Phys 735: Homework IV

due January 30, 2009

2. Scattering from a Square Well
A spinless particle of mass m, energy E scatters through angle θ in an attractive spherical square-well potential $V(r)$:

$$V(r) = \begin{cases} -V_0, & 0 < r < a \\ 0 & r > a \end{cases}$$

where $V_0 > 0$.

(a) *(5 p)* Establish a relation among the parameters V_0, a, m, and universal constants, which guarantees that the cross section vanishes at zero energy $E = 0$. This will involve a definite, but transcendental equation,which you must derive but need not solve numerically. For parameters meeting the above condition, the differential cross section, as $E \to 0$, will behave like

$$\frac{\partial \sigma}{\partial \Omega} \xrightarrow{E \to 0} E^{\lambda} F(\cos\theta).$$

(b) *(2 p)* What is the numerical value of the exponent λ?

(c) *(3 p)* The angular distribution function $F(\cos\theta)$ is a polynomial in $\cos\theta$. What is the highest power of $\cos\theta$ in this polynomial?

2. Resonance for a Square Well *(6 p)*
Use the exact solution for the spherical square well of depth V_0 to find the condition on the potential for the s-wave ($l = 0$) to produce a resonance at an energy $E_0 = (\hbar^2 k_0^2)/2m$. Can you relate your answer to bound state energies in the well?

3. Zero-Range Potential
Consider the $l = 0$ free Schrödinger equation and define a solution ψ by the boundary condition that the $r \to 0$ limit of the logarithmic derivative of $r\psi$ is a constant, i.e.

$$\lim_{r \to 0} \left[\frac{1}{r\psi} \frac{d}{dr}(r\psi) \right] = \text{a constant independent of energy} \tag{1}$$

The resulting problem mimics a potential, and is a solution for a *zero-range* potential. For such a 'potential', both the scattering amplitude and the bound state energy (if a bound state exists) are determined by a single parameter, namely the constant of Eq.(1).

Now, consider a zero-range potential which is known to have a single $l = 0$ bound state with energy

$$E_0 = -\frac{\hbar^2\alpha^2}{2m}$$

(a) *(5 p)* Find the bound state wave function $\psi_0(\vec{r})$ as well as the scattering solutions $\psi_k(\vec{r})$.

(b) *(6 p)* Show that the solutions are orthogonal and can be normalized to satisfy

$$\begin{aligned}
\int d^3r |\psi_0(\vec{r})||^2 &= 1 \\
\int d^3r \psi_0(\vec{r})\psi_k(\vec{r}) &= 0 \\
\int d^3r \psi_k^*(\vec{r})\psi_q(\vec{r}) &= \delta(\vec{k}-\vec{q})
\end{aligned} \tag{2}$$

Hint: It may be useful to recall that an integral of the form

$$\int d^3k e^{ikr}$$

has to be regularized and is defined by

$$\begin{aligned}
\int_0^\infty dk e^{ikr} &= \lim_{\epsilon\to 0^+} \int_0^\infty dk e^{ik(r+i\epsilon)} \\
&= -\lim_{\epsilon\to 0^+} \frac{1}{i(r+i\epsilon)} \\
&= i\left[\mathcal{P}\frac{1}{r} - i\pi\delta(r)\right]
\end{aligned} \tag{3}$$

where $\mathcal{P}$ denotes the Cauchy Principal value.