In this form, the optical potential enters the partial wave Lippmann-Schwinger equation given in Eq. (3.22). In practical calculations, the number of L values needed to represent the nuclear optical potential at the level of accuracy required through the partial wave components $U_{JL}(\mathbf{k}, \mathbf{k}')$ can be as large as 40 for a ^{40}Ca target at 200 MeV, and 80 for a ^{208}Pb target at the same energy. For high values of L an accurate calculation of Eq. (3.32) becomes increasingly difficult due to the oscillatory character of the Legendre polynomials $P_L(\cos(\theta))$. This problem can be alleviated through the use of the three-dimensional Born approximation to the scattering amplitude to account for the infinite set of L values satisfying the condition $L > L_c$, where L_c is chosen such that the Born approximation is accurate. We typically have as a condition for L_C where at this critical value 0.1%-0.5% difference occurs using the Born approximation.

3.3 The Scattering Observables

The most general form for the scattering amplitude for spin 0-spin $\frac{1}{2}$ scattering is given as

$$\langle \chi_{\frac{1}{2}}, \nu' | M(E) | \chi_{\frac{1}{2}}, \nu \rangle = -\mu (2\pi)^3 \langle \mathbf{k}', \frac{1}{2}, \nu' | T(E) | \mathbf{k}, \frac{1}{2}, \nu \rangle, \qquad (3.34)$$

where $\chi_{\frac{1}{2}}$ are the Pauli spinors [30, 31], $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final momentum. In elastic scattering $|\mathbf{k}| = |\mathbf{k}'|$. The projection of the spin state on the axis of quantization is given by ν and ν', and the reduced mass μ is defined relativistically as

$$\mu = \frac{\sqrt{E_{proj}(\mathbf{k})E_{target}(-\mathbf{k})E_{proj}(\mathbf{k}')E_{target}(-\mathbf{k}')}}{E_{proj}(\mathbf{k}) + E_{target}(-\mathbf{k})}. \tag{3.35}$$

The matrix M of Eq. (3.34), is an element in the spin space which is composed of the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ [30] and the unit matrix $\mathbf{1}$. Thus the most general form of M can be given as

$$M = A \cdot \mathbf{1} + \sum_{i=1}^{3} \sigma_i \cdot C^i = A \cdot \mathbf{1} + \vec{\sigma} \cdot \vec{C}, \tag{3.36}$$

where A, and C^i are complex functions of the momenta vectors. A set of three linearly independent vectors can be constructed from $\mathbf{k}$ and $\mathbf{k}'$, namely $\mathbf{k} \pm \mathbf{k}'$ and $\mathbf{k} \times \mathbf{k}'$. Since we also require parity conservation, only the term $\mathbf{k} \times \mathbf{k}'$ can contribute.

Under these assumptions (parity conservation and rotational invariance) the most general form of the scattering amplitude is thus given by

$$M = A \cdot \mathbf{1} + C\vec{\sigma} \cdot (\hat{\mathbf{k}} \times \hat{\mathbf{k}}'). \tag{3.37}$$

Using the normal vector $\hat{\mathbf{N}}$ (Eq. 3.5), we obtain for the most general form of M

$$M = A(k, \theta) + \vec{\sigma} \cdot \hat{\mathbf{N}} C(k, \theta), \tag{3.38}$$

where $k = |\mathbf{k}| = |\mathbf{k}'|$. The first term $A(k,\theta)$ cannot induce any change of the spin, $C(k,\theta)$ does. Thus $C(k,\theta)$ is sometimes called the spin-flip amplitude.

The amplitudes $A(k,\theta)$ and $C(k,\theta)$ are obtained from the partial wave solutions of the NA Lippmann-Schwinger equation as described in the previous section starting with Eq. (3.22). They are explicitly obtained as:

$$A(k, \theta) = \sum_{L=0}^{\infty} [(L+1) f_{L\,L+\frac{1}{2}}(k) + L f_{L\,L-\frac{1}{2}}(k)] P_L(\cos\theta) \tag{3.39}$$

and

$$C(k,\theta) = \sum_{L=0}^{\infty}(f_{L\ L+\frac{1}{2}}(k) - f_{L\ L-\frac{1}{2}}(k))P_L^1(cos\theta). \tag{3.40}$$

The functions $f_{L\ J}(k)$ are obtained from the partial wave NA t-matrix elements via

$$f_{L\ J}(k) = -\hbar c(2\pi)^2\mu T_{L\ J}(k,k), \tag{3.41}$$

where μ is given in Eq. (3.35).

Now we explicitly derive the expressions for the scattering observables which can be obtained in spin 0-spin $\frac{1}{2}$ scattering. We start from Eq. (3.38), and realize that we can choose a coordinate system such that the normal vector, $\hat{\mathbf{N}}$, points in the y direction. Thus one only has to consider $\sigma \cdot \hat{\mathbf{N}} = \sigma_y$. This means that one obtains the scattering amplitude for the scattering of nucleons of some initial spin state to an some final spin state by placing the operator $A + C\sigma_y$ between the Pauli spinors for these polarisation directions. The corresponding cross-section is then the absolute value of this amplitude squared. In the usual representation of the spin matrices, where σ_z is diagonal, we have the Pauli spinors:

$$\chi_{+x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_{-x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\chi_{+y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_{-y} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\chi_{+z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.42}$$

As an example, the cross-section for $+\hat{y} \to +\hat{y}$ scattering (polarisation out of the scattering plane) is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\theta, +\hat{y} \to +\hat{y}) &= |\chi^{\dagger}_{+y}(A + C\sigma_y)\chi_{+y}|^2 \\ &= \left| \frac{1}{\sqrt{2}} (\,1,\ -i\,) \left[A + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \right|^2 \\ &= |A + C|^2. \end{aligned} \tag{3.43}$$

For the other spin orientations one obtains

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\theta, +\hat{y} \to -\hat{y}) &= |\chi^{\dagger}_{+y}(A + C\sigma_y)\chi_{-y}|^2 \\ &= \left| \frac{1}{\sqrt{2}} (\,1,\ -i\,) \left[A + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \right|^2 \\ &= |0|^2, \end{aligned} \tag{3.44}$$

and similarly $\frac{d\sigma}{d\Omega}(\theta, -\hat{y} \to +\hat{y}) = |0|^2$. These relations show that the operator $|A+C\sigma_y|$ can rotate spins about the y axis, but cannot change $+\hat{y}$ into $-\hat{y}$. For completeness we also show

$$\frac{d\sigma}{d\Omega}(\theta, -\hat{y} \to -\hat{y}) = |\chi^{\dagger}_{-y}(A + C\sigma_y)\chi_{-y}|^2$$

$$= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} -i, & 1 \end{pmatrix} \left[A + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \right|^2$$

$$= |A - C|^2. \tag{3.45}$$

The unpolarised cross-section, $\frac{d\sigma}{d\Omega}(\theta)$, is a sum of the cross-sections for the final states and an average of the initial states. If we define the cross-section for an average of initial states as

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\theta, i \to +\hat{y}) &\equiv \frac{d\sigma}{d\Omega}(\theta, +\hat{y} \to +\hat{y}) + \frac{d\sigma}{d\Omega}(\theta, -\hat{y} \to +\hat{y}) \\ \frac{d\sigma}{d\Omega}(\theta, i \to -\hat{y}) &\equiv \frac{d\sigma}{d\Omega}(\theta, +\hat{y} \to -\hat{y}) + \frac{d\sigma}{d\Omega}(\theta, -\hat{y} \to -\hat{y}), \end{aligned} \tag{3.46}$$

we can then write the unpolarised cross-section as a combination of the two equations (all initial states to all final states)

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{2} \left[\frac{d\sigma}{d\Omega}(\theta, i \to +\hat{y}) + \frac{d\sigma}{d\Omega}(\theta, i \to -\hat{y}) \right], \tag{3.47}$$

which becomes using Eqs. (3.43-3.45)

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\theta) &= \frac{1}{2} \left[|A(\theta) + C(\theta)|^2 + |0|^2 + |0|^2 + |A(\theta) - C(\theta)|^2 \right] \\ &= |A(\theta)|^2 + |C(\theta)|^2, \end{aligned} \tag{3.48}$$

where there is assumed to be an implicit dependence on the elastic momentum, k.

The elastic cross-section, σ_{el}, is defined as an integration over all angles of the cross-section of Eq. (3.48)

$$\sigma_{el} = 2\pi \int_0^{\pi} (|A(\theta)|^2 + |C(\theta)|^2) \sin\theta d\theta. \tag{3.49}$$

We may also obtain σ_{tot} which is a combination of the elastic cross-section and the reaction cross-section, σ_{reac},

$$\sigma_{tot} = \sigma_{el} + \sigma_{reac}. \tag{3.50}$$

The total cross-section is found by using the optical theorem [30]. The M matrix obeys unitarity relations which give for spin 0-spin $\frac{1}{2}$ elastic scattering

$$\sigma_{tot} = -\frac{4\pi}{k}\mathrm{Im}(M(\theta = 0)) = -\frac{4\pi}{k}\mathrm{Im}(A(k,0))). \tag{3.51}$$

This equation implies that the C amplitude is zero at exact forward scattering which is true by definition, because $\hat{\mathbf{k}} = \hat{\mathbf{k}}'$. We can then find σ_{reac} by using Eq. (3.50).

In order to obtain the analyzing power, the spins of the outgoing projectiles are measured, while the incident beam may be unpolarised. If the difference between the $+\hat{y}$ and $-\hat{y}$ cross-section is taken and the result divided by the unpolarised cross-section, we obtain the analyzing power A_y

$$A_y = \frac{\frac{d\sigma}{d\Omega}(\theta, i \to +\hat{y}) - \frac{d\sigma}{d\Omega}(\theta, i \to -\hat{y})}{\frac{d\sigma}{d\Omega}(\theta, i \to +\hat{y}) + \frac{d\sigma}{d\Omega}(\theta, i \to -\hat{y})} \tag{3.52}$$

By using Eqs. (3.43-3.48), we can write this as

$$\begin{aligned} A_y &= \frac{\frac{1}{2}|A(\theta) + C(\theta)|^2 - |A(\theta) - C(\theta)|^2}{|A(\theta)|^2 + |C(\theta)|^2} \\ &= \frac{A^*(\theta)C(\theta) + A(\theta)C^*(\theta)}{|A(\theta)|^2 + |C(\theta)|^2} \\ &= \frac{2\mathrm{Re}(A^*(\theta)C(\theta))}{|A(\theta)|^2 + |C(\theta)|^2}. \end{aligned} \tag{3.53}$$

Equivalently, A_y can be measured by sending a beam of polarised protons along $+\hat{y}$ and measure the total cross-section at angles θ and $-\theta$ in the scattering plane.

From the definition of the normal vector $\hat{N}$, these measurements use $\hat{N}$'s of opposite directions and hence give rise to the same combinations $A + C$ and $A - C$.

The last independent measurement involves the rotation of the spin vector in the scattering plane, i.e. protons polarised along the $+\hat{x}$ axis have a finite probability of having the spin polarised along the $\pm\hat{z}$ axis after the collision [34]. Consider an incident polarised beam along $+\hat{x}$ and a vector which describes the polarisation in the z-direction of the scattered protons. The observable describing this 'rotation' of the spin in the scattering plane is called the spin rotation parameter, Q, and is defined as the difference of the cross-sections for $+\hat{z}$ and $-\hat{z}$ states, divided by the sum

$$Q = \frac{\frac{d\sigma}{d\Omega}(\theta, +\hat{x} \to +\hat{z}) - \frac{d\sigma}{d\Omega}(\theta, +\hat{x} \to -\hat{z})}{\frac{d\sigma}{d\Omega}(\theta, +\hat{x} \to +\hat{z}) + \frac{d\sigma}{d\Omega}(\theta, +\hat{x} \to -\hat{z})}. \tag{3.54}$$

As done earlier in this Section, we can explicitly calculate the different terms in Eq. (3.54):

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\theta, +\hat{x} \to +\hat{z}) &= |\chi^{\dagger}_{+x}(A + C\sigma_y)\chi_{+z}|^2 \\ &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1, & 1 \end{pmatrix} \left[A + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 \\ &= \frac{1}{2}|A + iC|^2, \end{aligned} \tag{3.55}$$

and

$$\frac{d\sigma}{d\Omega}(\theta, +\hat{x} \to -\hat{z}) = |\chi^{\dagger}_{+x}(A + C\sigma_y)\chi_{-z}|^2$$

$$
\begin{aligned}
&= \left| \frac{1}{\sqrt{2}} (\ 1,\ 1\ ) \left[A + C \begin{pmatrix} 0 & -i \\ & \\ i & 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \right|^2 \\
&= \frac{1}{2} |A - iC|^2. \qquad (3.56)
\end{aligned}
$$

Using the results of Eqs. (3.55,3.56), Eq. (3.54) can be written as

$$
\begin{aligned}
Q &= \frac{\frac{1}{2}|A(\theta) + iC(\theta)|^2 - |A(\theta) - iC(\theta)|^2}{|A(\theta) + iC(\theta)|^2 + |A(\theta) - iC(\theta)|^2} \\
&= \frac{i(C(\theta)A^*(\theta) - A(\theta)C^*(\theta))}{|A(\theta)|^2 + |C(\theta)|^2} \\
&= \frac{2\mathrm{Im}(A(\theta)C^*(\theta))}{|A(\theta)|^2 + |C(\theta)|^2}. \qquad (3.57)
\end{aligned}
$$

Notice that A_y and Q do complement each other. The A_y is a measure of any spin dependence out of the scattering plane, while Q is a measure of spin dependence in the plane. The following relation can be seen from Eqs. (3.53,3.57)

$$
{A_y}^2 + Q^2 \leq 1. \qquad (3.58)
$$

Spin observables are a tool used in probing the nuclear structure and force. As an example of experimental data using these observables we have plotted an elastic collision of a 200 MeV proton on calcium 40 (^{40}Ca (p,p)) in Fig. 3.2. Because the spin observables are normalized with the cross-section they only vary from -1 to 1 (no units), while the cross-section is measured in barns which is $10^{-28}\mathrm{m}^2$.

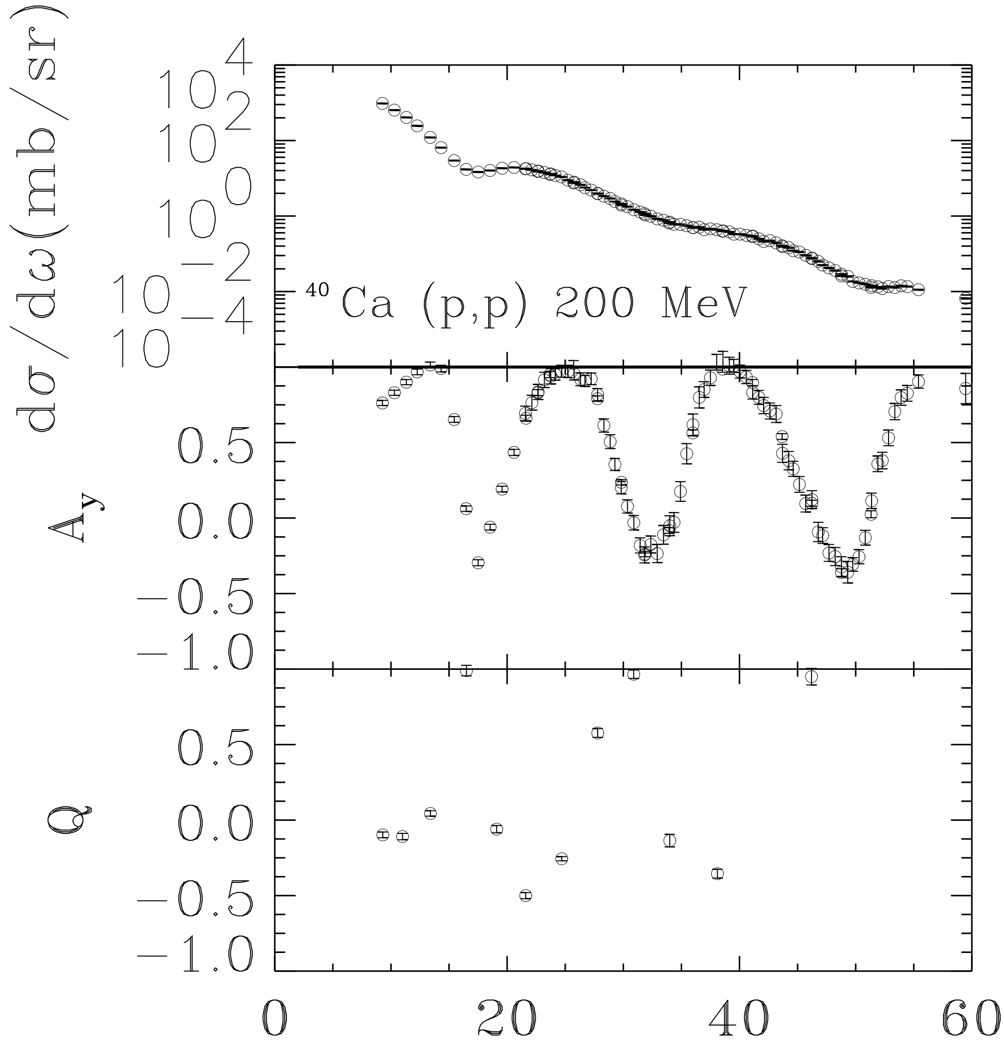

Figure 3.2: The angular distribution of the differential cross-section ($\frac{d\sigma}{d\Omega}$), analyzing power (A_y) and spin rotation function (Q) are shown for elastic proton scattering from ^{40}Ca at 200 MeV laboratory energy. The data are taken from Ref. [35].