and the identity operator in isospin space, one obtains

$$V_{NN} = V_1(\vec{r}, \vec{R}, \vec{p}, \vec{P}, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}) + (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})\ V_2(\vec{r}, \vec{R}, \vec{p}, \vec{P}, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)})\ . \tag{1.122}$$

We can now limit ourselves to study the implications of invariance an each term V_i separately, since all other symmetry transformations commute with the isospin operators. For convenience we drop the index i from now on.

4. Invariance under space translations is expressed by

$$U_a^\dagger \; V \; U_a \;=\; V \tag{1.123}$$

with $U_a \;=\; exp\;\left(\frac{i}{\hbar}\;\vec{P}\;\cdot\vec{a}\right)$. We get

$$\begin{aligned} U_a^\dagger \quad V \quad & (\vec{r},\vec{R},\vec{p},\vec{P},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)})\;U_a \\ &= V(U_a^\dagger\;\vec{r}\,U_a, U_a^\dagger\;\vec{R}\,U_a,\; U_a^\dagger\;\vec{p}\,U_a,\; U_a^\dagger\;\vec{P}\;U_a,\; U_a^\dagger\;\vec{\sigma}^{(1)}\;U_a,\; U_a^\dagger\;\vec{\sigma}^{(2)}\;U_a) \\ &= V(\vec{r},\vec{R}-\vec{a},\vec{p},\vec{P},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)})\;. \end{aligned} \tag{1.124}$$

The condition (1.123) then implies that V does not depend on $\vec{R}$:

$$V(\vec{r},\vec{p},\vec{P},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)})\;. \tag{1.125}$$

5. Invariance under proper Galilei transformations is expressed by

$$U_G^\dagger \; V \; U_G \;=\; V \tag{1.126}$$

with

$$U_G \;\equiv\; exp\;\left(\frac{i}{\hbar}\;\vec{P}\;\cdot\vec{v}_0 t\right)\;exp\;\left(-\;\frac{i}{\hbar}\;m\vec{R}\;\cdot\vec{v}_0\right) \tag{1.127}$$

with $\vec{v}_0$ being the c.m. velocity. It follows that

$$\begin{aligned} U_G^\dagger\;V\;U_G \;&=\; V(\vec{r},\vec{p},U_G^\dagger\;\vec{P}\;U_G,\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \\ &=\; V(\vec{r},\vec{p},\vec{P}-\vec{v}_0 m,\vec{\sigma}^{(1)},\vec{\sigma}^{(2)})\;. \end{aligned} \tag{1.128}$$

The condition (1.126) then implies that V is independent of $\vec{P}$:

$$V \;=\; V(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)})\;. \tag{1.129}$$

6. Invariance under space reflections implies in the normal way that

$$V(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \;=\; V(-\vec{r},-\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)})\;. \tag{1.130}$$

7. Invariance under the permutation of the two nucleons gives

$$V(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \;=\; V(-\vec{r},-\vec{p},\vec{\sigma}^{(2)},\vec{\sigma}^{(1)})\;. \tag{1.131}$$

Invariance under the combined transformations (6) and (7) gives

$$V(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \;=\; V(\vec{r},\vec{p},\vec{\sigma}^{(2)},\vec{\sigma}^{(1)})\;. \tag{1.132}$$

8. Invariance under time reversal means

$$\begin{aligned} V(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) &= V^*(\vec{r},-\vec{p},-\vec{\sigma}^{(1)},-\vec{\sigma}^{(2)}) \\ &= V(\vec{r},-\vec{p},-\vec{\sigma}^{(1)},-\vec{\sigma}^{(2)}) \end{aligned} \tag{1.133}$$

   Since V is assumed to be hermitian.

9. Invariance under spatial rotations is expressed by

$$U_R^\dagger \; V \; U_R \;=\; V \tag{1.134}$$

   with $U_R \;=\; exp\;\left(\frac{i}{\hbar}\;\vec{J}\;\cdot\vec{n}\;w\right)$, with $\vec{J}$ being the total angular momentum of the system, $\vec{J}=\vec{L}+\vec{S}$. Requiring rotational invariance means that

$$V(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \;=\; V(R\vec{r},R\vec{p},R\vec{\sigma}^{(1)},R\vec{\sigma}^{2)}) \;, \tag{1.135}$$

   where $R\vec{a}$ gives the rotated of the vector $\vec{a}$.

Let us first take into account the dependence of V on the spin variables. Here the procedure is not so straightforward as it was for the isospin, since spin, position and momentum vectors can be combined to build rotational invariant quantities. Using spin identifies similar to (1.120), one can show that V can be expressed as

$$V \;=\; V_\alpha \;+\; \vec{\sigma}^{(1)}\vec{V}_\beta^{(1)} \;+\; \vec{\sigma}^{(2)}\vec{V}_\beta^{(2)} \;+\; V_\gamma(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \;. \tag{1.136}$$

V_γ is linear in both $\vec{\sigma}^{(1)}$ and $\vec{\sigma}^{(2)}$ but contains only bilinear combinations of these two operators. From rotation and space-reflection invariance, V_α and V_γ must be scalars, $\vec{V}_\beta^{(1)}$ and $\vec{V}_\beta^{(2)}$ pseudovectors. Combination of space reflection and particle exchange [(6) and (7)] implies that

$$\begin{aligned} V_\alpha \;&+\; \vec{\sigma}^{(1)}\cdot\vec{V}_\beta^{(1)} \;+\; \vec{\sigma}^{(2)}\cdot\vec{V}_\beta^{(2)} \;+\; V_\gamma(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \\ &= V_\alpha \;+\; \vec{\sigma}^{(2)}\cdot\vec{V}_\beta^{(1)} \;+\; \vec{\sigma}^{(1)}\cdot\vec{V}_\beta^{(2)} \;+\; V_\gamma(\vec{r},\vec{p},\vec{\sigma}^{(2)},\vec{\sigma}^{(1)}) \;. \end{aligned} \tag{1.137}$$

Taking the average of these two expressions for V, one gets

$$V \;=\; V_\alpha + \vec{S}\;\cdot\vec{V}_\beta \;+\; V_\gamma(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) \tag{1.138}$$

where $\vec{S} \;=\; \frac{\hbar}{2}(\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)})$, $\vec{V}_\beta \;=\; \frac{1}{\hbar}(\vec{V}_\beta^{(1)},\vec{\sigma}^{(2)})$, and V_γ now being symmetric under the exchange of the spin operators. The vector we can use to construct $\vec{V}_\beta$ are $\vec{r},\vec{p}$ and

$\vec{L} = \vec{r} \times \vec{p}$, but only $\vec{L}$ is a pseudovector. Thus, $\vec{V}_\beta$ must then be $\vec{L}\times$ (scalar quantity). Then (1.138) reads

$$V = V_\alpha(\vec{r},\vec{p}) + \vec{S} \cdot \vec{L}\, V_\beta(\vec{r},\vec{p}) + V_\gamma(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) . \tag{1.139}$$

Since V_α and V_β are scalars, they can only be functions of $r^2, p^2, L^2, \vec{r}\cdot\vec{p}$ and $\vec{p}\cdot\vec{r}$. Since the operators $\vec{r}\cdot\vec{p}$ and $\vec{p}\cdot\vec{r}$ are non-hermitian, it is convenient to consider their hermitian combinations $(\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r})$ and $i(\vec{r}\cdot\vec{p}-\vec{p}\cdot\vec{r})$. The latter is a constant and can be dropped. The former can only appear quadratically in V_α and V_β due to time-reversal invariance. With

$$(\vec{p}\cdot\vec{r}+\vec{r}\cdot\vec{p})^2 = 2(r^2p^2+p^2r^2) - 4L^2 + 3\hbar^2 , \tag{1.140}$$

we get

$$V = V_\alpha(r^2,p^2,L^2) + \vec{S}\cdot\vec{L}\, V_\beta(r^2,p^2,L^2) + V_\gamma(\vec{r},\vec{p},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}) . \tag{1.141}$$

From the requirements on V_γ follows that it can only contains terms of the type

$$\begin{aligned} &\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}, (\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r}), (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}), \\ &(\vec{\sigma}^{(1)}\cdot\vec{L})(\vec{\sigma}^{(2)}\cdot\vec{L}) + (\vec{\sigma}^{(2)}\cdot\vec{L})(\vec{\sigma}^{(1)}\cdot\vec{L}), \\ &(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{r}) + (\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p}) + \mathbf{1} \leftrightarrow 2 . \end{aligned} \tag{1.142}$$

The last expression changes sign under time reversal and must be replaced by

$$[(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\vec{r}) + (\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p}) + 1 \leftrightarrow 2](\vec{p}\cdot\vec{r}+\vec{r}\cdot\vec{p}) . \tag{1.143}$$

It can be shown that (1.142) is de facto a function of the other quantities appearing in (1.141) and thus not independent. We have, therefore, for V_γ

$$\begin{aligned} V_\gamma &= (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, V_\gamma^{(I)}(r^2,p^2,L^2) \\ &+ (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, V_\gamma^{(II)}(r^2,p^2,L^2) \\ &+ (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\, V_\gamma^{(III)}(r^2,p^2,L^2) \\ &+ [(\vec{\sigma}^{(1)}\cdot\vec{L})(\vec{\sigma}^{(2)}\cdot\vec{L}) + (\vec{\sigma}^{(2)}\cdot\vec{L})(\vec{\sigma}^{(1)}\cdot\vec{L})]\, V_\gamma^{(IV)}(r^2,p^2,L^2) \end{aligned} \tag{1.144}$$

as most general form of V_γ compliant with all symmetry requirements.

Concluding, the most general, velocity-dependent, non-relativistic NN potential has the form (1.122) with V_i given by

$$\begin{aligned} V_i &= V_i^c(r^2,p^2,L^2) + \vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}\, V_i^\sigma(r^2,p^2,L^2) \\ &+ S_{12}\, V_i^T(r^2,p^2,L^2) + \vec{S}\cdot\vec{L}\, V_i^{LS}(r^2,p^2,L^2) \\ &+ [(\vec{\sigma}^{(1)}\cdot\vec{L})(\vec{\sigma}^{(2)}\cdot\vec{L}) + (\vec{\sigma}^{(2)}\cdot\vec{L})(\vec{\sigma}^{(1)}\cdot\vec{L})]\, V_i^{\sigma L}(r^2,p^2,L^2) , \end{aligned} \tag{1.145}$$

where S_{12} is the tensor force operator

$$S_{12} \;=\; \frac{3}{r^2}\;(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r}) \;-\; (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\;. \tag{1.146}$$

Remark: As shown in the Appendix of S. Okubo and R.E. Marshak, *Ann. Phys.* **4**, 166 (1958), the potential of (1.145) gives an S-matrix that on-shell is identical to the one obtained from a potential in which the term $V^{\sigma p}$ is dropped. Therefore, if one is only interested in NN scattering, it can be neglected. The same cannot be said for the bound states or for the off-shell S-matrix. Often the term $V^{\sigma L}$, is also neglected. Some arguments are given in Machleidt, Holinde, Elster, *Phys. Rep.* **149**, 1 (1987).

1.10 Simple Introduction to One Boson-Exchange Potential (OBEP)

The basic idea of OBE models is to represent the NN interaction as superposition of tree-diagrams (born terms) which represent the exchange of single mesons, namely scalar (s), pseudoscalar (ps), vector (v) bosons ($J^p \;=\; 0^+, 0^-, 1^-$, respectively), with masses up to 1 GeV between two nucleons. Mesons with masses larger than 1 GeV would only give very short-ranged exchange contributions and contribute in a region where the OBE model is no longer valid.

The couplings for the various mesons are given in terms of their interaction Lagrangian densities by

$$\mathcal{L}_{NN_{ps}} \;=\; g_{ps}\;\bar{\psi}\;i\gamma_5\;\psi\phi_{ps} \tag{1.147}$$

$$\mathcal{L}_{NN_{s}} \;=\; g_{s}\;\bar{\psi}\;\psi\;\phi_{s} \tag{1.148}$$

$$\mathcal{L}_{NN_{v}} \;=\; g_{v}\;\bar{\psi}\;\gamma_\mu\;\psi\;\phi_v^\mu \;+\; \frac{f_v}{4m}\;\bar{\psi}\;\sigma_{\mu v}\;\psi(\partial^\mu\phi_v^v - \partial^v\phi_v^\mu) \tag{1.149}$$

for pseudoscalar (π,η), scalar (σ,δ) and vector mesons (ρ,ω), respectively. m is the nuclear mass, ψ the nucleon and ϕ_α the meson field operators. For isospin $T=1$ means ϕ_α is to be replaced by $\vec{\tau}\cdot\vec{\phi}_\alpha$, with τ_i being the usual Pauli matrices. Furthermore, $\sigma_{\mu\nu} \;=\; \frac{i}{2}\;[\gamma_\mu,\gamma_\nu]$, where γ_μ are the usual Dirac-matrices (see, e.g., Bjorken-Drell). The coupling constants $g_\alpha(\alpha = s, ps, v)$ and f_v and the meson masses m_α are at least partially determined from high-energy experiments or symmetry relations. The Lagrangian density for vector mesons contains Dirac (g_v) as well as Pauli coupling (f_v). An OBE-potential

$V(\vec{q}\,',\vec{q})$ is obtained through the superposition of exchange contributions of the different mesons

$$V(\vec{q}\,',\vec{q}) \;=\; \sum_{\alpha=s,ps,v} V_\alpha(\vec{q}\,',\vec{q}) \tag{1.150}$$

with

$$V_\alpha(\vec{q}\,',\vec{q}) \;=\; \sqrt{\frac{m}{Eq'}}\,\sqrt{\frac{m}{Eq}}\;\bar{u}\;(-\vec{q}\,')\;\Gamma^{(2)}_\alpha\;u(-\vec{q})\;\mathbf{P}_\alpha\;\bar{u}(\vec{q}\,')\;\Gamma^{(1)}_\alpha\;u(\vec{q})\;. \tag{1.151}$$

The factors $\sqrt{\frac{m}{Eq'}}\,\sqrt{\frac{m}{Eq}}$ are the so-called minimal relativity factors, which take into consideration the relativistic unitarity condition (see K. Erkelenz, *Phys. Rep.* **13C**, 191 (1974)). They certainly contribute to the nonlocality of $V(\vec{q}\,',q)$. (Their effect has been studied in a simple model in Ch. Elster, E.E. Evans, H. Kamada, W. Glöckle, *Few-Body Systems* **21**, 25 (1996).

The meson propagators are usually given by

$$\mathbf{P}_\alpha \;=\; ((\vec{q}\,'-\vec{q}\,)^2+m_\alpha^2)^{-1} \tag{1.152}$$

and the vertex functions for the meson-nucleon vertices $\Gamma^{(i)}_\alpha (i=1,2)$ are given by

$$\Gamma^{(i)}_s \;=\; g_s \tag{1.153}$$

$$\Gamma^{(i)}_{ps} \;=\; g_{ps}\;i\;\gamma^5 \tag{1.154}$$

$$\Gamma^{(i)}_v(direct) \;=\; (g_v+f_v)\gamma^\mu \tag{1.155}$$

$$\Gamma^{(i)}_v(gradient) \;=\; -\;\frac{f_v}{2m}\;(\vec{q}\,'+\vec{q}\,)^\mu\;. \tag{1.156}$$

In order to take into account the finite extension of the nucleon and to be able to solve the dynamical equations, the coupling constants get modified with form factors. This is essentially achieved by replacing

$$g_\alpha \;\longrightarrow\; g_\alpha F_\alpha(\vec{q}\,',\vec{q}) \tag{1.157}$$

where $F_\alpha(\vec{q}\,',\vec{q})$ can be, e.g., of dipole type

$$F_\alpha[(\vec{q}\,',\vec{q}\,)^2] \;=\; \left(\frac{\Lambda_\alpha^2-m_\alpha^2}{\Lambda_\alpha^2+(\vec{q}\,'-\vec{q}\,)^2}\right)^{n_\alpha}\;. \tag{1.158}$$

The exponent n_α is usually taken as $n_\alpha = 1, \Lambda_\alpha$ is the cutoff parameter and usually of the order $1 - 2$ GeV. The positive energy Dirac spinors are given by

$$u^{(i)}(\vec{q}\,) \;=\; \sqrt{\frac{E_q + m}{2m}} \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{q}}{E_q+m} \end{pmatrix} \mid i\rangle \tag{1.159}$$

where $\mid i\rangle$ denote the Pauli spinors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Inserting (1.159), (1.153), (1.152) into (1.151) gives for the scalar contribution of the potential (Pauli spinors are omitted):

$$\begin{aligned} V_s(\vec{q}\,', \vec{q}\,) \;&=\; -g_s^2 \sqrt{\frac{m}{E_q'}} \sqrt{\frac{m}{E_q}} \frac{(E_q' + m)(E_q + m)}{4m^2} \frac{1}{(\vec{q}\,' - \vec{q}\,)^2 + m_s^2} \\ &\times\; \left(1 - \frac{\vec{q}\,' \cdot \vec{q} + i\vec{\sigma}_2 \cdot (\vec{q}\,' \times \vec{q}\,)}{(E_q' + m)(E_q + m)}\right) \left(1 - \frac{\vec{q}\,' \cdot \vec{q} + i\vec{\sigma}_1 \cdot (\vec{q}\,' \times \vec{q}\,)}{(E_q' + m)(E_q + m)}\right) \end{aligned} \tag{1.160}$$

This expression has, due to the $\vec{q}$ and E_q dependencies, a strong nonlocality. In order to arrive at expressions, which can be transformed to coordinate space, one changes variables to

$$\begin{aligned} \vec{k} \;&=\; \vec{q}\,' - \vec{q} \\ \vec{p} \;&=\; \frac{1}{2}\,(\vec{q}\,' + \vec{q}\,) \end{aligned} \tag{1.161}$$

and in addition has to introduce the following approximations:

1. On-shell approximation: $\quad E_q' \;=\; E_q$

2. Expansion of E in powers of $\frac{q^2}{m^2}$:

$$\begin{aligned} E \;=\; \left(\frac{1}{2}\,(\vec{q}\,' + \vec{q}\,)^2 + m^2\right)^{\frac{1}{2}} \;&=\; m + \frac{1}{4m}\,(q'^2 + q^2) \;+\; \cdots \\ &=\; m + \frac{p^2}{2m} \;+\; \frac{k^2}{8m} \;+\; \cdots \end{aligned} \tag{1.162}$$

3. Keeping only the lowest order in p^2 and k^2.

With these approximations, the scalar potential becomes

$$V_s^c(\vec{k},\vec{p}) \;=\; -\,\frac{g_s^2}{k^2+m_s^2}\;\left[1-\;\frac{p^2}{2m^2}\;+\;\frac{k^2}{8m^2}\;-\;\frac{i}{2m^2}\;\vec{S}\cdot(\vec{k}\times\vec{p})\right] \tag{1.163}$$

where $\vec{S} \;=\; \frac{1}{2}\;(\vec{\sigma}_1+\vec{\sigma}_2)$.

This expression still contains nonlocalities due to $\vec{p}^{\,2}$ as well as $(\vec{k}\;\times\vec{p})$ terms. The latter leads to the angular momentum operator $\vec{L} \;=\; -i\vec{r}\;\times\vec{\nabla}$ in r-space, whereas the former provides ∇^2 terms. After a Fourier transform, the coordinate space expression of the scalar potential is given by

$$\begin{aligned} V_s^c(r) \;&=\; -\,\frac{g_s^2}{4\pi}\;m_s\;\left\{\left[1-\;\frac{1}{4}\;\left(\frac{ms}{m}\right)^2\right]\;\mathcal{Y}(m_s r)\right. \\ &+\;\left.\frac{1}{4m^2}\;\left[\nabla^2\;\mathcal{Y}(m_s r)\;+\;\mathcal{Y}(m_s r)\;\nabla^2\right]\;+\;\frac{1}{2}\;Z_1(m_s r)\;\vec{L}\cdot\vec{S}\;\right\} \end{aligned} \tag{1.164}$$

where $\mathcal{Y}(x)=e^{-x}/x$ and $Z_1(x) = \;\left(\frac{m_\alpha}{m}\right)^2\;\;(1/x\;+\;1/x^2)\;\mathcal{Y}(x)$.

The treatment of the Schrödinger equation with a momentum dependent potential is given by O. Rojo, L.M. Simmons, *Phys. Rev.* **125**, 273 (1962). The expressions for the other potential terms shall only be given here:

$$V_{ps}^c(\vec{k}\;,\vec{p}\;) \;=\; -\,\frac{g_{ps}^2}{4m^2}\;\frac{(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})}{k^2+m_{ps}^2} \tag{1.165}$$

$$\begin{aligned} V_v^c(\vec{k},\vec{p}\;) \;&=\; \frac{1}{k^2+m_v^2}\;\left\{g_v^2\;\left[1+\frac{3p^2}{2m^2}\;-\;\frac{k^2}{8m^2}\;+\;\frac{3i}{2m^2}\;\vec{S}\cdot(\vec{k}\times\vec{p})\right.\right. \\ &-\;\left.(\vec{\sigma}_1\cdot\vec{\sigma}_2)\;\frac{k^2}{4m^2}\;+\;\frac{1}{4m^2}\;(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})\right] \\ &+\;\frac{g_v f_v}{2m}\;\left[-\;\frac{k^2}{m}\;+\;\frac{4i}{m}\;\vec{S}\cdot(\vec{k}\times\vec{p})\;-\;\vec{\sigma}_1\cdot\vec{\sigma}_2\;\frac{k^2}{m}\;+\;\frac{1}{m}\;(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})\right] \\ &+\;\left.\frac{f_v^2}{4m^2}\;\left[-\vec{\sigma}_1\cdot\vec{\sigma}_2\;k^2\;+\;(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})\right]\right\}\;. \end{aligned} \tag{1.166}$$

The structure of the expression (1.160) already suggests that one would prefer to work with OBE potentials in momentum space. Even the already approximated expressions (1.163), (1.165), (1.166) are still complicated functions of the momenta, though they

can be Fourier transformed analytically to coordinate space. The corresponding r-space expressions to (1.165) and (1.166) are

$$V_{ps}^{c}(r) \;=\; \frac{1}{12}\,\frac{g_{ps}^{2}}{4\pi}\,m_{ps}\,\left[\left(\frac{m_{ps}}{m}\right)^{2}\,\mathcal{Y}(m_{ps}r)\;\vec{\sigma}_1\cdot\vec{\sigma}_2\;+\;Z(m_{ps}r)\;S_{12}\right] \tag{1.167}$$

$$\begin{aligned}
V_{v}^{c}(r) \;=\;& \frac{g_{v}^{2}}{4\pi}\,m_{v}\,\left\{\left[1+\;\frac{1}{2}\,\left(\frac{m_{v}}{m}\right)^{2}\right]\,\mathcal{Y}(m_{v}r)\;-\;\frac{3}{4m^{2}}\,[\nabla^{2}\,\mathcal{Y}(m_{v}r)+\mathcal{Y}(m_{v}r)\nabla^{2}]\right.\\
&+\;\left.\frac{1}{6}\,\left(\frac{m_{v}}{m}\right)^{2}\,\mathcal{Y}(m_{v}r)\;\vec{\sigma}_1\cdot\vec{\sigma}_2\;-\;\frac{3}{2}\,Z_{1}(m_{v}r)\;\vec{L}\;\cdot\vec{S}\;-\;\frac{1}{12}\,Z(m_{v}r)\;S_{12}\right\}\\
&+\;\frac{1}{2}\,\frac{g_{v}f_{v}}{4\pi}\,m_{v}\,\left\{\left(\frac{m_{v}}{m}\right)^{2}\,\mathcal{Y}(m_{v}r)\;+\;\frac{2}{3}\,\left(\frac{m_{v}}{m}\right)^{2}\,\mathcal{Y}(m_{v}r)\;\vec{\sigma}_1\cdot\vec{\sigma}_2\right.\\
&-\;\left.4Z_{1}(m_{v}r)\;\vec{L}\;\cdot\vec{S}\;-\;\frac{1}{3}\,Z(m_{v}r)\;S_{12}\right\}\\
&+\;\frac{f_{v}^{2}}{4\pi}\,m_{v}\,\left\{\frac{1}{6}\,\left(\frac{m_{v}}{m}\right)^{2}\,\mathcal{Y}(m_{v}r)\;\vec{\sigma}_1\cdot\vec{\sigma}_2\;-\;\frac{1}{12}\,Z(m_{v}r)\;S_{12}\right\}\;.
\end{aligned} \tag{1.168}$$

Here the tensor operator S_{12} is given by (1.146) and
$Z(x) = (m_\alpha/m)^2(1+3/x+3/x^2)\mathcal{Y}(x)$.

Details on OBE potentials are given in the references quoted in Section 1.8.