On the Dirac Theory of Spin 1/2 Particles and Its Non-Relativistic Limit*

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By a canonical transformation on the Dirac Hamiltonian for a free particle, a representation of the Dirac theory is obtained in which positive and negative energy states are separately represented by two-component wave functions. Playing an important role in the new representation are new operators for position and spin of the particle which are physically distinct from these operators in the conventional representation. The components of the time derivative of the new position operator all commute and have for eigenvalues all values between $-c$ and c. The new spin operator is a constant of the motion unlike the spin operator in the conventional representation. By a comparison of the new Hamiltonian with the non-relativistic Pauli-Hamiltonian for particles of spin $\frac{1}{2}$, one finds that it is these new operators rather than the conventional ones which pass over into the position and spin operators in the Pauli theory in the non-relativistic limit.

The transformation of the new representation is also made in the case of interaction of the particle with an external electromagnetic field. In this way the proper non-relativistic Hamiltonian (essentially the Pauli-Hamiltonian) is obtained in the non-relativistic limit. The same methods may be applied to a Dirac particle interacting with any type of external field (various meson fields, for example) and this allows one to find the proper non-relativistic Hamiltonian in each such case. Some light is cast on the question of why a Dirac electron shows some properties characteristic of a particle of finite extension by an examination of the relationship between the new and the conventional position operators.

INTRODUCTION

A FREE Dirac particle of mass m is described by a four-component wave function Ψ satisfying the Dirac equation,

$$H\Psi=(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})\Psi=i(\partial\Psi/\partial t), \tag{1}$$

where $\mathbf{p}$ is the momentum operator for the particle, $\boldsymbol{\alpha}$ and β are the well known Dirac matrices (assumed here to be in their usual representation with β diagonal), and units have been used in which $\hbar$ and c are unity. The eigenfunctions of the Hamiltonian operator satisfy the equation:

$$(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})\psi=E\psi. \tag{2}$$

The eigenfunctions are of the form $u(\mathbf{p})e^{-\mathbf{p}\cdot\mathbf{x}}$. For each value of the momentum $\mathbf{p}$ there are four linearly independent spinors[1] $u(\mathbf{p})$ corresponding to the two eigenvalues $\pm(m^2+p^2)^{\frac{1}{2}}$ of the energy and the two eigenvalues ±1 of the z component of an operator $\boldsymbol{\Sigma}$ (defined later in Eq. (26)) related to the spin operator for the particle.

Except in certain trivial cases, for a given sign of the energy at least three of the components of $u(\mathbf{p})$ are different from zero. However, two of the four components go to zero as the momentum goes to zero, while at least one of the other two components remains finite. In the above representation, for positive-energy eigenfunctions, the last two (or *lower*) components vanish with vanishing momentum, while for negative-energy eigenfunctions, the first two (or *upper*) components vanish with vanishing momentum.

Of course any spinor Ψ may be split into the sum of two spinors Φ and X, having only upper and lower components respectively, in the following manner:

$$\Psi=\Phi+X,\quad \Phi=(1+\beta/2)\Psi,\quad X=(1-\beta/2)\Psi,$$

but the spinors Φ and X do not represent states of definite energy in the above representation.

In the non-relativistic limit, where the momentum of the particle is small compared to m, it is well known that a Dirac particle (that is, one with spin $\frac{1}{2}$) can be described by a two-component wave function in the Pauli theory. The usual method of demonstrating that the Dirac theory goes into the Pauli theory in this limit makes use of the fact noted above that two of the four Dirac-function components become small when the momentum is small. One then writes out the equations satisfied by the four components and solves, approximately, two of the equations for the small components. By substituting these solutions in the remaining two equations, one obtains a pair of equations for the large components which are essentially the Pauli spin equations.[2]

The above method of demonstrating the equivalence of the Dirac and Pauli theories encounters difficulties, however, when one wishes to go beyond the lowest order approximation. One then finds that the "Hamiltonian" associated with the large components is no longer Hermitian in the presence of external fields because of the appearance of an "imaginary electric moment" for the particle. Furthermore, all four components must again be used in calculating expectation values of operators to

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[1] See, for example, W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1944), Chapter 3. We shall use the term spinor for any four-component column (or row) matrix occurring in the Dirac theory.
[2] See, for example, W. Pauli, *Handbuch der Physik*, 2. Aufl., Bd. 24, Teil 1.

order v^2/c^2. Apart from these difficulties, however, there are some serious questions concerning the properties of corresponding operators in the Dirac theory and the Pauli theory even in lowest order. Thus, in the Dirac theory the operator representing the velocity of the particle is the operator $\boldsymbol{\alpha}$ whose components have only the eigenvalues ± 1. On the other hand, in the Pauli theory the operator representing the velocity of the particle is taken to be $\mathbf{p}/m$ whose components have eigenvalues embracing all real numbers. Furthermore, different components of the velocity operator in the Dirac theory do not commute and therefore are not simultaneously measurable with arbitrary precision, while different components of the velocity operator in the Pauli theory do commute and can therefore be measured simultaneously with arbitrary precision. One can well ask how the operators which purportedly represent the same physical variable in the two theories can have such completely different properties.

One can conclude from the above discussion that the relation between the Dirac theory and the Pauli theory is by no means clear from the usual method of descending from four- to two-component wave functions, and that further clarification of the connection between the theories would be desirable. In what follows we present an alternative method for passing from four- to two-component wave functions in the Dirac theory which alternative method clarifies many of the questions left open by the usual treatment. The method consists of a transformation to a new representation for the Dirac theory, putting the theory in a form closely analogous to the Pauli theory and permitting a direct comparison of the two. Specifically, we shall show that:

(1) For a free Dirac particle there exists a representation of the Dirac theory in which, for both relativistic and non-relativistic energies, positive-energy states and negative-energy states are separately described by two-component wave functions.

(2) There exists in the Dirac theory another position operator than the usual one; this operator has the property that its time derivative is the operator $\mathbf{p}/(m^2+p^2)^{\frac{1}{2}}$ for positive-energy states and $-\mathbf{p}/(m^2+p^2)^{\frac{1}{2}}$ for negative-energy states corresponding to the conventional concept of the velocity of the particle. It is this new position operator (which we call the *mean-position operator*) that in the non-relativistic limit is interpreted as the position operator of the Pauli theory.

(3) While the z component of the spin operator, $\boldsymbol{\sigma}=(1/2i)[\boldsymbol{\alpha}\times\boldsymbol{\alpha}]$, in the Dirac theory is not a constant of the motion, there exists in the Dirac theory another spin operator $\boldsymbol{\Sigma}$ (which we call the *mean-spin operator*) whose z component is a constant of the motion. In the non-relativistic limit, the operator $\boldsymbol{\Sigma}$ is the one which is interpreted as the spin operator in the Pauli theory.

(4) From a study of the transformation to the new representation further insight can be obtained into the question of why the Dirac particle has a magnetic moment and why it appears to show a behavior characteristic of a particle with finite extension of the order of its Compton wave-length.

(5) In the presence of interaction, such as with an external electromagnetic field, one can still make a transformation which leads to a representation involving two-component wave functions. The transformation, which previously could be made exactly, must now be made by an infinite sequence of transformations which process leads to a Hamiltonian which is an infinite series in powers of $(1/m)$, where m is the mass of the particle. This series is presumably semi-convergent in the sense that for given external fields, a finite number of terms of the series is a better-and-better approximation to the exact Hamiltonian, the larger the value of m. For strong interactions which strongly couple free-particle states of positive and negative energy, this representation is of little value; however, for sufficiently weak interactions, a finite number of terms of the series may be employed to obtain relativistic corrections to any order in $(1/m)$. In this way one can obtain the proper non-relativistic limit for the Hamiltonian representing a Dirac particle interacting with any type of external field.

THE FREE DIRAC PARTICLE

The essential reason why four components are in general necessary to describe a state of positive or negative energy in the representation of the Dirac theory corresponding to Eq. (1) is that the Hamiltonian in this equation contains odd operators,[3] specifically the components of the operator $\boldsymbol{\alpha}$. If it is possible to perform a canonical transformation on Eq. (1) which brings it into a form which is free of odd operators, then it will be possible to represent positive- and negative-energy states by wave functions having only two components in each case, the other pair of components being identically zero. We shall now show that there exists a canonical transformation which accomplishes just this end.

If S is an Hermitian operator, then the transformation,

$$\Psi'=e^{iS}\Psi, \tag{3}$$

$$H'=e^{iS}He^{-iS}-ie^{iS}(\partial e^{-iS}/\partial t), \tag{4}$$

[3] An odd operator in the Dirac theory is a Dirac matrix which has only matrix elements connecting upper and lower components of the wave function, while an even operator is one having no such matrix elements. Of the sixteen linearly independent matrices in the Dirac theory, the matrices 1, β, $\boldsymbol{\sigma}=1/2i[\boldsymbol{\alpha}\times\boldsymbol{\alpha}]$ and $\beta\boldsymbol{\sigma}$ are even, while the matrices $\boldsymbol{\alpha}$, $\beta\boldsymbol{\alpha}$, $\gamma^5=-i\alpha^1\alpha^2\alpha^3$ and $\beta\gamma^5$ are odd. The product of two even matrices or of two odd matrices is an even matrix, while the product of an odd matrix and an even matrix is an odd matrix. The matrix β commutes with all even matrices and anticommutes with all odd matrices in the Dirac theory. This last fact allows one to write any matrix as the sum of an odd and an even matrix in a simple way, namely:

$$m=\tfrac{1}{2}\{m+\beta m\beta\}+\tfrac{1}{2}\{m-\beta m\beta\},$$

where the first term on the right is the even part of the matrix and the second term is the odd part of the matrix m.

leaves Eq. (1) in the Hamiltonian form:

$$H'\Psi' = i(\partial\Psi'/\partial t). \tag{5}$$

(It must be remembered that in a canonical transformation of this type a physical variable whose operator-representative in the old representation is T has for its operator-representative in the new representation the operator:

$$T' = e^{iS}Te^{-iS}.) \tag{6}$$

Let us make a transformation of this type on Eq. (1) with S the non-explicitly time-dependent operator:

$$S = -(i/2m)\beta\boldsymbol{\alpha}\cdot\mathbf{p}w(p/m), \tag{7}$$

where w is a function of the operator[4] (p/m) whose form is to be determined such that H' is free of odd operators. With this choice of S one may readily show that

$$\begin{aligned}H' &= e^{iS}He^{-iS} = e^{2iS}H\\ &= [\cos(pw/m) + (\beta\boldsymbol{\alpha}\cdot\mathbf{p}/p)\sin(pw/m)]H\\ &= \beta[m\cos(pw/m) + p\sin(pw/m)]\\ &\qquad + \boldsymbol{\alpha}\cdot\mathbf{p}/p[p\cos(pw/m) - m\sin(pw/m)].\end{aligned} \tag{8}$$

We see from this that H' will be free of odd operators if we choose

$$w(p/m) = (m/p)\tan^{-1}(p/m), \tag{9}$$

and with this choice, we obtain

$$H' = \beta(m^2+p^2)^{\frac{1}{2}} = \beta E_p, \tag{10}$$

where E_p represents the *operator* $(m^2+p^2)^{\frac{1}{2}}$.

Equation (5), with H' given by (10), now has solutions such that the upper components represent positive energies and the lower components negative energies, for splitting Ψ' into its upper and lower components:

$$\Psi' = \Phi' + X',$$
$$\Phi' = (1+\beta/2)\Psi', \quad X' = (1-\beta/2)\Psi',$$

reduces Eq. (5) to the two uncoupled equations:

$$E_p\Phi' = i(\partial\Phi'/\partial t), \tag{11}$$

$$-E_pX' = i(\partial X'/\partial t). \tag{12}$$

In order for one to understand completely the nature of the transformation which we have performed, however, it is necessary to investigate in greater detail the manner in which the wave function and certain operators transform. A general wave function can be expressed in the form:

$$\Psi(\mathbf{x}) = \int u(\mathbf{p}')\exp(i\mathbf{p}'\mathbf{x}')d\mathbf{p}' = \Psi_+(\mathbf{x}) + \Psi_-(\mathbf{x}), \tag{13}$$

[4] A function $T(\mathbf{p})$ of the operator $\mathbf{p}$ is to be interpreted (in the coordinate representation) as defined by its Taylor expansion in powers of $(\mathbf{p}-\mathbf{p}_0)$, where $\mathbf{p}_0$ is any constant vector (c number), wherever the expansion converges, and by the analytic continuation of this series elsewhere; or, alternatively, by its integral-operator representation:

$$T(\mathbf{p})\Psi(\mathbf{x}) = \frac{1}{(2\pi)^3}\int\int T(\mathbf{p}')\exp[i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{x}')]\Psi(\mathbf{x}')d\mathbf{p}'d\mathbf{x}'.$$

$$\Psi_+(\mathbf{x}) = \int\frac{1}{2}\left[1 + \frac{\beta m + \boldsymbol{\alpha}\cdot\mathbf{p}'}{E_{p'}}\right]u(\mathbf{p}')\exp(i\mathbf{p}'\cdot\mathbf{x})d\mathbf{p}', \tag{14}$$

$$\Psi_-(\mathbf{x}) = \int\frac{1}{2}\left[1 - \frac{\beta m + \boldsymbol{\alpha}\cdot\mathbf{p}'}{E_{p'}}\right]u(\mathbf{p}')\exp(i\mathbf{p}'\cdot\mathbf{x})d\mathbf{p}', \tag{15}$$

where Ψ_+ represents the positive energy part and Ψ_- the negative energy part of Ψ. Now

$$\begin{aligned}e^{iS} &= \exp\left(\frac{1}{2m}\frac{\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\tan^{-1}\frac{p}{m}\right)\\ &= \cos[\tfrac{1}{2}\tan^{-1}(p/m)] + (\beta\boldsymbol{\alpha}\cdot\mathbf{p}/p)\sin[\tfrac{1}{2}\tan^{-1}(p/m)]\\ &= \left[\frac{E_p+m}{2E_p}\right]^{\frac{1}{2}} + \frac{\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\left[\frac{E_p-m}{2E_p}\right]^{\frac{1}{2}}\\ &= \frac{\beta\{\beta m + \boldsymbol{\alpha}\cdot\mathbf{p} + \beta E_p\}}{[2E_p(E_p+m)]^{\frac{1}{2}}},\end{aligned} \tag{16}$$

and hence

$$\Psi_+' = e^{iS}\Psi_+ = \left(\frac{1+\beta}{2}\right)\int\left[\frac{2E_{p'}}{E_{p'}+m}\right]^{\frac{1}{2}} \times\left[1 + \frac{\beta m + \boldsymbol{\alpha}\cdot\mathbf{p}'}{E_{p'}}\right]u(\mathbf{p}')\exp(i\mathbf{p}'\cdot\mathbf{x}')d\mathbf{p}' = \Phi', \tag{17}$$

$$\Psi_-' = e^{iS}\Psi_- = \left(\frac{1-\beta}{2}\right)\int\left[\frac{2E_{p}'}{E_{p'}+m}\right]^{\frac{1}{2}} \times\left[1 - \frac{\beta m + \boldsymbol{\alpha}\cdot\mathbf{p}'}{E_{p'}}\right]u(\mathbf{p}')\exp(i\mathbf{p}'\cdot\mathbf{x}')d\mathbf{p}' = X', \tag{18}$$

$$\Psi' = e^{iS}\Psi = \Psi_+' + \Psi_-'. \tag{19}$$

We may legitimately identify Ψ_+' with Φ' and Ψ_-' with X' since the presence of the factors $(1+\beta)/2$ and $(1-\beta)/2$ shows that these functions have only upper and lower components, respectively. Thus we see that indeed the transformation is such as to lead to a representation in which upper components correspond to positive energies and lower components to negative energies.

Furthermore, since

$$u(\mathbf{p}') = \frac{1}{(2\pi)^3}\int\Psi(\mathbf{x}')\exp(-i\mathbf{p}'\cdot\mathbf{x}')d\mathbf{x}',$$

we have

$$\Psi'(\mathbf{x}) = \int K(\mathbf{x},\mathbf{x}')\Psi(\mathbf{x}')d\mathbf{x}', \tag{20}$$

where

$$K(\mathbf{x},\mathbf{x}') = \frac{1}{(2\pi)^3}\int\left[\frac{2E_{p'}}{E_{p'}+m}\right]^{\frac{1}{2}} \times\left[1 + \frac{\beta(\beta m + \boldsymbol{\alpha}\cdot\mathbf{p}')}{E_{p'}}\right]\exp(i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{x}'))d\mathbf{p}'. \tag{21}$$

Since $K(\mathbf{x}, \mathbf{x}')$ is not a Dirac delta-function in its space dependence, the transformation of Ψ is an integral rather than a point transformation. In general, Ψ' at a given point is constituted from contributions depending on Ψ over a neighborhood of dimensions of the order of a Compton wave-length of the particle about the point. Thus a wave function which in the old representation corresponded to a state in which the particle was definitely located at one point, passes over in the new representation into a wave function which apparently corresponds to the particle being spread out over a finite region.

The key to understanding this rather unusual statement lies in the fact that in the new representation, the operator-representative for the position of the particle is no longer the operator $\mathbf{x}$, but the rather complicated operator:

$$\mathbf{x}'=e^{iS}\mathbf{x}e^{-iS}=\mathbf{x}-\frac{i\beta\boldsymbol{\alpha}}{2E_p}+\frac{i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}-[\boldsymbol{\sigma}\times\mathbf{p}]p}{2E_p(E_p+m)p}. \quad (22)$$

But if this is the case, then what is the operator-representative $\mathbf{X}$ in the old representation of the physical variable whose operator-representative in the new representation is $\mathbf{X}'=\mathbf{x}$? We find

$$\mathbf{X}=e^{-iS}\mathbf{x}e^{iS}=\mathbf{x}+\frac{i\beta\boldsymbol{\alpha}}{2E_p}-\frac{i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}+[\boldsymbol{\sigma}\times\mathbf{p}]p}{2E_p(E_p+m)p}. \quad (23)$$

In order to interpret this new "position" operator, we calculate its time derivative:

$$\frac{d\mathbf{X}}{dt}=i[H,\mathbf{X}]=\frac{\mathbf{p}}{E_p}\cdot\frac{\{\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}\}}{E_p}. \quad (24)$$

But since $\Lambda\mathbf{p}=(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})/E_p$ has the value $+1$ when applied to a positive energy wave function and the value -1 when applied to a negative energy wave function,[5] we see that the time derivative of $\mathbf{X}$ is just the "conventional" velocity operator $+\mathbf{p}/E_p$ for positive energy states and $-\mathbf{p}/E_p$ for negative energy states. The operator-representative of $d\mathbf{X}/dt$ in the new representation is

$$d\mathbf{X}'/dt=d\mathbf{x}/dt=i[H',\mathbf{x}]=\beta\mathbf{p}/E_p \quad (25)$$

with the same properties. We note also that since the operator $\mathbf{p}$ commutes with S, the operator-representative for the momentum in the new representation is still $\mathbf{p}$.

By comparing these results with certain well-known facts about the Dirac electron, we may understand the significance of our change in representation. In discussions of the Dirac theory one finds an analysis of the motion of a free Dirac electron which shows that the electron performs a very complicated motion, indeed.[6] Its velocity can be written as the sum of two parts: the first is essentially the conventional velocity operator

$$\frac{\mathbf{p}}{E_p}\frac{\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}}{E_p},$$

but to this is added a second term representing a rapidly oscillating motion ("Zitterbewegung") which ensures that a measurement of the instantaneous value of any velocity component shall yield the velocity of light. Our results above show that a corresponding division of the position operator for the particle is also possible, the first part $\mathbf{X}$ (in the old representation) representing a sort of mean position of the particle, and the second part $\mathbf{X}-\mathbf{x}$, oscillating rapidly about zero with an amplitude of the order of the Compton wave-length of the particle. While in the old representation the position operator $\mathbf{x}$ played the dominant role, in the new representation it is the position operator $\mathbf{X}'$, which we shall call the *mean-position* operator,[7] which plays the dominant role. Also, as will become obvious later when we consider the interaction of the particle with an external field, it is the *mean-position* operator which is identified with the position operator in the non-relativistic Pauli theory.

The modification in the interpretation of operators involved in our transformation does not end here, however, but new angular momentum operators also appear. In the old representation, the orbital angular momentum of the particle whose operator-representative is $[\mathbf{x}\times\mathbf{p}]$ and the spin angular momentum of the particle whose operator-representative is $1/2\boldsymbol{\sigma}$ are not separately constants of the motion,[6] although their sum is a constant of the motion. However, as one may readily verify, the operators $[\mathbf{X}\times\mathbf{p}]$ and

$$\boldsymbol{\Sigma}=\boldsymbol{\sigma}-\frac{i\beta[\boldsymbol{\alpha}\times\mathbf{p}]}{E_p}-\frac{[\mathbf{p}\times[\boldsymbol{\sigma}\times\mathbf{p}]]}{E_p(E_p+m)}, \quad (26)$$

whose analogues in the new representation are, respectively,

$$[\mathbf{X}'\times\mathbf{p}]=e^{iS}[\mathbf{X}\times\mathbf{p}]e^{-iS}=[\mathbf{x}\times p], \quad (27)$$

and

$$\boldsymbol{\Sigma}'=e^{iS}\boldsymbol{\Sigma}e^{-iS}=\boldsymbol{\sigma}, \quad (28)$$

[5] E. Schrödinger, Berl. Ber. 419 (1930); 63 (1931).

[6] See, for example, P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1935), Second Edition, Chapter XII. See also in this connection, reference 2, pp. 230–231.

[7] The operator, which we have designated the mean-position operator, has been discovered independently in other connections by several authors. M. H. L. Pryce (Proc. Roy. Soc. **150A**, 166 (1935)) found it useful to introduce this operator (as well as the operator which we later define as the mean spin angular momentum) in connection with the definition of coordinate and intrinsic angular momentum operators in the Born-Infeld theory, and again (Proc. Roy. Soc. **195A**, 62 (1948)) in a discussion of relativistic definitions of center of mass for systems of particles. In the latter paper, he also noted that this operator was connected with the usual position operator in the Dirac theory by a canonical transformation which is identical with the one performed above in this paper. T. D. Newton and E. P. Wigner (Rev. Mod. Phys. **21**, 400 (1949)) also found this operator in an investigation of localized states for elementary systems. The latter authors have also shown that this is the only position operator, with commuting components, in the Dirac theory which has localized eigenfunctions in the manifold of positive energy wave functions.

TABLE I. Table of operator-representatives of dynamical variables in old and new representations.

Dynamical variable	Operator-representative in old representation	Operator-representative in new representation
Position	$\mathbf{x}$	$\mathbf{x}'=\mathbf{x}-\frac{i\beta\boldsymbol{\alpha}}{2E_p}+\frac{i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}-[\boldsymbol{\sigma}\times\mathbf{p}]p}{2E_p(E_p+m)p}$
Momentum	$\mathbf{p}\equiv(\hbar/i)\nabla$	$\mathbf{p}'=\mathbf{p}$
Hamiltonian	$H=\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}$	$H'=\beta(m^2+p^2)^{\frac{1}{2}}\equiv\beta E_p$
Velocity	$\boldsymbol{\alpha}=\dot{\mathbf{x}}$	$\boldsymbol{\alpha}'=\boldsymbol{\alpha}+\frac{\beta\mathbf{p}}{E_p}-\frac{(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}}{E_p(E_p+m)}$
	β	$\beta'=(m\beta-\boldsymbol{\alpha}\cdot\mathbf{p})/E_p$
Orbital angular momentum	$[\mathbf{x}\times\mathbf{p}]$	$[\mathbf{x}'\times\mathbf{p}]$
Spin angular momentum	$\boldsymbol{\sigma}\equiv(1/2i)[\boldsymbol{\alpha}\times\boldsymbol{\alpha}]$	$\boldsymbol{\sigma}'=\boldsymbol{\sigma}+\frac{i\beta[\boldsymbol{\alpha}\times\mathbf{p}]}{E_p}-\frac{\mathbf{p}\times[\boldsymbol{\sigma}\times\mathbf{p}]}{E_p(E_p+m)}$
Mean position	$\mathbf{X}=\mathbf{x}+\frac{i\beta\boldsymbol{\alpha}}{2E_p}-\frac{i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p}+[\boldsymbol{\sigma}\times\mathbf{p}]p}{2E_p(E_p+m)p}$	$\mathbf{X}'=\mathbf{x}$
Mean velocity	$\dot{\mathbf{X}}=\frac{\mathbf{p}}{E_p}\cdot\left\{\frac{\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}}{E_p}\right\}$	$\dot{\mathbf{X}}'=\frac{\beta\mathbf{p}}{E_p}$
Mean orbital angular momentum	$[\mathbf{X}\times\mathbf{p}]$	$[\mathbf{X}'\times\mathbf{p}]=[\mathbf{x}\times\mathbf{p}]$
Mean spin angular momentum	$\boldsymbol{\Sigma}=\boldsymbol{\sigma}-\frac{i\beta[\boldsymbol{\alpha}\times\mathbf{p}]}{E_p}-\frac{[\mathbf{p}\times[\boldsymbol{\sigma}\times\mathbf{p}]]}{E_p(E_p+m)}$	$\boldsymbol{\Sigma}'=\boldsymbol{\sigma}$
Λ_p	$\Lambda_p\equiv\frac{\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}}{E_p}$	$\Lambda_p'=\beta$

are separately constants of the motion. We shall denote the physical variables of which these operators are the operator-representatives as the *mean-orbital angular momentum* and *mean spin angular momentum*, respectively, of the particle. It is again these variables which are conventionally identified with the orbital angular momentum and the spin angular momentum of the particle in the non-relativistic Pauli theory. For convenience, we have listed in Table I the operator-representatives in the old representation and in the new representation of the physical variables of principal interest in the Dirac theory.

DIRAC PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

We consider now the case where a Dirac particle interacts with an external field such as the electromagnetic field. Before proceeding to the general case, however, it is instructive to consider an elementary special case in order to clarify some points regarding the classification of states as either of positive or negative energy. Let us consider first a Dirac particle moving in a weak static electric field derivable from a scalar potential φ. The Hamiltonian is then

$$H=\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}-e\varphi.$$

For the case of a free particle, states were classified as of positive or negative energy according as they corresponded to values of $+1$ or -1, respectively, for the operator $(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})/E_p$ which was a constant of the motion. In the present case, however, this operator does not commute with the Hamiltonian and is therefore not a constant of the motion. If we regard the electric field as a perturbation, then one can say that the electric field induces transitions of the particle between the positive- and negative-energy states of a free particle. This is one way of viewing the physical situation.

On the other hand, one knows that for sufficiently weak fields the Hamiltonian above possesses a complete set of eigenfunctions with energy eigenvalues which may be classified according to whether they are positive or negative. There exists for these weak fields a clear-cut distinction between these two sets of stationary states[8] since they are separated by a relatively large energy gap of order $2m$. Furthermore, the wave functions corresponding to positive energies show a behavior of the particle appropriate to a particle of positive mass,[9] in that the particle tends to be localized in regions of low potential energy; while the negative-energy solutions show a behavior of the particle appropriate to a particle of negative mass, in that the particle tends to be localized in regions of high potential energy.

Either of the two descriptions of the behavior of the particle in a weak field given above is of course correct, although the distinction between what are called positive- and negative-energy states is different in the two descriptions. However, the question of terminology for positive- and negative-energy states being left to our own choice, we are free to choose our definitions in such a way as to give the more graphic (and perhaps more

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[8] It is assumed that the constant in the energy is chosen so that zero energy occurs approximately midway in the energy gap between the two sets of states.

[9] It is perhaps better to classify the states as states of positive or negative *mass* rather than energy, since the addition of a constant to the energy (by adding a constant to φ, for example) may upset energy classification but not the mass classification.

intuitively satisfying) description of the actual physical events which are being described. In this spirit we feel that the second description is to be preferred since it has a perfectly reasonable classical limit. It would be difficult indeed to picture classically the motion of a particle in a weak field in terms of transitions between free-particle motions with positive and negative mass.

Consider now what happens when the particle interacts with strong rather than weak fields. Under such circumstances, the division of states into those of positive and negative mass is no longer clear-cut, since the energy, separation of the two sets of states is reduced to a relatively small amount. Furthermore, the wave functions describing these states no longer appropriately describe the motion of a particle of fixed sign of mass according to our customary notions. In fact, if we try to interpret the wave function in these terms, we encounter certain well-known paradoxes—the Klein paradox, for example. While the energy of any stationary state will still have a definite sign, the statement that the particle is in a state of positive energy will no longer carry with it the validity of any intuitive conceptions as to the behavior of a classical particle with positive energy, and there will be little qualitative difference between certain states of positive energy and certain states of negative energy. Hence, in the presence of strong fields, the usefulness of a description in terms of positive and negative-energy states will be lost.

These same ideas may be carried over to the case of more general interactions. If there are any advantages to be accrued by the employment of two-component wave functions to describe states of positive and negative energy for a free particle, we may expect these advantages to be present still in the case of weak interactions, but they can scarcely survive in the case of strong interactions. To define *weak interactions* in a more quantitative fashion, we prescribe that the interaction terms have no time Fourier components comparable with or greater than m, so that no transitions between free-particle states with energy differences of this latter order of magnitude are possible; and that the interaction terms have no space Fourier components comparable with or greater than m, so that no transitions between free-particle states with momentum differences of this order of magnitude are possible. Under these conditions, if the initial state is one in which no high momentum states occur, then the momentum of the particle will remain small compared to m under the influence of the interactions. In these circumstances we have essentially a non-relativistic problem, and hence it will be in the domain of *non-relativistic problems* that a representation by means of two-component wave functions may be expected to be of value.

In the presence of interaction it no longer appears possible to make a single, simple, canonical transformation to a representation in which the Hamiltonian is free of odd operators, but instead one can always make a sequence of transformations, each of which eliminates odd operators from the Hamiltonian to one higher order in the expansion parameter[10] $1/m$. In this way one obtains in the new representation a Hamiltonian free of odd operators which is an infinite power series in powers of $1/m$. While it can hardly be expected that this series is convergent, the series is presumably an asymptotic or semi-convergent series in the sense that the sum of a finite number of terms of the series is a better-and-better approximation to the true Hamiltonian, the larger the value of m, provided the magnitude of the interaction remains fixed. The usefulness of the series is then contingent on the interactions being sufficiently weak compared to m in just the sense described above for non-relativistic problems.

Turning our attention now to the explicit calculation in the case where a Dirac particle is subject to interactions, we note first that the Hamiltonian operator can always be put in the form[3]

$$H=\beta m+\mathcal{E}+\mathcal{O}, \tag{29}$$

where $\mathcal{E}$ is an even operator and $\mathcal{O}$ is an odd operator, both of which may be explicitly time-dependent. We assume (and this is ordinarily the case) that $\mathcal{E}$ and $\mathcal{O}$ are of no lower order in $(1/m)$ than $(1/m)^0$. Consider now the canonical transformation generated by the Hermitian operator

$$S=-(i/2m)\beta\mathcal{O}. \tag{30}$$

We note that if the operator S may be regarded as small, then we can make an expansion in powers of $(1/m)$ of the Hamiltonian in the new representation:

$$\begin{aligned} H'&=e^{iS}He^{-iS}-ie^{iS}\left(\frac{\partial e^{-iS}}{\partial t}\right)\\ &=H+\frac{\partial S}{\partial t}+i\left[S, H+\frac{1}{2}\frac{\partial S}{\partial t}\right]\\ &\quad+\frac{i^2}{2!}\left[S,\left[S, H+\frac{1}{3}\frac{\partial S}{\partial t}\right]\right]+\cdots. \end{aligned} \tag{31}$$

Carrying this out explicitly for the Hamiltonian above and retaining terms only to order $(1/m)^2$, we obtain

$$\begin{aligned} H'=\beta m+\mathcal{E}+\frac{\beta}{2m}\mathcal{O}^2-\frac{1}{8m^2}\left[\mathcal{O},[\mathcal{O},\mathcal{E}]+\frac{\partial\mathcal{O}}{\partial t}\right]\\ -\frac{1}{2m}\beta\frac{\partial\mathcal{O}}{\partial t}+\frac{\beta}{2m}[\mathcal{O},\mathcal{E}]-\frac{1}{3m^2}\mathcal{O}^3+\cdots, \end{aligned} \tag{32}$$

[10] The expansion parameter may equally well be regarded as $1/c$ where c is the velocity of light. The actual dimensionless expansion parameters are the operators $(\hbar/mc)\nabla$ and $(\hbar/mc^2)(\partial/\partial t)$. From this, one sees that the successive terms of the series will decrease rapidly in magnitude if the interaction potentials do not vary appreciably in space over a Compton wave-length of the particle and in time over the period required for light to travel a Compton wave-length. This is equivalent to the restrictions imposed above on the space and time Fourier components of the interaction.

where we have made use of the fact that β commutes with all even operators and anticommutes with all odd operators in the Dirac theory. Remembering that the product of two even or two odd operators is an even operator and that the product of an odd and an even operator is an odd operator, we see that we have removed by this transformation all odd operators of order $(1/m)^0$ from the Hamiltonian although odd operators of order $(1/m)$ and higher remain and are given in the second line.

From this we see that by a sequence of further canonical transformations, the generator of the transformation at each step being chosen to be

$$S=-(i/2m)\beta \text{ (odd terms in Hamiltonian of lowest order in } 1/m), \quad (33)$$

we may successively remove odd terms from the Hamiltonian to any desired order in $(1/m)$. If we continue this process indefinitely, we obtain, as mentioned earlier, a Hamiltonian which is an infinite power series in $(1/m)$ completely free of odd operators.

Let us now apply this method to the case where the Dirac particle interacts with an external electromagnetic field. In this case the Hamiltonian is given by

$$H=\beta m-e\varphi+\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A}), \quad (34)$$

where φ and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field evaluated at the *position* of the particle. In accordance with the procedure outlined above, we then first make the canonical transformation generated by

$$S_1=-\frac{i}{2m}\beta\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A}),$$

obtaining then for the new Hamiltonian,

$$H_1=\beta m-e\varphi+\frac{\beta}{2m}(\mathbf{p}-e\mathbf{A})^2-\frac{e}{2m}\beta\boldsymbol{\sigma}\cdot\mathbf{H}-\frac{e}{4m^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times(\mathbf{p}-e\mathbf{A})+\frac{e}{8m^2}\operatorname{div}\mathbf{E}-\frac{ie}{2m}\beta\boldsymbol{\alpha}\cdot\mathbf{E}-\frac{1}{3m^2}[(\mathbf{p}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{H}]\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{H})+\cdots,$$

to terms of order $(1/m)^2$. Following with the two canonical transformations generated by

$$S_2=-\frac{i}{2m}\beta\left\{-\frac{ie}{2m}\beta\boldsymbol{\alpha}\cdot\mathbf{E}\right\},$$

and

$$S_3=-\frac{i}{2m}\beta\left\{-\frac{e}{4m^2}\beta\boldsymbol{\alpha}\cdot\frac{\partial\mathbf{E}}{\partial t}-\frac{1}{3m^2}[(\mathbf{p}-e\mathbf{A})^2-e\boldsymbol{\sigma}\cdot\mathbf{H}]\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A})\right\},$$

we eliminate odd operators from the Hamiltonian of order $(1/m)$ and $(1/m)^2$ respectively and obtain the Hamiltonian

$$H_3=\beta m-e\varphi+\frac{\beta}{2m}(\mathbf{p}-e\mathbf{A})^2-\frac{e}{2m}\beta\boldsymbol{\sigma}\cdot\mathbf{H}-\frac{e}{4m^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times(\mathbf{p}-e\mathbf{A})+\frac{e}{8m^2}\operatorname{div}\mathbf{E}+\cdots, \quad (35)$$

free of odd operators to order $(1/m)^2$. In the above $\mathbf{H}=\operatorname{curl}\mathbf{A}$ and $\mathbf{E}=-\nabla\varphi-\partial\mathbf{A}/\partial t$ are the magnetic and electric field strengths evaluated at what is now essentially the *mean-position* of the particle. By further canonical transformations we could remove odd operators of still higher order, but we shall limit our discussion of the Hamiltonian in the new representation to terms of order $(1/m)^2$.

Since the Hamiltonian (36) is free of odd operators, its eigenfunctions are two-component functions corresponding again to positive and negative energies. For positive energies, the Schrödinger equation is

$$\left\{m-e\varphi+\frac{1}{2m}(\mathbf{p}-e\mathbf{A})^2-\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{H}-\frac{e}{4m^2}\boldsymbol{\sigma}\cdot\mathbf{E}\times(\mathbf{p}-e\mathbf{A})+\frac{e}{8m^2}\operatorname{div}\mathbf{E}\right\}\Phi=i\frac{\partial\Phi}{\partial t}, \quad (36)$$

which will be recognized as essentially the Pauli equations for a non-relativistic particle of spin $\frac{1}{2}$ interacting with the electromagnetic field. The presence of the terms corresponding to the interaction of the anomalous magnetic moment of the particle with the magnetic field and the spin-orbit interaction is evident. The term proportional to $\operatorname{div}\mathbf{E}$ is a well-known correction[11] to the Pauli theory arising from the Dirac theory, and it is responsible for a relativistic shift of the S levels in the hydrogen atom (not to be confused with the Lamb-Retherford line shift).

The reason for the explicit appearance of these additional interaction terms (as well as further terms of higher order in $1/m$) in the new representation can now be understood in the light of our physical interpretation of the transformation to the new representation. In the old representation the Dirac particle interacted with the electromagnetic field only at its *position*. However, a particle which in the old representation was located at a point is in the new representation spread out over a region of dimensions of the order of a Compton wavelength in the space of its *mean-position variable*, $\mathbf{X}'=\mathbf{x}$. But in the new representation the interaction between the particle and the electromagnetic field is expressed in terms of the values of the electromagnetic-field quantities at its *mean-position*. Hence, one must expect a series of correction terms of the nature of a multipole

[11] C. G. Darwin, Proc. Roy. Soc. 118A, 654 (1928).

expansion of the field, since the particle is actually spread out over a finite region in the space of its *mean-position variable.* From this point of view the explicit appearance of the magnetic-moment interaction and the accompanying spin-orbit interaction in the Hamiltonian is to be expected. In fact, the term in div**E**, which has previously been regarded as of rather mysterious origin, can now also easily be understood since it comes from the fact that the electric charge in the new representation is also spread out over a finite region. In a static potential, the particle then moves according to a suitable average of the potential over this region. But in lowest order such an averaging process is known to lead to a term proportional to the Laplacian of the potential and this is just the character of the term in div**E**.

(In employing a finite number of terms in a Hamiltonian such as (35) it must be remembered that wave functions, transition matrix elements, and expectation values of operators computed from the Hamiltonian are only correct to terms of the order in $(1/m)$ to which terms in the Hamiltonian are retained. Thus in the case of (36) it would not be consistent to retain only terms of order $(1/m)$ in the Hamiltonian and then to employ terms of this order in second-order perturbation theory where they generate terms of order $(1/m)^2$.)

The method employed above for reducing the Hamiltonian to non-relativistic two-component form for the case of interaction with an external electromagnetic field can be employed generally for interaction of a Dirac particle with any type of external field, such as various types of meson fields. With meson fields one obtains, in different cases, various types of spin interaction with the meson field and also various types of spin-orbit coupling terms such as have been employed in discussions of spin-orbit coupling in nuclei. The discussion of the reduction to non-relativistic form in the case of the many-particle theory is left for a later communication.

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The Nucleon Magnetic Moment in Meson Pair Theories

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The contribution to the nucleon magnetic moment from an interaction of the nucleon with a spinor or scalar meson pair field is calculated. In both cases it is found to be logarithmically divergent.

THE covariant formulation of the pseudoscalar meson theory[1] together with the concepts of mass and charge renormalization[2] were applied by Case[3] to the computation of the anomalous magnetic moment of nucleons. He showed that finite results are then obtained. This is essentially due to the fact that one is now able to isolate and incorporate[2] the divergences into the mass and charge of the nucleon field thus leaving a convergent expression for the magnetic moment. This separation of the reactive terms represents an improvement over the previous treatment of this and related problems.[4] On the other hand, divergence difficulties are still encountered in the magnetic moment calculation based on the vector meson theory with tensor coupling.[5]

It is the aim of this note to report on the application of the renormalization program to the computation of nucleon magnetic moments by assuming a meson pair interaction (containing no derivatives of the fields) between the heavy particles. Two cases have been considered for the meson field, namely, the scalar (or pseudoscalar, which here amounts to the same) and the

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2 See F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

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4 Finite results for the nucleon magnetic moment were previously obtained by J. M. Jauch (Phys. Rev. **63**, 334 (1943)) in the conventional theory by using the lambda-limiting process; the nucleon field was treated non-relativistically.

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