Phys 735: Homework X

due March 13, 2009

1. Momentum space Dirac Equation

The normalized, coordinate-space plane wave solution of the Dirac equation is given as

$$\Phi_s^{(\pm)}(x) = e^{\pm i p_\mu x^\mu} u_s^{(\pm)}(\mathbf{p}), \tag{1}$$

where the space-time dependence is in the exponential, the 4D spin dependence in the spinor, and $\mathbf{p}$ is *not* an operator.

When this plane wave solution is substituted into the coordinate-space Dirac equation one obtains the *momentum-space Dirac equation* for the free particle spinors $u_s^{(\pm)}(\mathbf{p})$:

$$(\gamma^\mu p_\mu \mp m) u_s^{(\pm)}(\mathbf{p}) = 0 \tag{2}$$

(a) *(3pts)* Prove the relation (2) starting from (1) for $u_s^{(+)}(\mathbf{p})$ and $u_s^{(-)}(\mathbf{p})$.

(b) *(3pts)* The adjoint spinor $\bar{u}_s^{(\pm)}(\mathbf{p}) = u_s^{(\pm)\dagger}(\mathbf{p})\gamma^0$ satisfies the transpose Dirac equation

$$\bar{u}_s^{(\pm)}(\mathbf{p})(\gamma^\mu p_\mu \mp m) = 0 \tag{3}$$

Prove the above equation (3) for both cases.

2. Projection Operators

When manipulating solutions of the Dirac equation, it is often useful to combine groups of them into a simple operator. To do this, define the projection operators

$$\Lambda_\pm(p) = \frac{\pm\gamma^\mu p_\mu + m}{2m} \tag{4}$$

(a) *(3pts)* Show that these operators are projection operators, i.e. fulfill

$$\begin{aligned} \Lambda_\pm^2(p) &= \Lambda_\pm(p) \\ \Lambda_-\Lambda_+ &= \Lambda_+\Lambda_- = 0 \\ \Lambda_+ + \Lambda_- &= \mathbf{1} \end{aligned} \tag{5}$$

(b) *(3pts)* Show that these operators have the desired effect when applied on the Dirac spinors:

$$\begin{aligned} \Lambda_\pm(p) u_s^{(\pm)}(\mathbf{p}) &= u_s^{(\pm)}(\mathbf{p}) \\ \Lambda_\mp(p) u_s^{(\pm)}(\mathbf{p}) &= 0 \end{aligned} \tag{6}$$