Physics 735: Exam II (Take Home - Makeup)

March 24, 2009
Due March 25, 2009, 17:00 (5 pm)

1. New Diagonal Form of the Dirac Equation

We can introduce a Foldy-Wouthuysen transformation which will completely eliminate the lower component from the free positive energy solutions and the upper components from the free negative energy solutions.

The advantage of such a representation is that it allows us to regard the mixing of upper and lower components as a dynamical consequence of the interaction, whereas the free Dirac equation is fully diagonalized.

A unitary transformation which accomplishes this is

$$U = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} 1 & \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} & 1 \end{pmatrix} \tag{1}$$

(a) *[4 pts]* Show by direct computation that U is unitary, i.e. $U^\dagger U = \mathbf{1}$.

(b) *[2 pts]* Show that

$$\begin{aligned} U\ u(\vec{p}, s) &= \sqrt{2E} \begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix} \\ U\ v(-\vec{p}, -s) &= \sqrt{2E} \begin{pmatrix} 0 \\ -i\sigma_2 \chi^{(-s)} \end{pmatrix}. \end{aligned} \tag{2}$$

where

$$\begin{aligned} u(\vec{p}, s) &= \sqrt{E_p + m} \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \end{pmatrix} \chi^{(s)} \\ v(\vec{p}, s) &= \sqrt{E_p + m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \\ 1 \end{pmatrix} \left[-i\sigma_2 \chi^{(s)}\right]. \end{aligned} \tag{3}$$

Here $\chi^{(s)}$ is the usual Pauli spinor.

(c) *[5 pts]* Show that the transformed free Hamiltonian, $H_0 = \vec{\alpha} \cdot \vec{p} + \beta m$ is given by

$$U H_0 U^\dagger = \beta E = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & -E \end{pmatrix}. \tag{4}$$

In this form the free Dirac equation takes a form similar to the 2-component Klein-Gordon equation.

(d) *[15 pts]* Show that in this representation the Dirac equation with electromagnetic interactions (given usually with $H = \vec{\alpha} \cdot (\vec{p} - \vec{V} + \beta m + V^0)$ can be written

$$i\frac{\partial}{\partial t}\psi = \beta E\psi + (H_1 + \beta H_2 + \vec{\sigma} \cdot \vec{H}_3 + \beta\vec{\sigma} \cdot \vec{H}_4 + \vec{\alpha} \cdot \vec{H}_5 + i\beta\vec{\alpha} \cdot \vec{H}_6 + \gamma^5 H_7)\psi \tag{5}$$

Proceed by introducing

$$\begin{aligned} \vec{D} &= \frac{\vec{p}}{E+m} = \frac{-i\nabla}{E_\nabla + m} \\ \theta &= \sqrt{\frac{E+m}{2E}} \end{aligned} \tag{6}$$

and show that the functions $H_i (i = 1, 7)$ are given by

$$\begin{aligned} H_1 &= \theta(V^0 + D_j V^0 D_j)\theta \\ H_2 &= -\theta(\vec{V} \cdot \vec{D} + \vec{D} \cdot \vec{V})\theta \\ H_3^k &= \theta(i\epsilon_{ijk} D_i V^0 D_j)\theta \\ \vec{H}_4 &= -\theta(i\vec{V} \times \vec{D}) + i(\vec{D} \times \vec{V}))\theta \\ \vec{H}_5 &= \theta(-\vec{V} + (\vec{D} \cdot \vec{V})\vec{D} + \vec{D}(\vec{V} \cdot \vec{D}) - \sum_j D_j \vec{V} D_j)\theta \\ \vec{H}_6 &= \theta(iV^0\vec{D} - i\vec{D}V^0)\theta \\ H_7 &= \theta(i\epsilon_{ijk} D_i V_j D_k)\theta \end{aligned} \tag{7}$$

2. Massless Dirac particle in a 1D square well

A massless spin-1/2 particle moves in a one-dimensional square well potential of depth $V_0 \leq 0$ and a width a. Assume that the potential is a scalar potential (i.e. behaves as βV_0). Divide the real axis into three domains:

$$\begin{aligned} I: &\quad z < -a/2 \\ II: &\quad -a/2 \leq z \leq a/2 \\ III: &\quad z > a/2 \end{aligned} \tag{8}$$

(a) *(4 pts)* Write down the correct Dirac equation for the motion of this particle.

(b) *(4 pts)* Show that the equation found in (a) is invariant under parity transformation, i.e. the transformation $z \to -z$

(c) *(6 pts)* Solve the equation for the ground state and wave function of the trapped particle. Take the limit $V_0 \to \infty$, and sketch the solution for this case. Comment on any interesting feature which this solution might show.