Phys 726: Homework V

due November 6, 2009

On the Three-Body System

The Schrödinger equation for a three-body system with pairwise interactions is given by

$$(H_0 \ + \ \sum_{i=1}^{3} V_i) \ \Psi \ = \ E \ \Psi. \tag{1}$$

Here H_0 contains the kinetic energies of the three particles, and $V_1 \equiv V_{23}, V_2 \equiv V_{13}, V_3 \equiv V_{12}$ characterizes the interactions in the two-body subsystems.

In order to develop a scheme for solving Eq. (1), Faddeev suggested to decompose the three-body wave function Ψ into 3 components, nowadays called Faddeev components

$$\Psi = \sum_{i=1}^{3} \psi_i \equiv \psi(1,23) + \psi(2,31) + \psi(3,12) \tag{2}$$

and derived a set of 3 coupled equations for ψ_i, called Faddeev equations or later the Alt-Grassberger-Sandhas equations, which are more suited for practical applications. In the case of **three identical particles**, this system of coupled equations can be reduced to one single equation for one Faddeev component, which can be arbitrarily chosen to be $\psi_1 = \psi(1,23)$. In order to do this, one has to explicitly work with the permutation operators for 3 particles.

(a) *[2 pts]*
Show that once a solution for ψ_1 is obtained, the total wave function for the three-body system can be written as

$$\Psi = (1 + P) \ \psi \tag{3}$$

where the permutation operator P is given by

$$P \ \equiv \ P_{12}P_{23} \ + \ P_{13}P_{23}. \tag{4}$$

In addition, explain the choice of P.

(b) *[3 pts]*
Proof that in order for Ψ to be totally antisymmetric, one has to require that the Faddeev component ψ_1 is antisymmetric in the pair "1" $\equiv$ (23).
(Hint: Consider e.g. $P_{12}\Psi$ or $P_{13}\Psi$.)

(c) *[3 pts]*
The momenta in a three-body system are given by $\vec{k}_i$, and the total momentum is $\vec{K} = \sum_i \vec{k}_i$. Consider the kinetic energy term of Eq. (1)

$$H_0 = \sum_i \frac{k_i^2}{2m}, \tag{5}$$

where the 3 particles are assumed to have equal mass m. It is natural to work with Jacobi momenta, which are defined for the 3 different choices of the subsystem as:

$$\begin{aligned} \vec{p}_1 &= \frac{1}{2}\,(\vec{k}_2 - \vec{k}_3) \\ \vec{q}_1 &= \frac{2}{3}\,(\vec{k}_1 - \frac{1}{2}\,(\vec{k}_2 + \vec{k}_3)\,), \end{aligned} \tag{6}$$

which corresponds to $\psi_1(1, 23)$. The other two choices are:

$$\begin{aligned} \vec{p}_2 &= \frac{1}{2}\,(\vec{k}_3 - \vec{k}_1) \\ \vec{q}_2 &= \frac{2}{3}\,(\vec{k}_2 - \frac{1}{2}\,(\vec{k}_3 + \vec{k}_1)\,) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \vec{p}_3 &= \frac{1}{2}\,(\vec{k}_1 - \vec{k}_2) \\ \vec{q}_3 &= \frac{2}{3}\,(\vec{k}_3 - \frac{1}{2}\,(\vec{k}_1 + \vec{k}_2)\,) \end{aligned} \tag{8}$$

Show that the kinetic energy in Jacobi coordinates takes the form

$$H_0 = \frac{K^2}{2M} + \frac{p_\ell^2}{2\mu_\ell} + \frac{q_\ell^2}{2M_\ell} \tag{9}$$

with $\ell = 1, 2, 3$. In Eq. (9)

$$M = 3m\,; \quad M_\ell = \frac{2}{3}\,m\,; \quad \mu_\ell = \frac{1}{2}\,m\,, \tag{10}$$

where m is the mass of a single particle (e.g., the nuclear mass).

(d) *[3 pts]*
In order to change from one set of Jacobi coordinates to another, one has to express e.g. Set 1 as function of Set 2 etc. Derive relations for

$$\vec{p}_1(\vec{p}_2, \vec{q}_2), \quad \vec{q}_1(\vec{p}_2, \vec{q}_2) \tag{11}$$

and

$$\vec{p}_1(\vec{p}_3, \vec{q}_3), \quad \vec{q}_1(\vec{p}_3, \vec{q}_3) \tag{12}$$

(The correct answer can be found in Ref. [1]

(e) *[4 pts]*
Show that the matrix elements of the permutation operator P of Eq. (4) in the basis $|\vec{p}\vec{q}\rangle_1$ are given as

$$\begin{aligned}\langle \vec{p}\,'\vec{q}\,|\,P\,|\,\vec{p}\,''\vec{q}\,''\rangle &= \delta\left(\vec{p}\,' - \frac{1}{2}\,\vec{q} - \vec{q}\,''\right)\,\delta\left(\vec{q} + \vec{p}\,'' + \frac{1}{2}\,\vec{q}\,''\right) \\ &+ \delta\left(\vec{p}\,' + \frac{1}{2}\,\vec{q} + \vec{q}\,''\right)\,\delta\left(\vec{q} - \vec{p}\,'' + \frac{1}{2}\,\vec{q}\,''\right)\end{aligned} \tag{13}$$

References

[1] Ch. Elster, W. Schadow, A. Nogga and W. Glöckle, Few Body Syst. **27**, 83 (1999) [arXiv:nucl-th/9805018].