Introduction.

Syllabus Can we meet Fridays, instead of Thurs?

General thoughts.

This class is usually taught by theorists. I am not a theorist. But I use QM in my work a lot and am very comfortable with it. Point is I am not quite familiar with all the things that theorists find useful to concentrate on. Example Charlotte told me everyone should use del notation/ϵ_{ijk} I agree. Everyone should work out the group multiplication for all the angular momentum states. I don't agree so maybe we won't concentrate on that quite as much.

In fact last year we didn't make it to Chapter 3: (this year is different, this year we must.) Gives you an idea of the pace of the course. This course will cover approximately Ch-1-2 and some decent fraction of Ch 3. of Sakurai. Then there will be several other special topics

Therefore do not expect to move through Sakurai quickly. We will go very slowly through it. I recommend reading the entire 1st Chapter quickly, then for my "reading assignments" go back and carefully restudy.

How fast we can move will be partially dependent on you guys. I will be quizzing you along the way, both formally and informally.

I'd like to cover somethings related to my research, of Heavy Ion physics, but probably the most relevant thing is scattering which we won't get to in this course.

Didn't make it there last year though: Instead I'd like to teach a little about quantum entanglement and perhaps quantum computation.

Still, we will approach lectures a little differently. We will try having a day (Friday) where we do problems in groups.

The Web:

There is a lot out on the web, including solutions to many problems in Sakurai. Remember about cheating. Personally I don't care as long as you understand the solution.

I recommend Wikipedia for many subjects. I have been referring to it for preparation for this class. You can find quite a bit of detail on it. e.g. Mathematical Definitions.

I will put links occasionally, some for reference and some will be required reading.

At least at one point during the semester (possibly 2), I may want to meet with some or all of you individually, sometime after the first few weeks of the quarter. These will be 15-30 minute "conferences" and we will discuss your plans,your performance in the class, specifically any homework problems or midterm problems you may not have done so well on.

I will **call on people** specifically to answer questions sometimes. This will be part of your participation grade. I will go through the list of names in alphabetical order so I will let you know when your turn is up, and you should try to be in class those days.

Reading assignment Sakarai 1.1

I. Some Topics from Linear Algebra to Review

Note: First 1.1- ~2.2 (first 8 sections of Sak), formalism introduced. It is very mathematical, in large part, just **like** an extension of standard linear algebra to include complex vector spaces **(**where **matrices** are generalized to **operators)**. Thus I find it very useful to review some linear algebra.

1) Properties of (usually nxn) Matrices

System of linear equations: (e.g.)

ax + by = e

cx+dy = f

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y_2 \end{bmatrix}$$

represented by

Ax = y

(matrix multiplication)

2) Improved notation for original equations: ("Coordinate free rep" best?)

Aijxj = yi ie yi, xi → represent **vectors**

(row i then column j) !!!

makes it **easy to generalize to higher dim**:

- ANY number of dim, incl. > 3 (# eq + variables increase too)

Alternative view of above: A is transformation for any vector **x** to new vector **y.**

One way to solve (for x) such an equation is by *inverting* A.

Finding the ***inverse*** A^{-1} such= that $A^{-1}A = 1$ = I identity matrix

then $\mathbf{x} = \mathbf{A^{-1}y}$

For a matrix to be invertible it's determinant must not = 0.

3) Deterimants

From Wikipedia "Determinant",

"The fundamental geometric meaning of a determinant is a scale factor for measure when A is regarded as a linear transformation." It is the scale factor for *n*-D volumes before and after the transformations .

Property of Determinant: det(AB) = det(A)*det(B) (Easier to compute, e.g. if one has the "LU" [Lower/Upper] Decomposition)

Note oddity: central to the **properties of matrices**, → **determinant very important** in Linear Algebra, not ostensibly for **QM**. (focus on **instead algebraic properties**).

4) Non-zero Determinants

Conversely if

Ax = 0

for non-zero x, its determinant must be 0.

What's that called? (It has a non-zero/null ***kernel*** ie which x is part of).

Kernel: space for which this is true, Range

Lecture 1/5/10

Notes uploaded (pages)

Quiz today (?)

Reading for tomorrow: Sakuarai 1.1

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5) Orthogonality / Symmetric Matrices

If A is symmetric A = transpose (A) = A^T $A^T_{ij} = A_{ji}$

If A has columns that are orthogonal, A called orthogonal

If A has columns that are ortho*normal*, $A^T = A^{-1}$

6) Non- square Matrices, Row Vectors vs Column Vectors

Can view column vectors **x** = $\begin{pmatrix} a \\ b \end{pmatrix}$ as nx1 matrices themselves

Then x^T is **row vector $\mathbf{x^T}$ = (a b)**

Then with Normal Matrix Multiplication:

→ dot product: (inner product) **x· v** → $x^Tv = (a\ b)\begin{pmatrix} c \\ d \end{pmatrix} = ab + cd$ = number

→ **outer product** $\begin{pmatrix} a \\ b \end{pmatrix}(c\ \ d) = ?\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$ = matrix

(Any dimension vector works) as above (-----) (|)→ / (|)(-----)

n x m: (n : rows, m columns) (n ≠ m) →

-Remember any [n x m] [m x p] is allowed (no restrictions on n or p) !

-Important example: projection matrices

7) Projection on Subspaces (Often forgotten from L.A.:)

The projection of a vector x onto a subspace W defined by the orthonormal vectors $v_1..v_n$ is

$Proj_W$ x = $\mathbf{MM^Tx}$

where the m x n (m =dim of the vectors) **M = [v1|…|vn]**.

Example (1) (1,0,0), (0,1,0)

BY DEFINITION n < m in order for us to be taking a projection. (thus M is not a square matrix)

Why does M have this form? (Also good way to remember: a pneumonic)

pneumonic: trick to remember things -- word I use a lot

Think of projection onto a 1-D subspace: onto the line of a vector v:

$$vv^T x = (v \cdot x)\, \mathbf{v}$$

Example (1) (1,0,0), (0,1,0)

Back to above Example (2)

8) Gram-Schmidt Method for finding orthogonal bases

(Gram-Schmidt 'sche Orthogonalisierunsverfahren)

If we have *any* (e.g. non-othogonal) basis $v_1, \ldots v_n$ spanning a space, we can use it to find an orthogonal (or orthonormal) set $w_1 \ldots w_n$ recursively with the following steps…

1) $w_1 = v_1/|v_1|$.

2) $w_2 = v_2 - \mathbf{Proj}_{w1}v_2$ (/ | $v_2 - \mathbf{Proj}_{w1}v_2$| to normalize)…

…

n) $w_n = v_n - \mathbf{Proj}_E v_n$ (**/ | $v_n - \mathbf{Proj}_E v_n$|** to normalize)

Where n is the number of basis vectors = dim of space and E is the subspace spanned by the n-1 preceding vectors w_n.

9) Eigenfunction Equations / Bases

If

$$\mathbf{Ax} = \lambda\, \mathbf{x}$$

x is an eigenvector of A and λ is an eigenvalue

The latter are found by

$$(\mathbf{A}-\lambda \mathbf{I})\,\mathbf{x} = 0$$

or

$$\det(\mathbf{A}-\lambda \mathbf{I}) = 0$$

If A is symmetric the eigenbasis is ***orthonormal.*** (I double checked this, in class I wrote orthogonal, but in fact I should have written orthonormal--easy to forget this!)

10) Diagonalization

To "diagonalize" a matrix A we form the diagonal matrix

$$\mathbf{D = S^{-1}AS}$$

where S is the matrix whose columns are the eigenvectors of A

For symmetric matrices B this can be written

$$\mathbf{D = S^{T}BS}$$

which is another way of stating that symmetric A has orthonormal basis

II. Review of Introductory Quantum Mechanics (Wave Mechanics)

Just so we are on the same page…

Postulates of Liboff ***underlined***

Wave Functions ↔ States Complex-valued "Scalar Fields " of position vector **x**
Related to Probability to be at position **x**

Probability density $\propto$ dx (Ψ)(Ψ*)

$$\int dx\ (\Psi)(\Psi^*)$$

is $\propto$ the actual probability density as a function x. If normalized, = to prob density

E.g. 1-D Ψ(x)

* means complex conjugate (c.c.)

Digression: Quick Review of c.c.
This means subtract 2i * the imaginary part. "Replace every *i* with *–i* ". (?)

$z = ie^{ia} = i(\cos a + i \sin a) = -\sin a + i \cos a$
$z^* =? -ie^{-ia} = -i(\cos(-a) + i\sin(-a)) = -i \cos a - i(-i)\sin(a))$ yes of course: $(ab)^* = a^*b^*$

From Wikipedia "Complex Conjugate": " In general, if ϕ , is a holomorphic function whose restriction to the real numbers is real-valued, and ϕ (z), is defined, then $\phi(z^*) = (\phi(z))^*$ "

This can also be extended to $(\phi(z,w,u,\ldots))^* = \phi(z^*,w^*,u^*,\ldots)$

You should be familiar with complex analysis, although we won't use residues or anything like that.
Quick practice what's the c.c. of uv^zw^2
$u^*(v^*)^{z^*}w^*w^*$

Observables ↔ operators acting on wave functions, usually differential operators

These are perhaps the most important concerning wave mechanics:

x

$p_x = -i\hbar\ d/dx$

$$H_{free} = p^2/2m = (1/2m)\, d^2/dx^2$$

$$H = (1/2m)\, d^2/dx^2 + V(x)$$

Measurement of observable A forces ψ to become = to eigenfunction of operator A

Eigenfunction equation for H is called the Time Independent Shrodinger Equation (TISE)

$$H\psi(x) = E\psi(x)$$

Given the above differential form of H, this is just a 2nd order, linear ODE. Because of the Sturm-Liouville theorem, the general set of solutions for ψ will be a set of functions ψ_i "spanning" the entire space of functions of x. They are then called *complete*.

The theorem also states that they are orthogonal

$$\int dx\, (\Psi_a)(\Psi_b^*) = 0 \quad \text{unless } a = b$$

Thus any ψ(x) (any f(x)!) can always be expanded in terms of this basis of eigenfunctions ψ_i.

Expectation Value :

$$\langle C \rangle = \int dx\, \Psi^*\, C\, \Psi$$

Note often (e.g. if you can't remember the normalization constants) it is useful to perform the expectation value calculation as

$$\langle C \rangle = \frac{\int dx\, \Psi^*\, C\, \Psi}{\int dx\, \Psi^*\Psi}$$

In general in Wave Mechanics we can find the projection of any Wave Function Ψ (x) on any other χ (x)

$$\int \text{Proj}_\Psi\, \chi = dx\, \Psi^*\chi \quad \text{(more like dot product)}$$

This represents the probability if the state is in Ψ that it would be found in state χ e.g. if χ were the eigenfunction of another observable, the probability of measuring the eigenvalue of χ .

The Time-Dependent Shrodinger Eq Determines time evolution of the wavefunction

$$i\hbar d/dt\,(\psi) = H\,\psi$$

For a time-independent H, this implies that each eigenfunction of H (ψ_i) has time dependence given by the function f(t) = exp(-iEit).

It is instructive to remember how this solution comes about, as an important technique in solving differential equations
If H is time independent, H = H(x)

$i\hbar d/dt\,(\psi) = H(x)\,\psi$ ➔ g(t)y = h(x)y

Any time you can put a differential eq into this way, a ***separable*** solution will work.

$$\psi = \Psi(x)f(t)$$

$$\frac{i\hbar\,\partial_t f(t)}{f} = \frac{H(r)\Psi(r)}{\Psi}$$

$$\frac{i\hbar\,\partial_t f(t)}{f} = \text{const} = \frac{H(r)\Psi(r)}{\Psi}$$

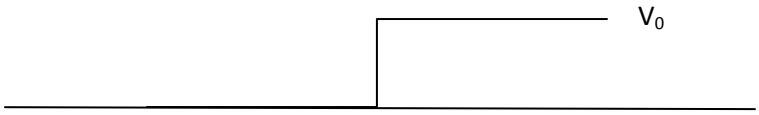
The rest of elementary Wave Mechanics then relies on solving the TISE, finding eigenfunctions for different forms of the potential term V(x) or in 3-D V(**x**).

E.g.

Free Particle: $V = 0$, $H = p^2/2m$

Solution: $\psi = A \exp(ikx - \omega t)$ $E_k = h^2k^2/2m$

Half free particle on a constant potential "step" (region of constant potential V_0)



Solution in the right hand region is

$$\psi = A \exp(-\kappa x) \text{ with } E_K = V_0 - h^2\kappa^2/2m$$

when E is less than V_0

Angular Momentum

The angular momentum operator can be defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = r \times i\hbar \nabla$$

In 3 dimensions it is often convenient to write the hamiltonian in terms of L, and Spherical coordiinates,

$$H = -\frac{\hbar^2}{2m}\frac{1}{r^2}r^2\frac{\partial}{\partial r} + \frac{\hbar^2\vec{L}^2}{2mr^2} + V(r)$$

The solutions to this Hamiltonian involve Spherical Harmonics $Y_{lm}(\theta, \phi)$

We'll talk a lot more about angular momentum later, and it won't be this ugly now. Point is before you dealt more w/ these solutions from a functional point of view now we will concentrate mostly on their algebraic properties

Approximation Methods: (WKB) **SKIP – we will cover this in class**

For non-constant V(x), even for the Simple Harmonic Oscillator ($V(x) = kx^2$), the form of the solutions $\psi(x)$ are not often easy to find.

The SHO has an easy trick to find it's Eigenvalue *Energies*, which we will discuss at length in this class. But the functional form of its ***wave functions*** "Hermite Polynomials" are not easy to remember!

However in the case of a **slowly varying V(x)**

For now we will ignore what constitutes "slowly varying" or how slowly it must vary. Let's just say there is some range of the input parameters to the problem for which this can be satisfied.

Then we can look for solutions that have a similar form as the constant potential solution. There we had $V = V_0$ as the solution

$$\psi_{WKB} = A \exp(i \int k(x)\ dx)$$

$$\text{w/}\quad k(x) = 2m\sqrt{E - V(x)}$$

We will have occasion to return to such a solution…

It will be useful in the future to see how this comes about.

First we assume the basic form $\psi = A\exp(iS(x)/\hbar)$

$$-i\hbar\, d^2S/dx^2 + (dS/dx)^2 = 4m^2 = \sqrt{E - V(x)} \qquad (0.1)$$

The crux is to expand the solution around $\hbar$ itself.

[Taylor series expansion: $f(x-x_0) = f(x_0) + (x-x_0)f'(x_0) + (x-x0)^2/2\,(f''(x0)) + \ldots$]

$$S = S_0(x) + \hbar\, S_1(x) + \hbar^2/2\, S_2(x) + \ldots$$

Example of approximation expansions using essentially any parameter often in Quantum Mechanics. In this case we say $\hbar$ is < 1,

Not true in some small length units: "semiclassical" approximation $\hbar \rightarrow 0$

In particle physics $\hbar = 1$ so this approximation is by definition unusable

Simple substitution of expansion into (0.1) and grouping all terms to the LHS we get an equation

$$F_0 + \hbar F_1(x) + \hbar^2 F_2(x) \ldots = 0$$

where $F_0 = (dS_0/dx)^2 - 4m^2 (E-V(x))^2 = 0$ for example.

The key is that every term in this series must vanish independently. Thus we can set each term = 0;
$F_0 = 0 \rightarrow dS_0/dx = k(x)$ k given above.

First order simple ODE→ And thus we have our solution form.

Time Independent Perturbation Theory

Last reminder of Intro QM/Wave Mechanics…

A very similar technique is used for a much more general approximation scheme.

If we can separate H into a form $H_0 + \lambda H'$ where H_0 is one that we know the eigenfunctions for

H' is called the **perturbation (→)** to the Base Hamilitonian.

e.g. the most common thing (e.g. in Quantum Field Theory Scattering Calc's –note that in that case we are talking about a completely different mathematical Hamiltonian.) is to think of the entire V(x) as a "small" so that H_0 in that case is the **free particle Hamiltonian.**

ie $H = p^2/2m + V(x) \rightarrow p^2/2m + \lambda V'(x)$ (it would be interesting to see the SHO handled this way—perhaps we will come back to the idea when we learn about the SHO.)

Anyway making a similar expansion to the WKB this time in λ

$$e.g. \quad \psi = \phi_0 + \lambda \phi_1(x) + \lambda^2/2 \ \phi_2(x) ; \qquad En = E0 + \lambda E1 + \ldots$$

And setting each factor of λ n = 0 independently and taking the expectation value of each equation we find that we can represent the MODIFICATIONS to the eigenfunctions of H0 in terms of a "mixing" of the other eigenfunctions due to the perturbation H'.

$$\Psi = \psi_n + \sum_{in} \frac{H'_{in}}{E_n^0 - E_i^0}$$

Where mixing is just a projection of one Wave Function Ψ_i on another:

$$H'_{ni} = \int dx\, \psi_n{}^* H' \psi_i$$

The new Energies (MODIFICATIONS to each states energy) are

$$E_n{}^{new} = E_n{}^0 + \langle H' \rangle$$

Lecture 1/8/2010:

-Problem Set Due Next Fri? (Reduced: TBA tonight)

-Class is NOT Canceled Wed: Prakash will cover.

-Quiz results: solutions: Index notation?

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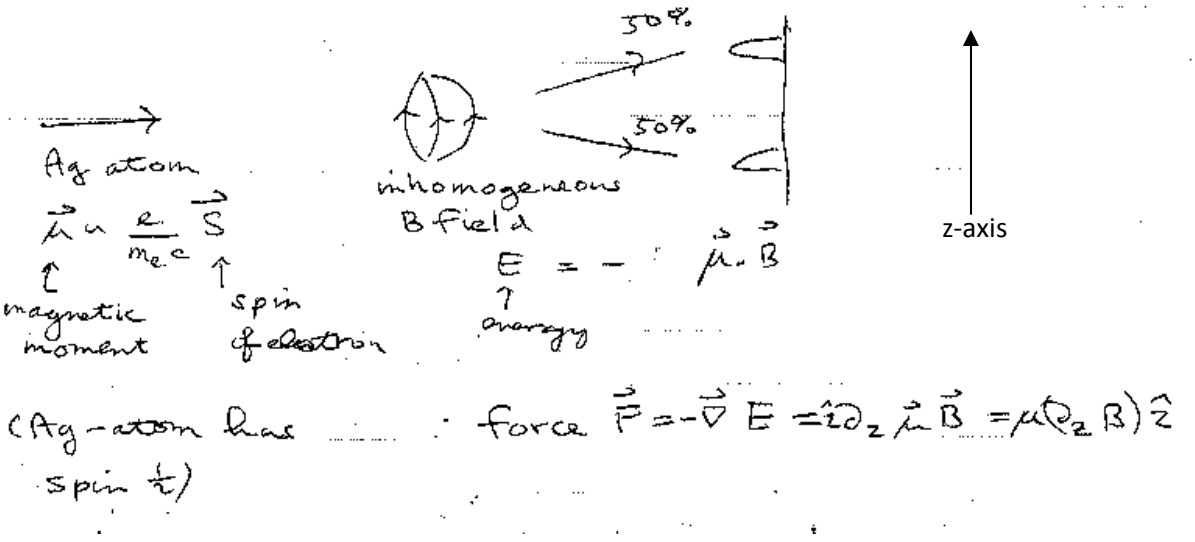
II. Sakurai Section 1.1

A) Stern-Gerlach Experiment



drawing courtesy of Jacobus Verbaarschot (Stony Brook Physics Dept).

Interesting Aside:

From **Wikipedia: Magnetic Dipole Moment**

http://en.wikipedia.org/wiki/Magnetic_dipole_moment#Effects_of_an_external_magnetic_field_on_a_magnetic_moment

the case of a current loop model is

$$\vec{F}_l = \nabla\left(\vec{m}\cdot\vec{B}\right)$$

In the case of a pair of monopoles are used (i.e. electric dipole model)

$$\vec{F}_d = \left(\vec{m}\cdot\ \nabla\right)\vec{B}$$

and one can be put in terms of the other via the relation

$$\vec{F}_l = \vec{F}_d + \vec{m}\times\left(\nabla\times\vec{B}\right)$$

In all these expressions $\vec{m}$ is the dipole and $\vec{B}$ is the magnetic field at its position. Note that if there are no currents or time-varying electrical fields $\nabla\times\vec{B}=0$ and the two expressions agree.

Interesting idea: electron (all spin ½ fermions) is composite particle made of two monopoles.

Analogy: Fractionally charged quarks confined in proton.

Two differences from Classical Expectation

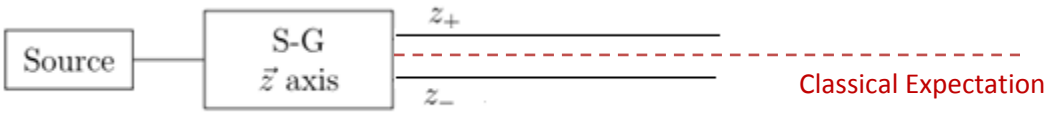
1) Existence of μ → Spin : Sakurai ------ <|) → Liboff: [(if unquantized "spin" expected]

2) Quantization of μ → Quantization of Spin (at least in the z direction) -- ---__----__----

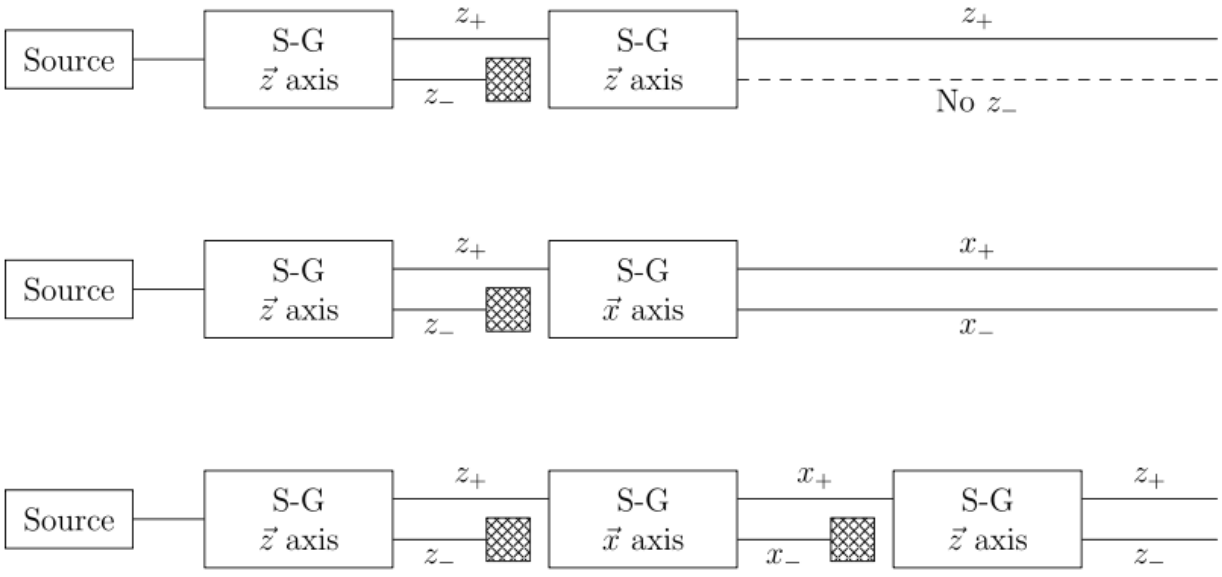
Quantization oft spin was $S_z = +/- \hbar / 2$

can be determined from the distance btw the 2 peaks (must have been tiny!) (probably why no lab demonstration)

Basic SG:



Second More Interesting SG Setup…



this drawing (Wikipedia: Stern-Gerlach)

Main Points:

a) In 1) Same value of z is selected

b) In 2) Of course x must have the same effect if by itself (azimuthal asymmetry—assuming gravity wasn't an issue!). But more importantly, if somehow **z** was chosen as vector it **didn't affect measurement** of **x** direction. (except for intensity) (?)

c) Measurement of x DID affect the outcome of the second z measurement—it made it like the first z measurement.

Lecture 1/11/2010

Pset posted (small changes made Sat) Index notation reading is optional for this week, but in reality required for most

Prakash: Wed:

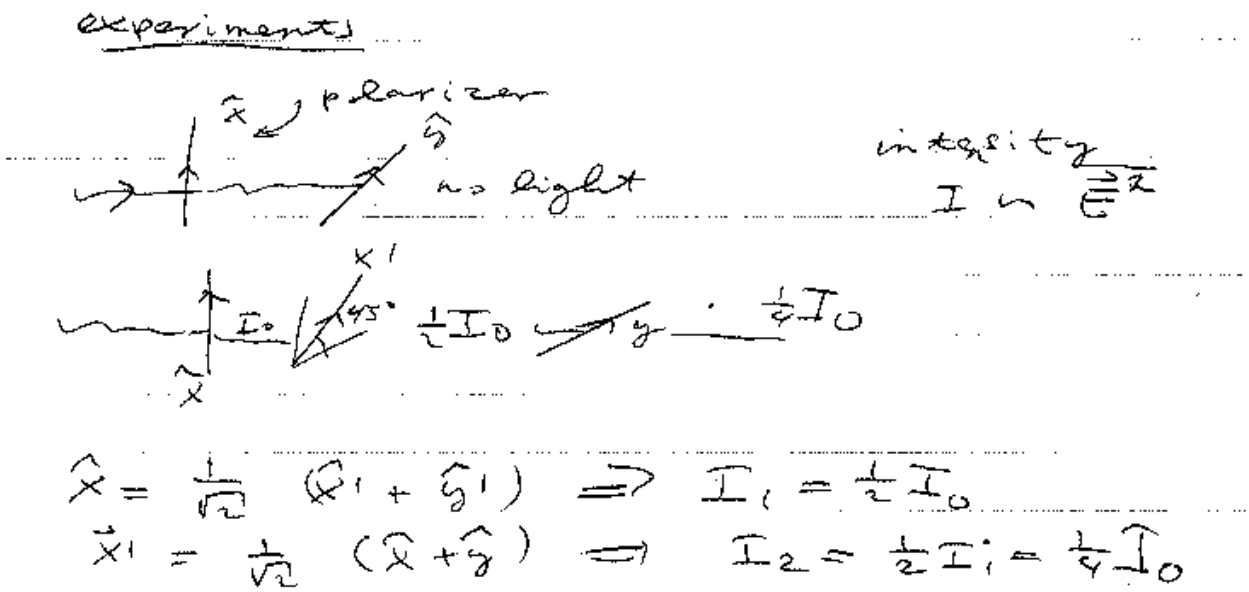
II.B) Analogy w/ light: (Demonstrations)

B.0) The situation is like light polarization and the behavior of light when using filters… Have you guys all seen what happens such experiments?

The light has an electric field E that's always transversely polarized. If traveling in the z direction. It can be described via the vector field

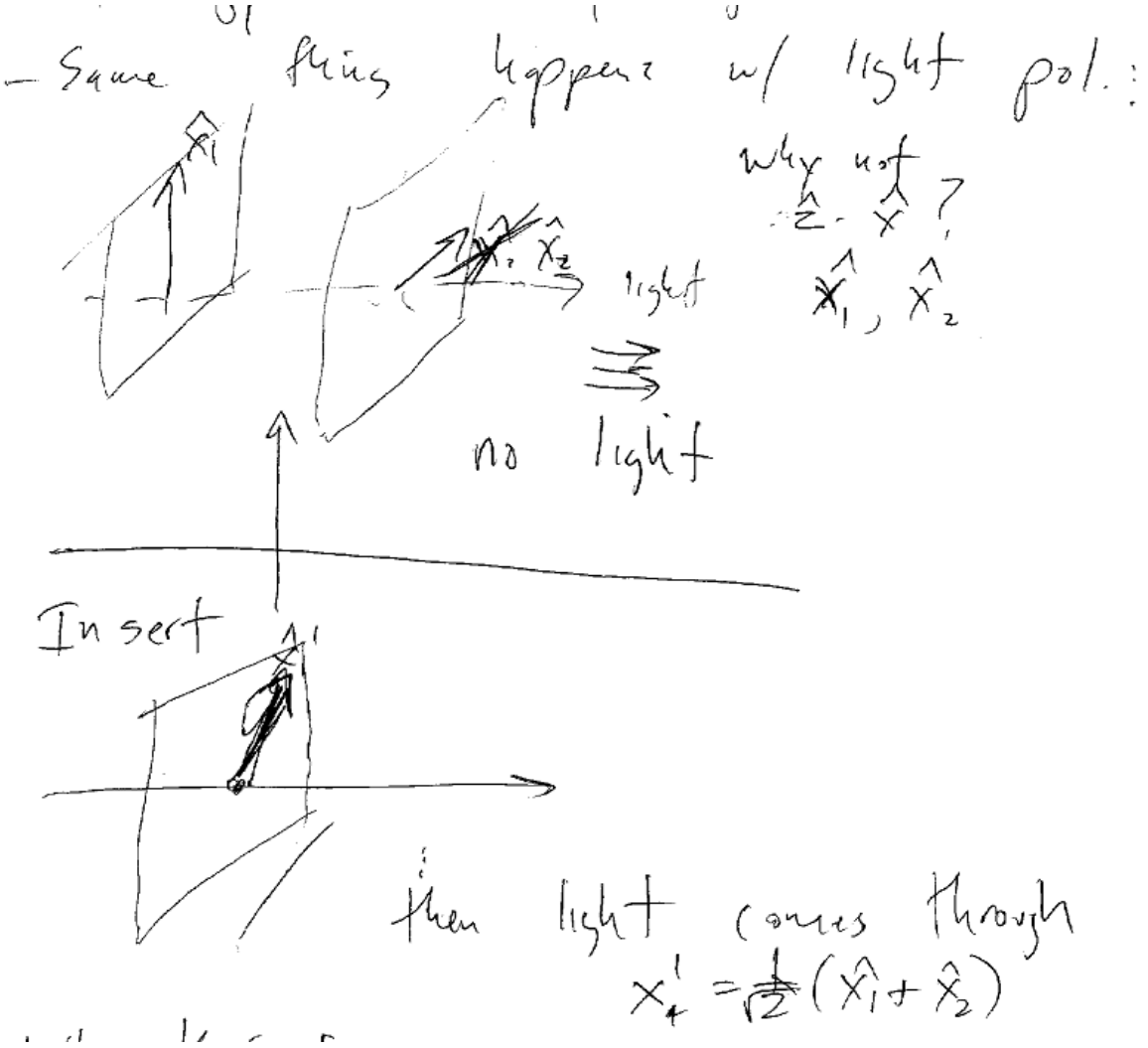
$$E = E0 \cos(kz-\omega t)\,(c\,\hat{x} + be^{i\alpha}\,\hat{y})$$

c, b = 0 → pure y, x (x_1,x_2) → c=± b → x' y' (x_1',x_2')



(Verbaarshot)

My version:

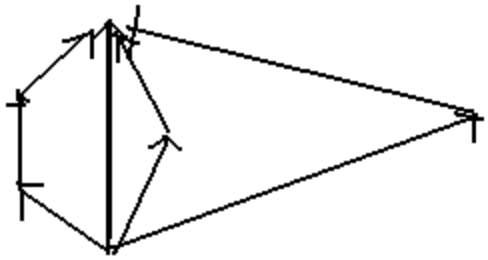


First take α – 0: for this we would need a laser (or CCD)? because incoherent light can take all polarization states. And let's stop using **y** because it can be confusing (it somehow makes you want to relate y polarized light somehow to Sy, which as we shall see, the analogy does not intend that at all) instead let's use x1, x2.

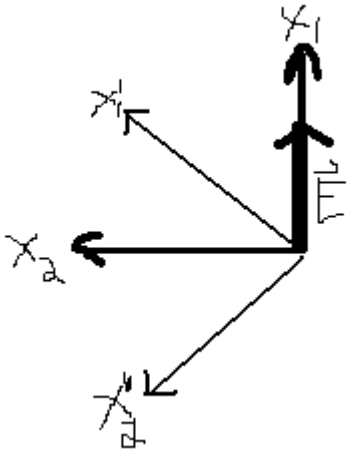
Let's analyze why this is true for light…

My reason for why **this** is true **Vectors have** the property of **superposition** a: any vector can be thought of as an infinite set of vector combinations :

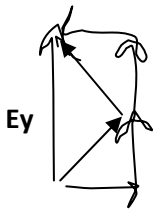
Can be thought of:

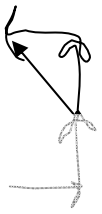


So for our light (help ful to draw it out)



The explanation is **in** of the **filter experiment** is the **x' filter removes** minus portion of the 0 y (x_2) component





notice I reversed the usual convention for x-y (x_1-x_2) w/

Weird interpretion : the electric field **Ex** is **still "there"** in the y-direction even **after x filter**, **but** it just **has** a **value** of **0**.

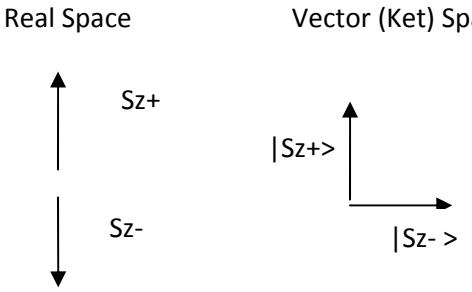
But when we talk about **Quantum Mechanical systems**, this weird way of looking at it becomes → "the **normal way** of thinking". (**to think**)

B.1) QM Explanation of SG

Thus the idea behind the **QM explanation** of the situation **of** the **SG** experiment is that the→ **Ag atom's spin states** re made up of **abstract 2-D vectors** called **ket's**.

Denoted w/ ket notation |Sz+> , |Sz->

Don’t get confused by the various directions here: Remember



So the **correspondence to filter** is

|Sz+>, |Sz- > ➔ $\hat{x}_1, \hat{x}_2$

|Sx+>,| Sx-> ➔ $\hat{x}'_1, \hat{x}'_2$

Draw→

ie There is a **2-D “internal” space spanned by the 2-D basis |Sz+>, |Sz - >!**

Sx +- Sy± states just linear combinations of these

C) Important Differences btw SG(QM)/Light

C.1) States And Collapse of the "State"

A **Major difference in interpretation** is that for the SG Measurement of Sx Quantum Mechanics says there is a "Destruction of the State"/"Collapse"

Difference operationally hard to define now: Hard to define difference actually-- This can be phrased in terms of the **"incompatibility of Sz, Sx observables"**

-Light we can be able to devise an experiment to "measuring" x_i'and x_i components simultaneously. Where as by Quantum Postulate → Sz, Sx cannot be simultaneously measured.

[Difference operationally hard to define now:]

For now focus on what is meant by collapse:

One of the most important points to notice about these experiements, not just that one of 2 values of Sz are chosen, but that **afterwards** the **same** value is chosen ie state is selected. (point of "more interesting SG Exp…" part (a) above.

-Went from "unpolarized"(isotropic $\hat{s}$) to "polarized" (Sz)

called "Collapse of the Wave Function"

Generalize: "Collapse of the Wave Function" → **Collapse of the State** (now that we aren't doing Wave Mechanics only anymore):

Postulate: Measurement causes collapse

This is something very non-**physical and its nature is still debated**. The major questions about what this implies pertain to "Shrodinger's Cat" type questions—what actually constitutes measurement?. As Far As I Know (AFAIK) they are still unresolved.

Copenhagen Interpretation (this course) measurement causes change, likened mathematically to projection operator being applied

Many worlds interpretation --all possibilities are realized ? (Leonard Suskind: book)

Penrose others: very non traditional ideas…

Related to nature of QM entanglement, etc… To me: interesting frontier of physics (if experiments can be devised) "Recent" paper claimed measurement position/momentum simultaneously.

Back to difference w/ light: Hard to define difference actually—It can be phrased in terms of the **"incompatibility of Sz, Sx observables"**

-Light we can be able to devise an experiment to "measuring" x_i'and x_i components simultaneously. Where as by Quantum Postulate → Sz, Sx cannot be simultaneously measured.]

We will define this more formally later this week…

C.2) (Light Pol/vs QM) Differences btw Polarization/Directions

The analogy with light is very nice and really helps understand some of the basic ideas of quantum mechanics. But the analogy w/ light can only be taken so far:

-Light waves must be transversely polarized.: Direction of motion matters.

-**SG: Direction of particle motion is irrelevant**

These particles do not need to be. In fact it is easy to mistakenly think that the direction of the particle beam is somehow analogous to the direction of the light going through the filter. In fact the direction the particles are moving is somewhat irrelevant. x and y (Sx, Sy) in the SG are equivalent even though one direction is || to the beam and one is perp.

So there is **No polarization of the SG particles in Real** (our Euclidean) **3-D space**. But they are polarized in our "internal" spin space.

Mag field defines special direction

C.3) More Observations about Directions in this Experiment

This also makes the point that the actual directions themselves **are not so relevant** in the SG experiment. Sz was special, quantized, but Sx (or Sy it turns out—we can do the same experiment with the Sy it will be the same) isn't.

Mag field defines special direction→ NOT beam direction!

It means that of the three spacial directions, only 1 gets quantized, and the other two are then just linear combinations of the quantization.

So …

Question: True/False: The SG exp. proves that nature has a preferred direction (z)? A: False

BUT: True/False: The SG exp. proves that nature prefers ONE direction ? A: True

–it does prefer ONE direction out of the 3 directions inherent in 3-D space, considering the Ang Momentum of Quantum Particles. For any given particle locally 3 space?

How about if we define an axis z before the experiment--before it's measured what is its spin state?—Can't we define it in terms of this axis? Yes , this chapter instructs us that we can—regardless of whether we put a SG magnetic field there or not in fact. But of course that still doesn't imply that our chosen z direction itself is preferred—rather just that ONE direction is preferred.

IE Better to think more abstractly here: The major point of these experiments is that there is **ONE special space axis** not necessary that we call it z, but still there is 1. It's probably best (though not required) to call it the "**ALIGNED**" axis, meaning aligned w/ the magnetic field.

In fact, the experiments imply not just that ALIGNED is special, but that there are 2 special space directions**. ALIGNED and ANTI-ALIGNED**.

Correspondingly the other 2 space axes/ 4 spacial directions w.r.t. these two directions have hardly any special significance, they are just linear combinations of the first 2!!!! In the internal spin space, the fact that in real space, those other directions/axes correspond to linearly independent vectors/axes is irrelevant.

We shall see that Quantum Mechanics is constructed in a way that automatically causes this weird situation to have a mathematical explanation.

B.5) The role of S_y in Sakarai 1.1:

Finally we haven't talked to much about S_y.

Sakurai's point is that treating **Sy REQUIRES us to use complex spaces.**

That we can continue our analogy w/ S_y by simply choosing the circularly polarized state with

$\beta = 1$, $\alpha = +/- \pi /2$

$$E = E0 \cos(kz-\omega t)\,(\hat{\boldsymbol{x}}_1 + e^{i\alpha}\,\hat{\boldsymbol{x}}_2)$$

Ie

$$E = E0 \cos(kz-\omega t)\,\hat{\boldsymbol{x}}_1 + E0 \sin(kz-\omega t)\,\hat{\boldsymbol{x}}_2$$

This is very nice: it makes the analogy so much nicer and very natural.

Just to summarize the whole analogy now:

|Sz+>, |Sz- > ➔ $\hat{x}_1, \hat{x}_2$

|Sx+>,| Sx-> ➔ $\hat{x}'_1, \hat{x}'_2$ **w/ real coefficients (e.g. α = 0)**

|Sy+>,| Sy-> ➔ $\hat{x}'_1, \hat{x}'_2$ **w/ complex phase (α = ± π/2)**

Sx: b = ± c would give x' considered so far (45) but other combo's will be considered in homework problems.

Important point again: |Sy+> is not orthogonal to |Sx+>: just one is complex one is not

However in the end there IS **one thing special** about Sx and Sy in relation to Sz: they are "maximally incompatible" with z and with each other. Perpendicular in real space maps to maximal destruction of one chosen direction by measurement of the perp dir.

However, I pondered whether it was true—could we use some other mathematical device, perhaps higher dimensional matrices, for our initial state vectors and avoid complex numbers. The answer is yes, but it would be very complicated.

One way to see this before even approaching our SG, is to start with answer the following more simple question. Complex numbers are much like 2-D vectors. Can we actually define an algebra using 2-D vectors that have all the properties of complex numbers. Let me show you one I thought of—it's good practice w/ complex numbers and also an example of thinking about "constructing" algebras that fulfill your needs, something that is done in advanced physics all the time (e.g. dot product in General Relativity defined w/ metric—String Theory, etc…) :

The idea is to take our *z = a+bi* → $\vec{z} = \begin{pmatrix} a \\ b \end{pmatrix}$

actually this way requires a special definition of "inner product" like in relativity one needs a "metric" to put in the minus sign (remember only dealing w/ real numbers)

Actually Wikipedia shows us a better way…

n n

Matrix representation of complex numbers [edit]

While usually not useful, alternative representations of the complex field can give some insight into its nature. One particularly elegant representation interprets each complex number as a 2×2 matrix with real entries which stretches and rotates the points of the plane. Every such matrix has the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where *a* and *b* are real numbers. The sum and product of two such matrices is again of this form, and the product operation on matrices of this form is commutative. Every non-zero matrix of this form is invertible, and its inverse is again of this form. Therefore, the matrices of this form are a field, isomorphic to the field of complex numbers. Every such matrix can be written as

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which suggests that we should identify the real number 1 with the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the imaginary unit *i* with

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

http://en.wikipedia.org/wiki/Complex_number#Matrix_representation_of_complex_numbers

(apparently matrix representation is better than my vector way).

But you see that it is very complicated, requires many more external postulates. ("Definition" of multiplication is BLAH). We get so much for free just from how complex numbers "work". And we will see that the same is true of the whole ***Hilbert Space*** technology we've been talking about above.

B.4) One more Interesting q: (not covered in class):

Why is there still a spread? (beyond points made previously)

(Remember Classical Expectation **------ <|)** Notice the two spreads are drawn in Sakurai fairly similarly.sized. Perhaps it is just to indicate that the magnetic field/measurements themselves aren't perfect, but I wonder if Sakurai didn't intend for it some other significance. Maybe related to the inherent quantum uncertainty in Sx, Sy—or more probably, of the inherent uncertainty of the actual wave function of the atoms and the imperfection of the original assumptions/approximation that we could talk about classical trajectory.

Lecture 1/12/2010

For problem 1.2: just view σ_i as being complex matrices with the following definitions

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\vec{\sigma}$: Vector of Matrices

For problem 1.12: can use the result of 1.9 without proof.

IMPORTANT: I WILL UPDATE THESE NOTES IN EVENING OF 1/12 to better reflect what we covered and what we skipped.

III. Quantum Formulism 1: Abstract Vector or Hilbert Spaces

A) Hilbert Space

Hilbert Space: Implies complex numbers (another digression: Argand Plane**)**

ket => abstract "vector" representing a physical state

$$|\psi\rangle$$

It is ***complex***. in some sense, meaning it has some imaginary portion or component, although it may **not** be **generally specified in all cases exactly what this means:** the only thing that is specified is how these imaginary parts change information that we extract from them. E.g. we do **not** say in general that a **|ket>** can (ie it **can't**) **be written like z = a+ib.** with **"separable"** real and imaginary parts: It **cannot** be written like that in general.

Multiplication by a constant

$$a\,|\,\psi\rangle = |\,a\psi\rangle = |\,\psi\rangle\,a$$

number can multiply a ket from **either direction**

Sometimes people we will use this notation. May seem **pointless (?)** in this situation—(As Sak states,) "only the direction matters". so pointless if in reference to a state by itself. (Could have relevance in relating one state to another—ie **defining the proportion of one ket vs another e.g. that add up a third ket**).

Many properties that may **seem obvious,** but should be stated for mathematical completeness

Addition → "closure"

$$|\psi\rangle + |\varphi\rangle = |\psi + \varphi\rangle \in \mathcal{H}$$

(conceptual meaning…)

$$|\psi\rangle + |\varphi\rangle = |\varphi\rangle + |\psi\rangle$$

addition commuting

$$|\varphi\rangle + |\psi\rangle + , |\chi\rangle$$

…

(This just means we can add any number of kets together..)

$$\begin{aligned} (|\psi\rangle + |\varphi\rangle) + |x\rangle &= |\psi\rangle + (|\varphi\rangle + |x\rangle) \\ |\psi\rangle + |0\rangle &= |\psi\rangle \\ |\psi\rangle + |-\psi\rangle &= 0 \end{aligned} \tag{1.4}$$

The last two relations state the existence of a 0-vector and the existence of a negative vector with respect to $|\psi\rangle$.

$$\begin{aligned} \mathbf{1}\,|\psi\rangle &= |\psi\rangle \\ a(b\,|\psi\rangle) &= (ab)\,|\psi\rangle \\ (a+b)\,|\psi\rangle &= a\,|\psi\rangle + b\,|\psi\rangle \\ a(|\psi\rangle + |\varphi\rangle) &= a\,|\psi\rangle + a\,|\varphi\rangle \end{aligned} \tag{1.5}$$

"Pneumonic": word that means "trick to help you remember"

Pneumonic: best way to remember *properties*→ think of ***ket's as column vectors*** e.g $|\psi\rangle$ ➔ $\begin{pmatrix} \vdots \\ 2 \\ 1 \\ 0 \\ -6 \end{pmatrix}$

All of the above properties become **completely obvious.** The reason we like to state them this way though is because they will apply **to other mathematical objects besides vectors** (e.g. wave functions). But most of the time it will be very beneficial to think of them as real column vectors. THEY ARE NOT FORMALLY column vectors. Later we will define a FORMAL representation of them as vectors, but they will not be vectors yet.

The **only difference:** this space (so far) has with normal vector spaces that you may be used to, is its **complex-ness**

It is **perfectly acceptable:** in a Hilbert Space for any of the above **constant factors** (numbers multiplying) kets to be **complex**

III.B) Basis

The vector representation reminds us that the space is assumed to **spanned by an complete or linearly independent set** (we can also envision this set as orthogonal if we like, although we will redefine orthogonality in terms of kets in a second) of basis vectors.

$$|\psi\rangle = \sum_{n=1}^{\infty} |\varphi_n\rangle\, a_n$$

which in the pneumonic of course you can remember as something like**, for our above example**

$$\begin{pmatrix} \vdots \\ 2 \\ 1 \\ 0 \\ -6 \end{pmatrix} = 2\begin{pmatrix} \vdots \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 6\begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \ldots$$

ie it's good to think about the "basis vectors" as (1,0,0,…) etc… (if the basis is orthonormal)

Q: What were examples of basis ket's for the Stern Gelach experiment? Sz+/---Actually it could've been the x' (Sx) or even Sy eigenkets too

III.C) Discrete or Continuous (skipped in class for now)

In the above examples we are thinking of kets that could be written as a "***discrete"*** sum over some some ***discrete basis*** (discrete, but **possibly infinite in number**) , e.g. the basis members alternatively could be labeled by a (possibly infinite) set of integers. (this in **math**ematics is apparently known as **"countable" or "denumerable").**

We will also consider spaces where ***the basis is infinite in a slightly different way***

Ones whose space must be described by **"continuous" set of basis vectors:**, meaning two basis vector *labels* can be infinitesimally close (like any rational number x, vs. $x + \epsilon$, taking the limit where $\epsilon \rightarrow 0$) I will refer to these as "**continuous kets**". Any linear combination of these types of "continuous" kets, most importantly the expansion over the orthonormal basis kets, as the limit of the above vector sums as an integral

$$c_1 |\psi\rangle + c_2 |\alpha\rangle \Rightarrow \int c(w) |w\rangle \, dw$$

Actually a standard general label for these continuous kets is ξ ,ie if you see **|ξ > it may imply automatically** a **continuous ket**.

In **Sakurai these types of kets are never explained 100% fully.** Meaning some of the properties of our kets that "come for free" for the discrete type of kets, will seem to require extra "postulates" in order to e.g. calculate physical observables. However, if ever in doubt **Messiah:** explains in more detail ~all the (little) aspects of these continuous kets/spaces .

END SKIPPED 1/12

III.D) Bra space

We will also define a **"dual" space** to the kets. In this case "dual" just means there is a **1 to 1 mapping** of all kets.

The dual to any ket |ψ > will be denoted <Ψ |

and called a **bra (or bra vectors).**

Mapping: The mapping will be such that the following is true:

<aψ| = a* <ψ|

meaning "the dual to |aψ> is a*<ψ|

(**Now we see at least one use for the <aψ | ("all inside") notation.)**

Main External Postulate necessary to make formalism work for Quantum Description

III.E) Inner Products

These we will be able to mathematically multiply bras and kets through a "multiplication" called the **braket (or bracket!) multiplication == Inner Product**. Denoted as

<ψ|χ>

And interpreted the **same as an inner product for vectors—**that is same as the **dot product.**

Operationally we 'll say that bra multiplication will have the algebraic properties of the dot product between two vectors: e.g. very importantly it will be **distributive:**

$$\langle\psi|\chi+\phi\rangle = \langle\psi|\chi\rangle + \langle\psi|\phi\rangle$$

We know how to formulate the dot product for vectors as a $v^T v$ (row*column vect) in linear algebra.

Vector then we can say this **inner product** can be represented by **a row vector times column vector:**

$$(0 \quad 3i \quad 1 \quad \ldots)\begin{pmatrix} \vdots \\ 0 \\ 2 \\ 1+i \\ -5 \end{pmatrix}$$

which, because the we can **always write our column vector** as the **sum over the unit vector basis (1,0,0…), (0,1,0,0,…)** ,it's easy to "prove" that this means that

$$|\psi\rangle = \begin{pmatrix} \vdots \\ a \\ b \\ c \\ d \end{pmatrix} \quad \textbf{corresponds to} \quad \langle\psi| = (\ldots \quad a^* \quad b^* \quad c^* \quad \ldots)$$

which in turns helps you remember

$$\langle\psi|\chi\rangle = \langle\chi|\psi\rangle^*$$

As the vector **representation implies,** is a **kdirect result of our "First Main Postulate" Above** as we will later find out. **For now, you may think of it as being postulated separately**, as Sakurai states.

The above implies

$\langle\psi|\psi\rangle$ is real

Further we <u>postulate a positive definite metric :</u>

$$\langle\psi|\psi\rangle \geq 0$$

This will be **because we want this product to be ∝ Probability** as with our wave functions in wave mech.

But also, it implies we can define a **norm of the ket,** which we can interpret geometrically as a length.

$$\| \psi \| = \sqrt{\langle \psi \mid \psi \rangle}$$

These norms obey the **same behavior as geometric vector lengths (pneumonic):**

$$\begin{aligned} | \langle \varphi \mid \psi \rangle | &\leq \| \varphi \| \, \| \psi \| \\ \| \varphi + \psi \| &\leq \| \varphi \| + \| \psi \| \end{aligned}$$

And as our **vector pneumonic tells us**, we can have kets which are orthogonal to each other

|χ> and |ψ> are orthogonal if <χ|ψ> = 0

(all vectors are orthogonal to the null vector).

Obviously we can "normalize" any ket by dividing by it's norm.

|ψ> / ||ψ||

Thus we can always define an orthonormal basis if we have *any* basis—using a generalized Gramn-Schmidt method!

now that defined orth & inner product

In terms of such an orthonormal basis

$$| \psi \rangle = \sum_{n=1}^{\infty} | \varphi_n \rangle \, a_n$$

The ∞ in that relation should be replaced by N where N means the maximum value of n (ie "all n")

SKIPPED THE FOLLOWING IN CLASS 1/12 (WILL COME BACK TO) you should read though to help prepare for problem set? : (till "END OF SKIP BELOW")

So we have the relation.

$$\langle \varphi_m \mid \varphi_n \rangle = \delta_{mn}$$

δ_{mn} is the Kroneker delta (1 if m = n , 0 if not)

We will have occasion to use this Kronecker delta a lot . It is very useful.

For example we can easily prove the following relation for the projection of a general state ket onto one of our base kets: which we will usually refer to as

$$\langle \varphi_m \mid \psi \rangle = \sum_{n=1}^{\infty} \langle \varphi_m \mid \varphi_n \rangle \, a_n = \sum_{n=1}^{\infty} \delta_{mn} a_n = a_m$$

MATH DIGRESSION: (Reminder)

There is a convention for using the Kronecker delta, part of something I mentioned the first day, which also makes it more useful (this whole "system" of notation is collectively known **as index notation):**

Instead of writing the sum:

$$\sum_{n=1}^{\infty} \delta_{mn} a_n = \delta_{mn} a_n$$

A **repeated index** (in this case n) just **implies a summation** over all values of n. **In index notation a_m just means a vector.** Type equation here.

Q: How do we write the dot product of two· normal vectors $\vec{v} \cdot \vec{u}$? $v_i u_j \delta_{ij} = v_i u_i$

Normal Linear Algebra: Matrix notation for Matrix B ➔ just B_{ij}.

Q: How do we write a matrix times a vector $B\vec{x}$ in index notation? think if B_{ij} as jth element of vector B_i ➔ $B_{ij} x_m \delta_{mj} = B_{ij} x_j$

(QUICK:)

Another common index notation element that we will see for **3-D vectors only** (Sakarai uses it without definition?) is of course **ϵ_{ijk}** : the **fully antisymmetric tensor or Levi –Civita Symbol**. Reminder:

$$\begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1 \\ \text{all others} = 0. \end{cases}$$

Most commonly used to define the cross product of two 3-D vectors **a x b = c ➔ $c_i = \epsilon_{ijk} a_i b_j$**

---------- **END Math Digression**

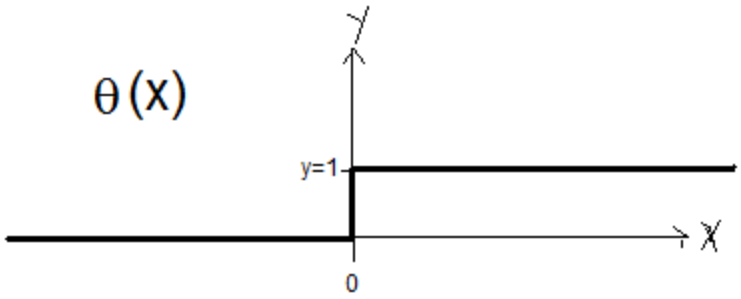
Continous kets. Note one thing about our "continous" ket's: the idea of the norm is a little weird because we say that the **norm of each ket is infinite! For continuous kets $|\xi>$ we define our orthonormal condition as**

$$<\xi_1 | \xi_2> = \delta(\xi_1 - \xi_2)$$

Dirac Delta function… (0, except at 1 where it is infinite, such that $\int_X \delta(x)dx = 1$ if X contains 0.)

Thus we can think of our continuous kets as having **infinite norm**. Sometimes another way to call these types of vectors besides "continuous" is **"infinite norm" kets. According to Messiah, this is related to the fact that "[technically |ξ> itself is not a member of our Hilbert Space]"** but rather **only its "eigendifferential" is.** Eigendifferential is something like $|\xi\rangle d\xi$ --see Messiah if you're interested.

MATH DIGRESSION: One way to define Dirac Delta δ(x) d/dx of θ(x) (unit step function)



END OF SKIP 1/12----------

Final Exercise on Inner Products: show that <ψ|χ> = <χ |ψ >* implies our first Main Postulate: <aψ| = a*<ψ| in the form of <aψ|χ> = a*<ψ|χ>?

We don't really NEED to prove this, because we started out ***postulating*** it to be true. However the point is, one *could* postulate the <ψ|χ> = <χ|ψ>* first, and then prove the other case. This means those **two postulates are equivalent.**

III. F) Operators (→ Observable)

Note: idea w/ these two physical connections: all mathe

The basis we were just talking about before our digression will be defined in terms of Operators which already places which, as in wave mechanics, we will associate **physical observables** with.

An **Operator is a generalized "function" or transformation** (in general) taking **one ket into another**.

$$A \mid \psi\rangle = \mid A\psi\rangle = \mid \psi'\rangle$$

Continuing with our **pneumonic** can think of **operators as (complex) matrix's (plural of matrix = matrices)** , acting on ket "vectors". Thus as before **everything we will state about Operators can be easily remembered** since it will be the same as for matrix-es for vectors—including the above statement. (A matrix is a transformation of 1 vector into another of course!)

Note the notation again (inside the |>)

There are several properties of these operators.

First for this quarter we will only consider linear operators:

$$A(a \mid \psi\rangle + b \mid \varphi\rangle) = aA \mid \psi\rangle + bA \mid \psi\rangle \ .$$

(think of f(x) = x2 → f(a+b) != f(a) +f(b) [instead it's that +2ab!] ie f is not a linear transformation)

Other properties:

X = Y if and only if X|ψ> = Y|ψ> for ALL POSSIBLE |ψ> !!!

Addition of OPERATORS (as opposed to vectors) Commutative and Associative. (Unlike Multiplication)

X+Y = Y+X ; (X + Y) + Z = X +(Y+Z)

Multiplication: (meaning: consecutive transformations) Associative But not commutative (just like Matrix pneumonic)

Operators do not commute! In general XY is different than YX thus we define the

Commutator: XY-YX = [X,Y]

thus [X,Y] is another non-null operator. If it = 0, then we are free to reverse the order of multiplication. We will also define

Anti commutator {X,Y} = XY+YX

Or course both the commutator and anti-commutator are **different operators themselves, distinct from X or Y**.

Exercise: Suppose we have 3 operators labeled A_1, A_2 , and A_3, prove that $\epsilon_{ijk}\{A_j A_k\} = 0$.

Proof: by commutative property of addition, $\{A_j,A_k\}$ is ***symmetric*** in j, k it is equal with j and k reversed {X,Y} = {Y,X}. It is instructive to REALLY GRASP why this implies that $\epsilon_{ijk}\{A_j,A_k\} = 0$. One way: it's

because for each value of i, $\epsilon_{ijk}\{A_j,A_k\}$ is a sum over nine terms which are 9 different anti-commutators. First, by definition of ϵ, the 3 terms that have j = k are automatically 0. For each of the 6 remaining terms we can match them up and create 3 pairs, each of which as the same two indices, but in reversed order e.g. one term will be $\epsilon_{i12}\{A_1,A_2\}$ 1-2, the other will have 2-1 -- $\epsilon_{i21}\{ A_2,A_1\}$. Since {A1,A2} = {A2,A1} these two terms are obvious equal but due the ϵ having reversed indicies, opposite in sign and therefore each pair adds to 0.

Better way: (?) by our j,k symmetry due to commutativity of addition, $\epsilon_{ijk}\{A_j,A_k\} = \epsilon_{ijk}\{A_k,A_j\}$. We are free to choose any letters for our indices, so obviously on the RHS we can re-name the indices j→k , k→j. Then it becomes manifestly obvious that the RHS = -(LHS) (still 9 → 6 term sums, but each non-zero term will obviously always get the opposite sign. The only number (besides possibly infinity sometimes perhaps) that = the negative of itself is 0.

Identity operator $\mathbb{I}$ **or 1**. (special operator) This is defined as the operator whose transformation doesn't change a ket at all. $\mathbf{1}|\psi\rangle = 1|\psi\rangle$. The same as multiplying by a constant 1, however a true operator. (Like the identity matrix, which has the same effect, but is different from the number 1)

Inverse: Operators can have one: if they multiply together to make the identity operator

$$\mathbf{AB = 1}$$

then $B = A^{-1}$ and we call B and A-1 the inverse of A. For a product of operators.

$$(AB)^{-1} = B^{-1}A^{-1}$$

This **follows from** thinking of the **operators as transformations**.

Exercise: Explain this (what I mean)

(matrix $(AB)^{-1} = B^{-1}A^{-1}$ also—pneumonic)

However, **operators do not ALWAYS have an inverse** (just like all matrices don't).

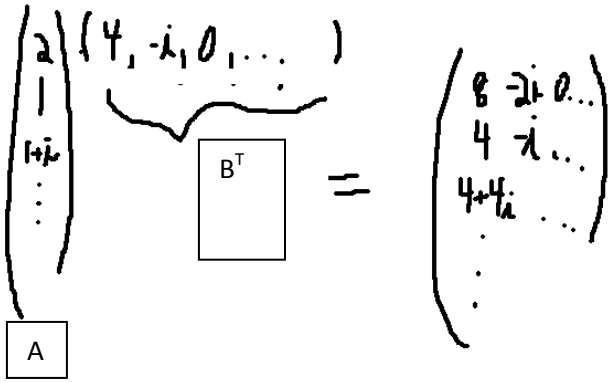
Example of an operator: An interesting "more explicit" operator: " outer product":

SKIPPED 1/12: (to "END OF SKIP"): ----------

There is a basic form of an operator that can be made purely in terms of kets/bras. It is called the **outer product.**

$$|\psi\rangle\langle\chi|$$

The **pneumonic is very helpful here**: analogous to the outer product of vectors now **column x row** (like when we were constructing **MM^T x for projections Proj E x** for single-vector space E. Except this is AB^T)



We can **indeed** form **$|\alpha\rangle\langle\alpha|$** and this will indeed look like **1-D MM^T exactly**: thus it is easy to guess that that this operator will indeed represent a **projection in the space onto the ket vector $|\alpha\rangle$.**

Back to the general $|\psi\rangle\langle\chi|$ ($\psi \neq \chi$) : We can have that this **operator acts on** some other **ket $|\phi\rangle$** and by the **power of associativeness,** we automatically form a bracket, which indeed can be interpreted as inner product (e.g. a number)

$$|\psi\rangle\langle\chi| \quad |\phi\rangle = |\psi\rangle(\langle\chi|\phi\rangle) \quad = \quad \langle\chi|\phi\rangle \ |\psi\rangle = c\,|\psi\rangle$$

Question: which associative law proves this? number-ket mult? operator?, a new associativity?: yes we've defined a new associativity for outer products.

One thing to note about an **outer product** operator that can be written this way is:

Outer product is NOT INVERTIBLE

(**-Messiah** Why? I can't see an easy way to prove this generally. Can this be derived from vector/matrix **pneumonic**? likely -- It is **easy to see for orthonormal unit basis vector -- Exercise: explain why? hint: how many non-zero elements will it have for the unit basis vector)**

----------end skip

Operators act on both kets & bras…

How will operators act on bras?

Question: what is the result of a row vector * Matrix? (is it a column vector, matrix, what is it?)

Just as with matrices, operators can **act from** the **right or** the **left** (just like matrix pneumonic)

$$\langle\psi|A|\chi\rangle = \langle\psi|(A|\chi\geq) = (\langle\psi|A)|\chi\rangle$$

Think of A acting on either χ or ψ. **However** it is important to realize that **A is not performing the same transformation in both directions!**

Important: when acting to the left on a bra, an operator is not performing the transformation (in bra space) that is "dual" to the transformation that occurs in ket space when A acts to the right on the ket!!! To remember this think about matrices: even though it they can multiply a row vector, in general, $\mathbf{b}_{row}\mathbf{C} \neq (\mathbf{C}\mathbf{b}_{column})^T$ This only works for special cases of C (symmetric).

Another way to say this is $\langle A\psi| \neq \langle\psi|A$:

Exercise: explain in words why the "inside the bra/ket" notation in the expression above expresses this

This can be seen for our outer product operator (and is also fully implied by our ket-bra notation borrowing the same associative convention as above)

$$\langle\phi|\ (|\psi\rangle\langle\chi|) \quad = \quad \langle\phi|\psi\rangle\ \langle\chi| \ = \text{const}\ \langle\chi|$$

We get a bra state in the direction of $\langle\chi|$. Comparing to our above case where the outer product acted on a ket to the right we obtained a ket in the direction of $|\psi\rangle$. We constructed our outer product such that in general $\langle\chi|$ was not the dual of $|\psi\rangle$

Definition: Expectation Value (for α): $\langle\alpha|B|\alpha\rangle$

========== END OF LECTURE 1/12

III G.) Adjoints of Operators

Since $\langle\psi|A \neq \langle A\psi|$, that is to say $\langle\psi|A$ is NOT the dual vector to $A|\psi\rangle$, it is important to find indeed **what is** the bra that is dual to $A|\psi\rangle$?

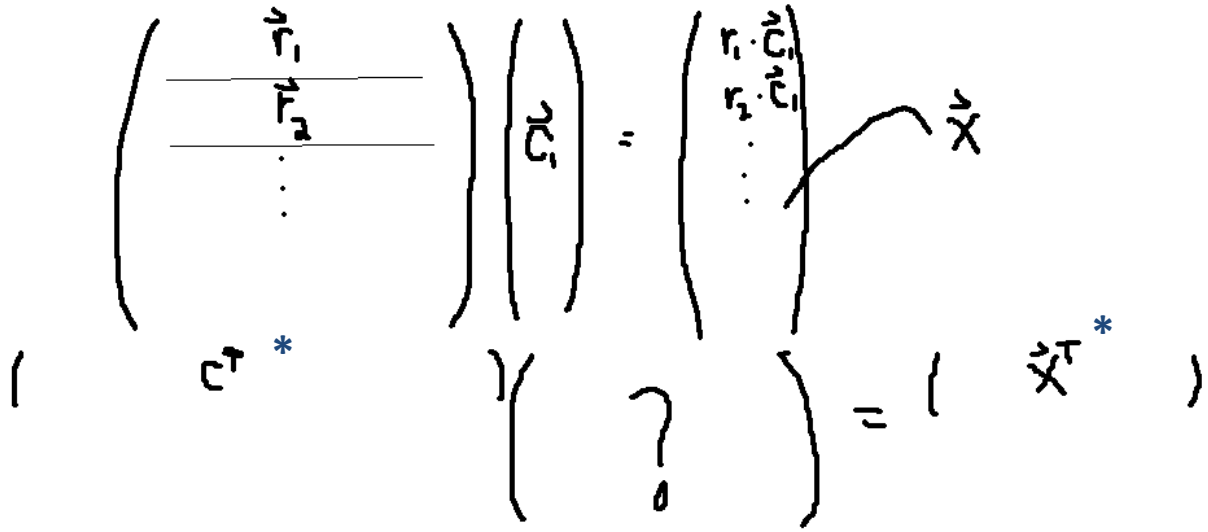
Let's analyze the situation with our **matrix pneumonic comparing the two transformations:**

A |ψ> = |χ>

<ψ|B = <χ|

In other words,now B **IS** the transformation we are looking for, the one in the bra space that is dual to what A does in the ket space. The comparison is made for matrix /vector pneumonic below, forming A with row vectors $\mathbf{r}_i$ as shown, we can ask, what does B have to look like? First from observing this, we can see obviously see that the same A will not work in the second case so B ≠ A.

Why????



What we need for A is a matrix that is the transpose and complex conjugate of **A (A^T*)** . "Conjugate Transpose" We will denote **this as A† for matrices. (ie X^T* = X†**

"Following the pneumonic", we will **define** something called the **adjoint or Hermitian Conjugate** for our **operators** in our ket space (not just the matrices in our pneumonic) and denote them the same.

$$\langle \varphi \mid A \mid \psi \rangle = \langle \varphi \mid A\,\psi \rangle = \langle A^{\dagger}\varphi \mid \psi \rangle$$

This means ie that

dual conjugate to A|ψ> is <ψ|A†

Properties of the Adjoint:

(XY)† = Y† X† (like inverse or matrix transpose)

(cX)† = c*X†

(A+B)† = A† + B†

same as $^{T}*$ for matrices. Easy to forget last one—exercise –"prove w/ our pneumonic"? **Good practice** for index notation→ we add the matrix together element by element: $(A+B)_{ij} = A_{ij} + B_{ij}$ Because of our complex conjugate rule * applies to each term in the equation. To perform the Transpose, in index notation, we simply reverse the order of the indices, so taking the transpose of the LHS we get $((A+B)^*)_{ji}$ but by the element by element logic, this is just Aji*+Bji *. which is $A^\dagger + B^\dagger$

Finally based on the **definitions, multiplications, etc…,** outlined **above** one property of thing about our **special outer product operator** is

$$(|\psi\rangle\langle\chi|)^\dagger = |\chi\rangle\langle\psi|$$

Why? Apply $|\psi\rangle\langle\chi|$ to $|\phi\rangle$: we get $c|\psi\rangle$ where $c = \langle\chi|\phi\rangle$. By our rules, the dual to $c|\psi\rangle$ is $\langle\psi|c^* = c^*\langle\psi|$. But c* can obviously be written as $\langle\phi|\chi\rangle$, since $c = \langle\chi|\phi\rangle$. So the dual $c^*|\psi\rangle$ can be written $\langle\phi|\chi\rangle\langle\psi|$.

Question: what is a special about our projection operator $|\alpha\rangle\langle\alpha|$ with respect to the Adjoint? it is the adjoint of itself! We have a name for such operators…

III.H) Hermitian Operators

An operator is A called self adjoint or most of time also **Hermitian if**

$$A = A^\dagger$$

We will require that the operators we want to represent observables will be Hermitian in Quantum Mechanics, for reasons we will soon see. Apparently there is a slight **conceptual difference between** being **Hermitian** and being **self-adjoint**. The technical definition of Hermitian: H is Hermitian if

$$\langle H\psi|\psi\rangle = \langle\psi|H\psi\rangle$$

For **this class** we will refer to **Hermitian and self-adjoint** as meaning **exactly the same** thing: $A=A^\dagger$

Exercise: If A and B are Hermitian (Self Adjoint) is the product AB? only if they commute

$$(AB)^\dagger = B^\dagger A^\dagger = BA$$

III.J) Eigenkets/Eigenstates

Following the example of wave mechanics, you knew we were going to have eigen-somethings…

If we have the following relation

$$\boldsymbol{B}|\boldsymbol{\beta}\rangle = b\,|\beta\rangle \qquad \text{(III.1)}$$

then we **call the ket |β> the eigenket** of the operator **A**. We will sometimes follow the convention in Sakarai and label our eigenstates by their eigenvalue (which assumes that there are no **degeneracies**)

Eigenvalue degeneracy : means two or more linearly independent eigenkets correspond to the same eigenvalue

Actually the **notation** we have been using for basis kets $|\varphi_n\rangle$ is nice because it avoids this issue—let's use a combination of both $B|b_n\rangle = b_n|b_n\rangle$. Sakarai's notation has a problem of needing multiple primes as we will soon see...this is a **much nicer convention than using multiple primes** (e.g. a''').

Hermitian Operator has real eigenvalues: important proof

First notice that even if |β> is an eigenket of B, **if B is NOT Hermitian**, then **<β| might NOT be an eigenbra of B**. It is only the eigenbra of **B†**. The eigenvalue for B† in bra space is must be b* for how we defined the dual space. So the relation corresponding to eq III.1 is

$$\mathbf{\langle\beta|b^* = \langle\beta|B^\dagger}$$

For a Hermitian operator B , this of course becomes

$$\mathbf{\langle\beta|b^* = \langle\beta|B}$$

Now that this is established let's switch to our new eigenvalue notation and go over the proof in Sakarai that the eigenvalues for such a Hermitian B must be real. Consider when B is bracketed by two different eigenstates: $\mathbf{\langle b_n|B|b_m\rangle}$: operating in either direction with b we see that

$$\mathbf{\langle b_n|B|b_m\rangle = \langle b_n|b_n^*|b_m\rangle = \langle b_n|b_m|b_m\rangle}$$

considering the two right hand quantities only

$$\mathbf{\langle b_n|b_n^*|b_m\rangle = \langle b_n|b_m|b_m\rangle}$$

the b's are just numbers, so moving through the bra's and moving the right hand side (RHS) to the LHS we get the same relation as in Sakarai

$$\mathbf{(b_n^*-b_m)\langle b_n|b_m\rangle = 0}$$

The key point as to why this proves that the eigenvalues must be real, is because if m = n, $\langle b_m|b_n\rangle$ cannot be 0 since it equals $\langle b_n|b_n\rangle$--it will be 1 if the $|b_n\rangle$'s are normalized. Thus $b_n^*-b_n = 0$ or

$$\mathbf{b_n^* = b_n}$$

which of course means b_n is real. This is true for any n, so it means that ALL eigenvalues must be real.

If $m \neq n$, then, using the realness of our b_n's means that for non-degenerate eigenvalues, b_m-b_n will by definition of non-degeneracy be non-zero. Thus the factor $\langle b_m | b_n \rangle = 0$.

NOTE that for degenerate eigenvalues/vectors, the relation implies that $\mathbf{b_n = b_m}$ for some m and n, while the corresponding states are different. This only means that the states corresponding to the same eigenvalues do not necessarily have to be orthogonal to each other, although they still must be orthogonal to all others. This means that they will form a subspace that we can always make orthonormal using the Gram-Schmidt Method. **Thus the proof in a sense still demonstrates that we can always form an orthonormal basis, with eigenstates.**

(Q) Why do we want the eigenvalues to be real? Because we will say that the eigenvalues of Hermitian operator will correspond to the values the observable which that operator represents. (Observables have to be real).

Expansion in terms of eigenkets.

The above **proof also shows that Hermitian operators** have **orthogonal eigenbasis**, which also means we can just normalize it and make it →**orthonormal**…

We know **from our linear algebra review**, if $A^T = A$ (the **matrix is symmetric**) it has an **orthonormal** eigenbasis. As in that case however, it does not imply a **complete** eigenbasis though. (think of a 4-D matrix 1)

How about completeness?

Are all observable/operator's eigenket's complete? Sakurai avoids the question: "under assumption" Shrodinger Equation→ Sturm Liouville? (says yes for wave functions—wave functions of course are not themselves going to be considered kets in our notation.) **Postulated in Sakarai…**

III. J) Projection operator Λ_n

We already stated that $|\alpha\rangle\langle\alpha|$ would be a projection. We will refer to such an operator, with a Λ.

It's projection quality is easiest to see in terms of our orthonormal basis (whether the basis is an eigenbasis of some observable or not!) .

Thus the easiest projector to consider is $\Lambda_{bm} = \mathbf{|b_m\rangle\langle b_m|}$ (note here we are not using the index notation implied sum)

Act on our expansion in the eigenbasis…

$$|\psi\rangle = \sum_n c_n |\varphi_n\rangle \rightarrow (\boldsymbol{eigenbasis\ b}) \rightarrow = \sum_n a_n |b_n\rangle$$

(we use a_n because b_n is the eigenvalue, for $|b_n>$, not the expansion coefficient.

$$|b_m >< b_m| \left(\sum_n a_n |b_n > \right)$$

$$= a_n \sum \langle b_m | b_n \rangle |b_m\rangle$$

$$= \sum \delta_{mn} a_n |b_m\rangle$$

$$= a_m |b_m\rangle$$

Remembering how to quickly construct the matrix for this operator looks like in our pneumonic from our discussion of the non-invertibility of outer product operators

(ie remembering $|\alpha>$ (a, b, c) $|\phi_n>$ == $|a_n>$ (1,0,0,))

Question: describe this matrix. Like the case where we had two different eigenvectors ket-bra 'd, it will be a **matrix with only 1 non-zero element of value 1. but this time only along the diagnonal.**

Suppose we add all N pneumonic projector matrices? What matrix is this? Identity matrix. This leads us to consider such a sum of ket-bra projectors.

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III. K) Completeness—As an operator.

Thus indeed, even with our ket-bra forms of the Λ's, it is easy to see that if we sum all the projectors for a given orthonormal basis, and multiply it by our expansion for $|\psi>$ we will get back $|\psi>$

Q: explain in words. If we have the sum of all projectors (see below for expression), and each projector individually when acting on our expansion ($|\psi> = \sum a_n|b_n>$) yields a new ket = equal to $a_n|b_n>$, then obviously the sum of all projections of a ket is just equal to the ket itself again.

Thus the

$$\sum_n \Lambda = \sum_n |b_n >< b_n| = \mathbb{I} \equiv \mathbf{1}$$

We can think of this as a **special form of 1 operator :**

For our continuous kets, since by definition of continuous kets, the set of basis kets can be enumerated by numerical labels ξ that are "infinitesimally close to each other". Thus for continuous kets we need must express our completeness sum as an integral over all states…

$1 = \int d\xi \; |\xi><\xi|$

This is a **very important tool as it will allow us to prove and calculate many things using only the bra/kets**

Examples:

1) Prove that any hermitian operator B with eigenvalues b_n can be written as $\sum b_n \, \Lambda_{b_n}$

B = 1(B)1

for each 1 insert completeness (we must use different indices for each new sum we insert in order to resolve them.

$$\left(\sum_n (|b_n><b_n|)\right) B \left(\sum_m (|b_m><b_m|)\right)$$

$$\left(\sum_n \sum_m (|b_m><b_m|)\, B (|b_n><b_n|)\right)$$

$$\left(\sum_n \sum_m (|b_m><b_m|)\, b_n |b_n><b_n|\right)$$

$$\left(\sum_n \sum_m b_n (|b_m><b_m||b_n><b_n|)\right)$$

$$\left(\sum_n \sum_m b_n (|b_m> \delta_{mn} <b_n|)\right) = \sum_n b_n \Lambda_{b_n}$$

The above proof is done in section 1.5 of Sarakai.

This form of an operator is actually very important, let's look at a concrete example (discussed in Sakarai):

Example: Spin states: in terms of projector sums using the kets/bras of the Sz+/- states from 1.1, and using what we said about the Sz eigenvalues from 1.1. What is an explicit ket-bra form for Sz?:

Answer: $\hbar/2|+><+| - \hbar/2\,|->\,<-|$ (I've formed a sum with all projectors and put the corresponding eigenvalues in front of each.)

Q: In terms of the basis kets of Sx (|Sx±>) what should the operator Sx look like.? Same.
$\hbar/2$|Sx+><Sx+| - $\hbar/2$ |Sx- > <Sx- |

IMPORTANT—this ket-bra representation, though written with Sx kets, can still operate directly on Sz kets e.g. we can still project | Sz+> on |Sx - > : |Sx - ><Sx - | |+>….

This can just be done using the reprepresentations of the Sx states we already guessed in Sakarai section 1.1. e.g.

<Sx-| = 1/√2 (<+| - <-|)

PS we actually never stated formally the "distributive" property of bra-ket'ing (also applying to outer product multiplication): but the following is always true: <ψ|χ+ϕ> = <ψ|χ> + <ψ|ϕ> & |ψ><χ+ϕ| = |ψ><χ| + |ψ><ϕ|

So Sx = = $\hbar$ /2 *1/√2 (|- > - |+>) 1/√2 (<-| - <+|) - $\hbar$/2 *1/√ 2 (|-> + |+>) 1/√ 2 (<-| + <+|)

"diagonals " cancel → =

Worked out here:

$$S_x = \frac{\hbar}{2}\left(|S_x+\rangle\langle S_x+| - |S_x-\rangle\langle S_x-|\right)$$

Note: S_x can operate on $|+\rangle, |-\rangle$ (S_z)
How? $|S_x+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$$\frac{\hbar}{2}\left[\frac{1}{2}(|+\rangle + |-\rangle)(\langle+| + \langle-|) - (|-\rangle - |+\rangle)(\langle-| - \langle+|)\right]$$

$$\frac{\hbar}{2}\left[\frac{1}{2}|+\rangle\langle+| + |-\rangle\langle+| + |+\rangle\langle-| + |-\rangle\langle+| - \frac{1}{2}(|-\rangle\langle-| + |+\rangle\langle+| - |+\rangle\langle-| + |-\rangle\langle+|)\right]$$

$$= \frac{\hbar}{2}\left(\frac{1}{2}(2|-\rangle\langle+| + 2|+\rangle\langle-|)\right)$$

This is the same form in Sakurai: e.g. Problem 1.8. The same can be done to get S_y

2) We will define the trace of an operator tr(Z) as

$$tr(Z) = \sum_n < b_n|Z|b_n >$$

show that the trace is independent of which orthonormal basis we use to perform the bracketing.

Proof:

$$= \sum_n < b_n|\ 1(\ Z)1\ |b_n >$$

For the 1's now insert the "completeness sum" for a different orthonormal basis, suppose for concreteness they are the eigenstates $|c_n>$ of an operator C.

$$\sum_n \left(< b_n| \left(\sum_m |c_m >< c_m| \right) Z \left(\sum_j |c_j >< c_j| \right) |b_n > \right)$$

always move the summation symbols to the left first…

$$\sum_n \sum_j \sum_m < b_n|c_m >< c_m|\ Z|c_j >< c_j|b_n >$$

<bn|cm>, <cj|bn> just numbers so

$$= \sum_n \sum_j \sum_m < b_n|c_m >< c_j|b_n >< c_m|\ Z|c_j >$$

$$= \sum_n \sum_j \sum_m < c_j|b_n >< b_n|c_m >< c_m|\ Z|c_j >$$

$$= \sum_j \sum_m < c_j| \underbrace{\sum_n |b_n >< b||c_m}_{1} >< c_m\ |\ Z|c_j >$$

$$= \sum_j < c_j| \sum_m |c_m >< c_m|\ Z|c_j >$$

$$= \sum_j < c_j|\ Z|c_j >$$

3) As in Sakarai, we can also use this completeness to easily show that the sum of the norm's squared of the expansion coefficients : e.g. for orthonormal $|\psi> = \sum a_n|b_n>$, then $\sum |a_n|^2 = 1$

Lecture 1/20/2010 Announce: Homework due Fri 1/22 website still needs updated.

IV. (FORMAL) Matrix Representations

Once we've chosen a basis, as Sakarai explains in section 1.4, we can FORMALLY define a matrix representation of our discrete type kets/bras:

Warning: This formal definition of the "**Matrix Representation"** of the bras/kets, operators but conceptually, this is **slightly different** than my pneumonic of remembering them as vectors:

The difference is

pneumonic:

$|\psi\rangle$ → (any column vector)

$|\varphi_n\rangle$ → always unit (…,1,0,0,…) vectors

A (operator) → any (unspecified) Matrix A

Formal matrix def:

FIRST !!!! : CHOOSE $\langle\varphi_n|$: … then,

$\langle\varphi_n|\Psi\rangle$ → n^{th}component of vector representation of $|\psi\rangle$ ($*^T$ for $\langle\psi|$)

$\langle\varphi_n|A|\varphi_m\rangle$ n,m matrix element of matrix representation of A

Really the difference (other than some extra notation) is mainly **choosing a basis first.**

In other words,for the pneumonic I want you to think of the ket itself as a vector to remember the properties of kets. FORMALLY, the formal vector representation of a ket does not **define** it, it is just one "description" of it. The point is that the **ket /bras that are the fundamental things**.

Nonetheless it will generally be **sufficient for this class, to prove and evaluate things using the matrix representation, for "discrete" kets.** For now, we will also say that the **matix rep. just doesn't apply to continuous kets.**

There is a good summary (essentially of what is in Sakarai) of matrix representation on page 3 of the document at http://www.isv.uu.se/thep/courses/QM/lecturenotes-1.pdf

Please take a look at this document. In these notes, the convention for the basis kets is ($|a^{(n)}\rangle$ which will correspond to our $|b_n\rangle$.

Another way to state it is simply using our index notation:

One we've chosen the basis $|b_n\rangle$ which we want to represent the operator A as a matrix in, (note the $|bn\rangle$'s are not the eigenstates of A) then:

The Matrix M^A which represents A will be defined as (through definition of its *elements*)

$$(M^A)_{ij} \equiv M_{ij}^A \ \langle b_i|A|b_j\rangle$$

An arbitrary ket $\langle\alpha|$ will be represented by a column vector with *elements:*

$$\mathbf{v}_i = \langle b_i|\alpha\rangle$$

while the bra will be represented by the row vector

$$(\mathbf{v}^{T*})_i = \langle\alpha|b_i\rangle$$

Note that this means that a bra in its own eigenbasis's matrix representation will be the a unit row vector: e.g. $|b_1\rangle$ will be represented by (1,0,0….) $|b_2\rangle$ by (0,1,0,….) Similarly for the column vectors that represent the kets $|b_n\rangle$. This is just as my suggestion for the pneumonic—however as opposed to the pneumonic, where I suggested **always** thinking of orthonormal basis vectors as unit vectors, if we choose a different basis in the FORMAL matrix representation, e.g. for example say we want to represent $|b_1\rangle$ in the basis of the eigenbasis of A, $|a_n\rangle$, then the vector representing $|b_1\rangle$ is no longer unit vector (1,0,0…). . **e.g. In the $|Sz\pm\rangle$ basis, $\langle+|$ = (1,0) but if we choose the eigenbasis of $|Sx\pm\rangle$ then we wouldn't get unit vectors for the eigenvectors of Sz.**

How about notating the vector v itself in ket/bra notation: $\vec{v} = \overrightarrow{\langle b|\alpha\rangle}$ **?**

Practice 1:

4-D Basis $|\lambda\ Dn\rangle(\equiv |dn\rangle)$ = (eigenbasis op D)

what is MR of $1/\sqrt{3}\ (|d2\rangle + \sqrt{2}|d4\rangle)$

what is MR of Operator K =$-5|d3\rangle\langle d2|$

Practice 2:

Exercise: For Spin states, What is the matrix representation of Sz? BEFORE YOU ANSWER I FIRST MUST TELL YOU WHAT BASIS (OR YOU MUST CHOOSE & STATE IT!!!!)—OK so if we use the $|\pm\rangle$ ket's of Sz, operator Sz = $\hbar/2|+\rangle\langle+| - \hbar/2\ |-\rangle\langle-|$

forming the brackets for each matrix element: e.g. $\langle+|\ Sz|-\rangle = 0$, etc… we get:

$$S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

What is the matrix rep of Sx? FIRST I MUST TELL YOU WHAT BASIS!!!! (same) 1) In the basis of $|Sx\pm\rangle$ states: it's actually the same

$$\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

IN THE BASIS of the Sz states. We can actually do it. But first we should construct the Sx operator in ket and bras. BUT WE ALREADY HAVE IT!!!!

We already said $\mathbf{Sx} = \hbar/2|Sx+><Sx+| - \hbar/2\ |Sx- > <Sx- |$ and we already said it this can act on +/-kets /bras including eigenstates of any **Operator, including Sz.**

In this sense the ket-bra representations are "independent" of what basis you decide to work in.

Thus to get the matrix representation,in the basis of the Sz states we need to know how evaluate inner products like <Sx-|+>.

$\hbar/2$ (|+>< -| + |- ><+|)

Using all this info to evaluate each matrix element of Sx in the in the Sz± basis (meaning bracket the above with all permutations of <±|, |±>), we get:

NEXT YEAR DO Sy instead?

$$S_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

It should now be easy to do Sakarai Problem 1.5 b following this prescription: It's just an easier version of doing this.

We can do the same thing to get Sy

$$S_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

Two things: we will use these three matrices (since normally we will be working in the Sz± basis) a lot. They will also be referred to with the index notation Sx, Sy, Sz = S_1 S_2 S_3.

Note that for the problems Sakurai 1.11 & 1.12, it is very useful to think in terms of the Pauli Spin Matrices:

Sz=S_3 = $\hbar/2$ σ_3

so

$$\sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

etc… there are a lot of useful properties of these matrices listed on page 165 of Sakarai. e.g. $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$ (some of which are repeated in this chapter in terms of the S matrices.) It will definitely be useful for the problem set to use some of these properties.

Equally or probably **more** important as the matrix rep of operators (which are actual matrices, hence the name "matrix rep") is the matrix representation of the state kets. These will be the vectors with n components $\langle\varphi_n|\Psi\rangle$ **We should always represent the kets in the matrix representation using the same basis kets we use to rep the operators, so following the above examples,we will want to represent an arbitrary state |α > in the |±> basis. The matrix rep vector of |α> will have 2 components <+|α> and <-|α>:**

$$\doteq \begin{pmatrix} <+|\alpha> \\ <-|\alpha> \end{pmatrix}$$

As a concrete example we could think of the state |α> = |Sx-> : .

$$\textbf{Matrix Rep(|S->)} \doteq \begin{pmatrix} \mathbf{1/\sqrt{2}} \\ \mathbf{-1/\sqrt{2}} \end{pmatrix}$$

Digression: Help on Problem 1.11) Sakurai

Here is the road map of doing this problem using the hint:

1) Write the matrix representation of H in the |1>,|2> orthonormal basis –this will be a 2x2 matrix whose elements are the constants in front of the outer products:, e.g. H_{11}

2a) From here you could just approach the problem like the quiz problem of finding eigenvalues/vectors in Linear Algebra. The vector $\begin{pmatrix} a \\ b \end{pmatrix}$ you get obviously represents a|1>+b|2>. Messy: λ 's will be some fn's of H_{11}, H_{12}, etc… but DONE… you need not follow the next steps…

2b) **BUT INSTEAD** to use the hint though, then the exact same way you did last week in problem 1.2, write that 2x2 matrix H in the 1.2 form.

The answer looks like this: $\mathbb{I}\,\overline{H} + \Delta H\,\sigma_3 + H_{12}\sigma_1$ (for a clever but not so hard to think of choice of $\overline{H}, \Delta H$)

Thus the $\vec{a}$ of 1.2 in this case is the vector (H_{12},0,Δ H)

3) The connection to hint about the eigenvector of $S\cdot n$ is that $S\cdot n \propto \sigma\cdot a$. If H had only the $\sigma\cdot a$ term, you obviously could just use the given hint answer with the substitution |+> → |1> and |-> → |2> and for γ β such that $\hat{a} = \hat{n}$

4) But from Linear Algebra: Eigenvectors of matrix A are same as Eigenvectors of matrix (A+xI) (try it!). So 1.2 form should still have given hint eigenvector form, so then find γ, β from step 3 in terms of constants H_{11} etc…

$\cos^2\beta = \Delta H^2/(\Delta H^2 + H12^2)^2$

For 1.12: use 1.11 results: γ → β

Lecture 1/22/2010 --- Homework next week will again be due on Fri: let me know about other class's midterms—we need to schedule ours.

V. Measurement (Part 1)

A) Postulates

Three Postulates concerning Measurements in Quantum Mechanics:

Postulate 1) The only possible values for a measurement of an observable B, will be the possible eigenvalues of the Hermitian operator representing B. (How to find the operator B if don't already know the eigenvalues we want e.g. through first measuring them! == empirically determining them, will be discussed later, and is not specified by this postulate).

Postulate 2) Before measurement for a quantum system in the state $|\alpha>$, the probability to measure eigenvalue bn of B will be given by $|<b_n|\alpha>|^2$ which defines the probability distribution P(n) for each state n. During measurement an eigenvalue bn will be randomly chosen, according to the probability distribution P(n). $P(\xi 1 < \xi < \xi 2) = \int_{\xi 1}^{\xi 2} |c(\xi)|^2 d\xi = \int_{\xi 1}^{\xi 2} |<\phi|\xi>|^2 d\xi$

Postulate 3) Immediately after measurement, the system will "collapse" into a new state that is completely in the direction of the eigenstate $|b_n>$ or in some cases, when there is something called degeneracy, to an "eigen" sub-space defined below, corresponding to the chosen b_n.

Notice the postulates are not specific about how to mathematically represent this measurement

Can measurement be represented by operating with the projection operator? The plain projection operator acting on a state, will give back a state that is no longer properly normalized. Thus we could think of this as a way to mathematically represent measurement, but we would have to specify that the state afterwards benormalized again.

This is easy to see by thinking of Successive measurements… Successive projection operators might keep reducing the normalization of the state. One might be tempted to equate this with our Stern-Gerlach experiment, where each time we "block a state" we are removing half of our Ag beam, and thus successively reducing the intensity of the beam. **It is important to realize this is NOT EXACTLY THE CASE.** Because if we only consider what is happening to 1 Ag atom alone, after it "survives" one filter, it still has probability of 1 to go either way in the next filter. (In thinking about the beam intensity/"flux" of Ag, as a whole though this may not be a such a bad model.)

Thinking about whether it doesn't survive the filter, **This is related to another point that contains the essence of Quantum Mechanics: without the S-G there, in fact, no definite state is chosen. One must be careful.**

B) Expectation Values

If we know the probability of all outcomes, we can calculate what the outcome will be on average:

Average weighted by the probabilities:

$$\langle b_n \rangle = \sum_n P(b_n) b_n$$

Sum C P(C) This is the most important relations to apply to science. Use it all the time in experimental physics…

Since by our postulates above *$P(b_n)$ =* $|\langle b_n|\alpha\rangle|^2$, for any general state α then **it is easy to see, thinking of our projector form of the operator B, that this expectation value can also be written**

$$\langle \alpha | B | \alpha \rangle$$

This we already know from wave mechanics. We will discuss how the wave mechanics version of the expectation value fits in to our new formalism this week.

C) Compatatible/incompatible observables.

If [A,B] == 0, then A and B, along with their observables, are called compatible

Else they are called incompatible.

Good examples are angular momentum matrices. (we will demonstrate with our Pauli matrices σ_i)

From wave mechanics, L2 and Lz are compatible, while Lz, Lx are incompatible. Similarly for our spin matrices we can define the operator S2

$$S^2 = SxSx+SySy+SzSz = \hbar^2/4(\sigma_1^2 + \sigma_2^2+\sigma_3^2)$$

Here are some useful properties of the σ matrices, you can check w/ the matrices themselves:

$$\sigma_i \sigma_j = i\epsilon_{ijk} \sigma_k + \mathbb{I}\, \delta_{ij}$$

e.g. (σ 1 σ 2 **sheet**) --try it with the matrices themselves e.g. σ_x^2

which also implies

$$\{\sigma_i,\sigma_j\} = 2\,\delta_{ij}$$

$$[\sigma_i,\sigma_j] = i\epsilon_{ijk}\,\sigma_k$$

One thing that is interesting from these relations is that

$$S2 = \hbar^2/4(\mathbb{I} + \mathbb{I} +) = 3\,\hbar^2/4$$

There fore it is obvious that S2 commutes with every operator. This includes the S_i.. It is compatible with any of them. On the other hand each Si is incompatible with any of the others.

Starting 1.11/1.12. While we're on the subject of the σ matrices and their properties, let's talk about a few hints for the problems 1.11:

D) Non-commuting (Incompatible) Operators cannot have common eigenstates.

Suppose $|\phi_n>$ were an eigenvalue of both A and B. Then

[A,B]$|\phi$ n> = anbn$|\phi$ n> -bnan$|\phi$ n> = 0. But by first definitions of operators this implies [A,B] = 0.

E) Compatible operators share eigenstates:

The proof of this is as follows, for [B,C] = 0

$$C|b_k\rangle = \mathbb{1}\,C\,\mathbb{1}|b_k\rangle = \sum |b_m\rangle\langle b_m|\,C\sum |b_i\rangle\langle b_i|\,|b_k\rangle$$

$$= \sum_m \sum_i |b_m\rangle\langle b_m|C|b_i\rangle\langle b_i|b_k\rangle \quad (\langle b_i|b_k\rangle = \delta_{ik})$$

$$= \sum_m |b_m\rangle\langle b_m|C|b_k\rangle$$

Because B and C commute, it is easy to show that the term in the red box has only non-zero diagonal terms. That is in the basis $|b_n>$, $C_{ij} = C_{ij}\delta_{ij}$ (no implicit sum). If this is the case then the sum is removed, and the last line becomes

$$= \langle bkl|C|bk\rangle\,|b_k\rangle = \text{number x } |bk\rangle$$

which of course means that |bk> is an eigenket of C, with eigenvalue <bk|C|bk>.

This proof of Cij = $C_{ij}\delta_{ij}$ is similar to the Hermitian => real eigenvalues proof:

$$\langle b_n | [B,C] | b_k \rangle = 0$$

$$= \langle b_n | BC - CB | b_k \rangle$$

$$= (b_m - b_k) \cdot \langle b_n | C | b_k \rangle$$

thus as long as bm-bk are not the same, (which implies m and k are not the same). Then the term outside the parenthesies (the term in the box) must be zero.

F) Degenerate Operators and Eigenspaces:--

We will call cases where bm = bk for any m ≠ k the degenerate case: B will be called a degenerate operator that has *degeneracies*, degenerate eigenvalues meaning more than one the same. So if B has degenerate eigenvalues, then this proof isn't sufficient, but only only for the degenerate states. For the "non-degenerate" states, and obviously for any operators that don't have any degeneracies, it is sufficient to prove the initial statement.

F.1) Degenerate operator H: Some eigenvalues $\lambda_n^H \equiv h_n$ are the same (hn = hm for m≠ n). (see text above)

F.2) Degenerate eigenspace: (sometimes **called just an eigenspace,** since eigenvector implies 1-D space**): sub space spanned by all these |hn's> for which this is true. (Multiple degeneracies→multple degenerate eigenspaces)**

As before, it just makes things more convenient to assume non-degenerate case: the entire proof still works for those states bn that do have distinct eigenvalues. And for the ones that don't it is easy to see that C still always takes a degenerate eigenstate into another state that is still in the degenerate subspace. By definition this subspace is spanned by these eigenstates, so thinking of the linear algebra of either the pneumonic or the actual FORMAL matrix rep, it should be easy to believe we can always find a linear combination of the |bn>'s which "diagonalizes" C. Diagonalization in fact is what we call when Cij = Cijδ ij! Such diagonalization is the subject of the next section.

Prob 1.17: Essentially: if $[H,A_1] = [H,A_2] = 0$, but $[A_1, A_2] \neq 0$ prove that H must have degenerate eigenspaces.

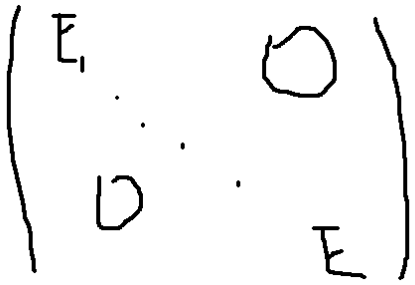
Problem 1.17 in fact is really all about our discussions about degenerate eigenspaces**…**

- Reminders:

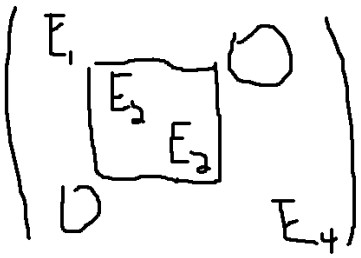
-We also said we can always represent any operator H as in the form $H = \sum h_n \, |h_n\rangle\langle h_n|$ --

if H is the "Hamiltonian", we can call the eigenvalues En ie hn ≡ En (although we haven't actually gotten this far yet in the formalism—but you already know this from (Wave Mechanics.) $H = \sum h_n \, |h_n\rangle\langle h_n| = \sum E_n \, |h_n\rangle\langle h_n|$

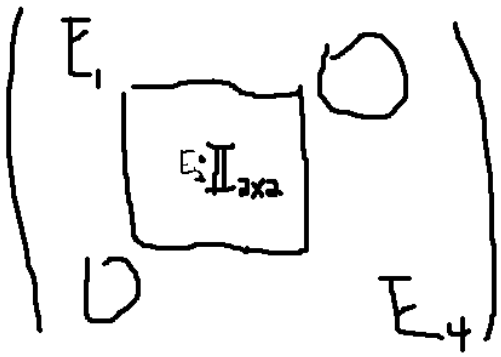
For this problem 1.17 especially, and to understand what we're talking about with degenerate subspaces, it is good to think about the matrix representations of operators. The above form makes it obvious that the matrix representation in the |hn> basis is diagnonal



If there's a degenerate subpace it means some number are the same: (point out grouping) for simplicity lets think about a concrete 4 D example.



Now for this problem it is also very useful to think about "block diagonal" form of any matrix: it means matrix within a matrix, along the diagonals.

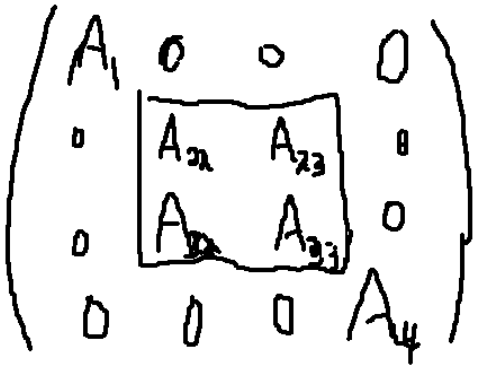


From yesterday: if we have another operator A, that commutes w/ H, then by our $\boxed{(h_m - h_n)\langle h_n|A|h_m\rangle = 0}$ relation, ➔ it must have

a) common eigenstates for all the non degenerate eigenstates, which means

b) these matrix elements in the |hn> rep are also diagonal.

c) Even for degenerate eigenspaces, in fact the relation actually also means that this matrix rep is "BLOCK DIAGONAL" (but NOT fully diagonal)



See this by actually considering matrix elements 1 by 1: from our relation in fact any elements A_{mn} must be 0 unless m and n are BOTH in the set of degenerate indices: in this case 2 or 3. This is exactly equivalent to our two statements from yesterday: statement A) that each eigenSPACE corresponding to each distinct eigenvalue are still proven to be orthogonal by our boxed relation above [in the case there are 3 eigenspaces: two 1-D spaces, corresponding to E_1 & E_4 and one 2-D eigenspace corresponding to E_2 (E_2=E_3).] and statement B) that A acting on any eigenket |hn> produces another ket which is still within the same eigenspace.

Finally to do the problem it is important to realize that when multiplying block diagonal matrices, you are just multiplying the "blocks". E.g. to see "why" A and H commute it can be traced down to the commutation of each block, e.g. that the "middle" 2x2 block of matrix A must commute with the corresponding block of H? Think about why that must be, and what would have to happen for it to fail.

Lecture 1/25/2010

More on degenerate eigenspaces, eigenvalues: Sakurai Prob 1.17. See powerpoint slides below

Review of End Class Friday

Phys 611, 01/25/2010

Justin Frantz

Proof $[C,B]=0$ share eigenstates:

$C|b_n\rangle = \langle b_n|C|b_n\rangle |b_n\rangle$

$= \text{number} \cdot |b_n\rangle$

thus: $|b_n\rangle$ is eigenket of C What's its eigenval?

For this, we needed to prove:

$C_{nk} = \langle b_n|C|b_k\rangle = \delta_{nk}\langle b_n|C|b_n\rangle$

$\Rightarrow (b_n - b_k)\langle b_n|C|b_k\rangle = 0$

Best to think of M_R

$M_C \doteq \begin{pmatrix} C_{11} & & & \\ & C_{22} & & \\ & & \ddots & \\ & & & C_{nn} \end{pmatrix}_{M_R}$ nondegenerate

$M_G \doteq \begin{pmatrix} G_{11} & G_{12} & & \\ G_{21} & G_{22} & & \\ & & \ddots & \\ & & & G_{nn} \end{pmatrix}$ degenerate (still "block diagonal")

Implication: back to original proof!

for degenerate b_n

$C|b_n\rangle = C_{nn}|b_n\rangle + \sum_{k\,\text{degen}} C_{nk}|b_k\rangle$

confined to degenerate subspace

e.g. $G|b_2^{(1)}\rangle = G_{11}|b_2^{(1)}\rangle + G_{12}|b_2^{(2)}\rangle$

Still confined to 2,3 subspace

Help 1.17: Make sure you understand my solution

Based on $[D_1^{\text{diag}}, D_2^{\text{diag}}] = 0$ always (talking about matrix's)

but $[D_1^{\text{diag}}, N^{\text{non diag}}] \neq 0$ unless (check yourself)

$D_1^{\text{diag}} \propto \mathbb{I} = \text{const}\,\mathbb{I}$

So thinking of MR of H, A^1, A^2 in 1.11:

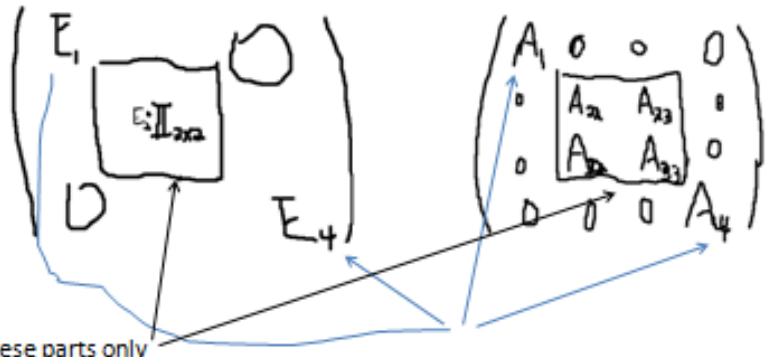
Simplest case 2×2:

$$\underbrace{\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}}_{H} \quad \underbrace{\begin{pmatrix} A^1_{11} & A^1_{12} \\ A^1_{21} & A^1_{22} \end{pmatrix}}_{A^1} \quad \underbrace{\begin{pmatrix} A^2_{11} & A^2_{12} \\ A^2_{21} & A^2_{22} \end{pmatrix}}_{A^2}$$

these will only commute if $E_1 = E_2$

these can easily be chosen for $[A^1, A^2] \neq 0$

Generalize to mixture degen. + non degen (all possibilities)



These parts only commute if $E_{22} = E_{33}$

These parts are Diag * Diag = always commute

One can be even more general and think of arbitrary number of nxn blocks of arbitrary size... then it is proven generally for *all* discrete operators with the same relations btw H, A^1, A^2

The big picture here is to realize that the above "mathematical" proof involving the matrix rep's of the operators is really what is behind the same proof for the operators
And in the same way, the relations involving compatible operators sharing eigenkets and degenerate eigenspaces really boil down to how their matrix rep's look

F) cont.

Notation for degenerate eigenstates (use > 1 label!): when we do have degeneracy it is obviously convenient to label the states according to the eigenvalues of both C and B, to uniquely specify which state we are talking about.

$$|b_n,c_j\rangle$$

e.g. from Wave Mechanics our L2 operator will have the same eigenvalue $\hbar$ l(l+1) for 2l+1 states: those will be further labeled with the Lz eigenvalues $\hbar$ m_l.

$$\mathbf{|l,m\rangle \rightarrow \quad L2|l,m\rangle = \hbar\, l(l+1)\, |l,m\rangle;\quad Lz\ |l,m\rangle = \hbar\, m_l\ |l,m\rangle}$$

for my notation where I label each eigenvalue h_n with the integer n it is actually not necessary: the state is still fully specified. But in Sakurai's notation, and actually as you have seen above for e.g. ang momentum/H atom states, other common labeling schemes, it is necessary.

F.3 Rule: If we find all such operators which commute with another w**e will call this the maximal set.** Then the labels for all those operators will resolve all the degeneracy. NOT well posed statement (we can find an infinite number of commuting observables of form c*) thus not very important for now.

G) Refinement to Measurement Postulate #3

Measurement of Degenerate Eigenvalue λ^G_{same} for degenerate observable G causes "partial collapse" to Degenerate Eigenspace $\{|\lambda^G_{same}\rangle\}$ a subspace of the full Hilbert space.

In this case the probability of measurement of value λ^G_{same} will be equal to the norm squared of the projection of the orginal state $|\psi\rangle$ onto the subspace. We have not discussed how to write a projection operator for a eigen subspace in bra-ket notation only the projection operator for a single eigenket. However from linear algebra, it should be 100% clear how to represent the Matrix Representation of such a projection Operator: It is just our Proj matrix from our linear algebra review

$$\text{Proj}_W\, x = \mathbf{MM^T x}$$

ie MR (Degenerate Eigen space Proj op) is just $\mathbf{MM^T}$. Remembering how M is constructed, you should be able to figure out a way to represent this in ket /bra format. Therefore I expect that you should already be able to derive an expression that represents this probability. More on this later.

Note that the partial collapse aspect is one of the essential properties of Quantum Mechanics. It is a VERY IMPORTANT feature of quantum systems and actually is what provides the basis

P-Set #1 Essay discussion Updated for 2010:

-Cat $|\Psi\rangle$ = (Alive + Dead): whether you believe or not→ matter of opinion (no wrong answer)

-however, I don't believe, I think most physicists are skeptical at best…(this is something philosophers like to debate, not physicists as much.) Think of this: why isn't the cat a valid observer—observation depends on IQ? Roger Penrose (famous mathematician) has theory of thought having quantum roots which is supposed to explain this, so my guess is there must be some way to validly pose the "thought == measurement" theory. Thus I will not discount it completely. However for this class, we will never rely on this argument.

But the reasons for me not believing Cat = Alive + Dead have to do with details of the cat being a large complex system, and I don't believe the quantum mechanics of simple states like in the SG expereiment apply to it as a whole without some further specifications. That is I don't believe one can so simply connect the cat to the simpler 2-state system. I do believe that the 2-state system is in the "Alive + Dead" superposition, and in that sense the Alive + Dead way of thinking is the more correct quantum mechanical intuition.

Screen or B field… (I will accept all answers as long as they were justified)

-FIRST almost no one said "after"—(eyes see?)—this would be equivalent to thought==measurement :

-Good! Important point: not just a question of "does the falling tree make noise if no one hears it"? We don't want to answer that question in this class. We are only concerned w/ things we can test in science—by definition that is an untestable question.

-I'm not 100% sure of answer myself: For this course, we are **only concerned with BOTH Mag + Filter/Screen**. So as Sakurai never does, I will not be able to give you a definite answer for now—we will discuss the situation a little further though, and after that we will be able to say more. Also we will see in the next section **how we can find out for sure by expt—which is really the most important determiner.**

-The **safest answer is the screen**. **True, the B field by itself should do SOMETHING.** But even for the single atom, if there is no screen, the two possible states $|\pm\rangle$ should have an Quantum Mechanical interference effect, much like when a single electron beam goes through a double slit collimator and exhibits an interference pattern that is equivalent to a superposition of it going through both slits. So as with the electron, it may depend on actual **wave function** considerations of the changes in trajectory (meaning the space wave function from wave mechanics, which of course still is part of the atom's quantum description—for example, will the beam split be small or large--something that is ignored in this thought experiment) as to whether such an interference will occur, and thus whether a full collapse has occurred in the magnetic field alone **occurring in the SG at the B field or screen**

So taking the interference effect into account, my best answer is that the B-field by itself (without the screen) is a "partial" measurement that causes a partial collapse: the state

is collapsed into a superposition of +/- for whatever the direction of the Bfield was, but it does not necessarily choose one or the other: it can in fact be left in a state that is the superposition of both.

Part of this question has to do with whether initial state are a "pure state" or a "mixed state" something we will discuss later in the course.

H) Consequence of Measurements (Summary)

For Degenerate Observables: For an observable that has a corresponding operator which is degenerate, measurement of (*only*) that observable which results in the degenerate eigenvalue being measured will only cause a "partial collapse" of the state into the degenerate subspace. The exact state within the subspace is not generally affected by the measurement, and is still uncertain.

For compatible observables: During two successive measurements of two different but compatible observables, the second measurement can not change the state in such a way that the previously measured eigenvalue, if measured again, will not be measured again.

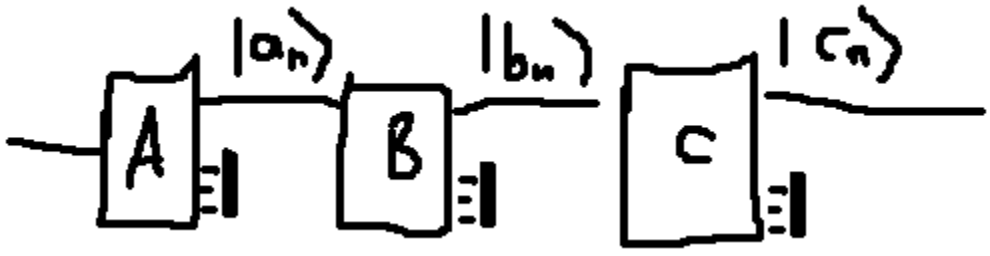
"Measurements of Compatible Observables do not interfere"

Incompatible Observables: on the other hand: During two successive measurements of in compatible observables, the second measurement causes a "re-collapse" of the state which changes the probability of subsequent measurements.

"Measurements of Incompatible Observables DO interfere"

I) Example of Measurement Interference

Consider the following set of S-G –like apparatuses applied in succession: 3 observables A, B, C, and a filter for only picking one of the eigenstates.



If before going through measurement filter A, our quantum state (think of the state of an atom flying through all three, like in SG) is some arbitrary state $|\alpha\rangle$, then we can write the total probability to have the set of measurements indicated in the drawing (an, bn, cn) as $P_{tot} = |\langle\alpha|a\rangle|^2 P_{a\to b\to c}$ where $P_{a\to b\to c}$ is

$$P_{a\to b\to c} = |\langle cn|bn\rangle|2|\langle bn|an\rangle|2$$

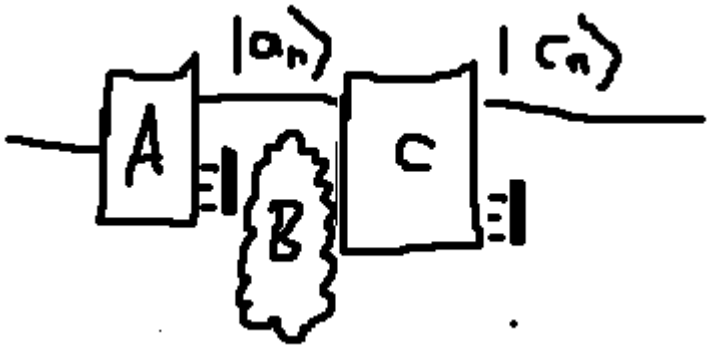
(

Could've labeled it ak, cj, bn→means the same—thus let's just use a, b,c . We can rewrite $P_{a\to b\to c}$ as $\langle c|b\rangle\langle b|a\rangle\langle a|b\rangle\langle b|c\rangle$ Thus, the sum over all measured b routes to get one particular a→c combination , $P_{a\to c}^{\ b}$ as

$$P_{a\to c}^{\ b} = \sum_b \langle c|b\rangle\langle b|a\rangle\langle a|b\rangle\langle b|c\rangle$$

Notice there is only one sum over all the b states.

Now compare to if we remove b completely from the picture



(we can just imagine the presence of B)

For here Pa→c is just $|\langle c|a\rangle|2$ which can be rewritten inserting 2 closure sums over b (the concrete expression of our b imagination), and therefore two *different* sums over b

$$Pa\to c = \sum_n \sum_m \langle c|b_n\rangle\langle b_n|a\rangle\langle a|b_m\rangle\langle b_m|c\rangle$$

Notice: it is not the same as $P_{a\to c}^{\ b}$ N!!!!

Neat. Huh good demonstration of the essence of the weirdness that is at the heart of Quantum Mech. But really this is not very hard to understand what's going on here or really very "surprising" what we already stated about measurements in general.

Why it's not hard to understand: Thinking of routes "through b states" is very helpful to understanding this:

> **Q explain in terms of what b states are "gone through" in each case: what routes?** every time it's going through all states of |b> ***at once***. N routes. As opposed to going 1 route N times. Think of a traffic: it only 1 road is open out of N. the resulting traffic will certainly not flow as efficiently.

Why it's not too surprising: it is just a result of the collapse of that occurs from the measurement of b). All it is saying is that $|a\rangle = \sum c|b\rangle \neq |b\rangle$

How to use this result to answer our question of where the collapse in the Stern-Gerlach takes place definitively – in the B field or when it hits the screen/filter?

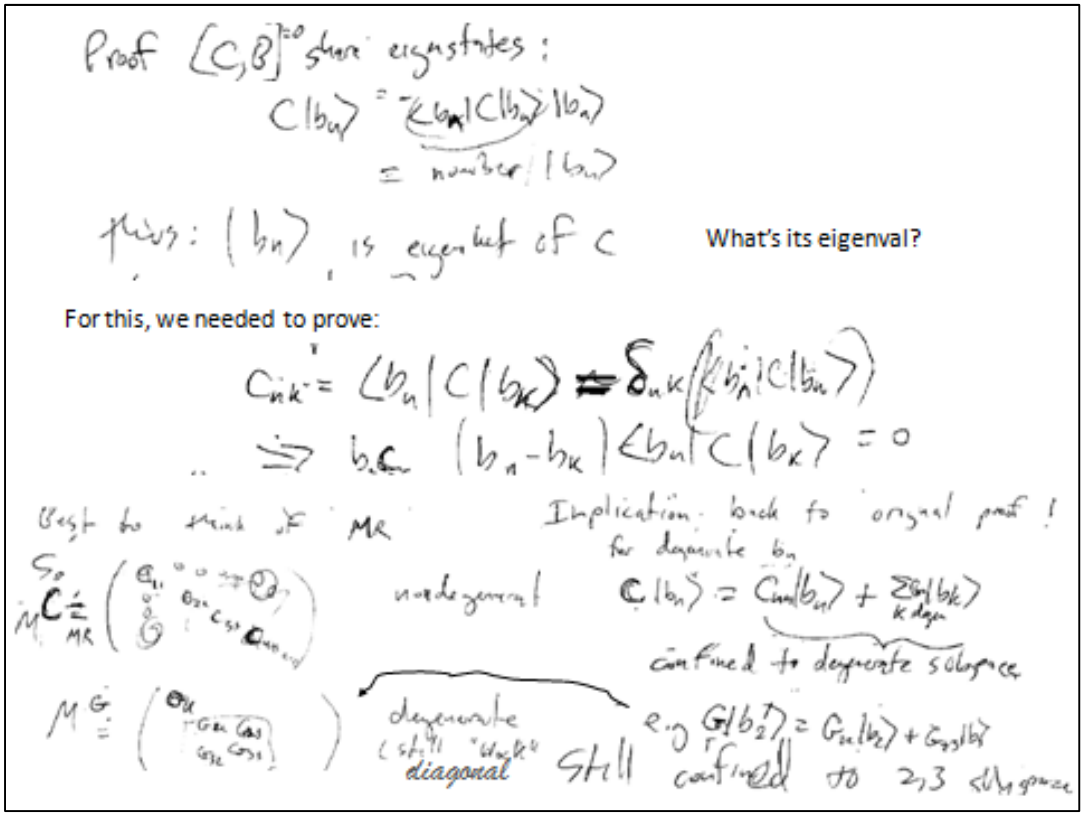
Actually it's not this relation per se that could prove it for us, but using 3 successive SG we could at least gain some valuable insight: Suppose we repeat our first "MORE INTERESTING" SG experiment w/ 3 successive SG's

SG1: z field filter SG2: x field filter SG3: z field filter

Now just remove the filter part of SG2 leaving only the x B field. Our argument before was that we were "removing half of zero" w/ the x filter, which caused a new state with both Sz components again. So if the magnetic field caused z field to again have both components then it is likely that the x field alone did cause a collapse. Though it is still not clear whether the collapse was in the x+ or x- direction.

Note that in class I did think we could also determine whether or not the collapse was occurring by comparing probability sums just like the above case w/ A, B, C. Although seemingly in need of the actual filter I believe there may be a way around not having an actual filter, by averaging over all possible filter configurations, and comparing that sum to going without the filter, but only having the field. I still think this may be possible using a single atom beam (ie one Ag atom at a time), but I do not have time to figure it out—if you've thought of a way, please let me know…

Lecture 1/26/2010 Handouts from yesterday in notes. Problem set: mistake: last problem 1.23 a) Reading assignments.



Review : Mention Diagonalization of Degenerate subspace/Labeling

J) Generalized Uncertainty Relation: (Last thing about Incompatible Operators)

We all remember the Heisenberg Uncertainty relation $\Delta x \, \Delta p \geq \hbar/2$

One of the most important points of this course is impress upon you that such relation is not just restricted to p and x, but is

true for any two incompatible observables.

Thus it should be **derivable from just our most general formalism.**

In Sakurai the following relation is proved

$$\langle \Delta A^2 \rangle \langle \Delta B^2 \rangle \geq \frac{1}{4} |\langle [A,B] \rangle|^2$$

OK we can take sort of take the sqrt of both sides: point is it's still **an** "uncertainty" relation….

However it is just as easy (perhaps more) to prove:

Let A and B be hermitian operators. With Schwartz' inequality follows

$$\begin{aligned} |\langle\psi \mid (A-\bar{A})(B-\bar{B}) \mid \psi\rangle| &= |\langle (A-\bar{A})\psi \mid (B-\bar{B})\psi\rangle| \\ &\leq \| (A-\bar{A})\psi \| \, \| (B-\bar{B})\psi \| \\ &= \Delta A_\psi \, \Delta B_\psi \end{aligned}$$

Note that by convention what we call ΔA equals √($\langle(\Delta A)^2\rangle$ which is equal to || **|ΔA ψ >** || (meaning norm of ket |ΔAΨ>) Obviously $\bar{A}$ means <A> which is just a number.

Now turning to the LHS, we know that we know that for any complex z, $|\mathrm{Im}(z)|^2 \leq |z|^2$

$$|\, Im\langle\psi \mid (A-\bar{A})(B-\bar{B}) \mid \psi\rangle| \leq |\langle\psi \mid (A-\bar{A})(B-\bar{B}) \mid \psi\rangle| \ .$$

One also has

$$\begin{aligned} |\, Im \ \langle \ \psi \mid (A-\bar{A})(B-\bar{B}) \mid \psi\rangle| &= \frac{1}{2} \, |\langle\psi \mid (A-\bar{A})(B-\bar{B}) \mid \psi\rangle - \langle (A-\bar{A})(B-\bar{B})\psi \mid \psi\rangle| \\ &= \frac{1}{2} \, |\langle\psi \mid (A-\bar{A})(B-\bar{B}) - (B-\bar{B})(A-\bar{A}) \mid \psi\rangle| \\ &= \frac{1}{2} \, |\langle\psi \mid [(A-\bar{A}),(B-\bar{B})] \mid \psi\rangle| \\ &= \frac{1}{2} \, |\langle\psi \mid AB-BA \mid \psi\rangle| = \frac{1}{2} \, |\langle\psi \mid [A,B] \mid \psi\rangle \end{aligned}$$

so...

$$\frac{1}{2} \, |\langle\psi \mid [A,B] \mid \psi\rangle| \leq (\Delta A)_\psi (\Delta B)_\psi \ .$$

The proof in Sakurai is more complicated. I'll let you read it and let me know if there is anything you don't understand. But it does use 1 useful relation we may use in the future.

An Anti-hermitian operator ($A^\dagger = -A$) has purely imaginary expectation values (use $\langle\psi|A|\chi\rangle = \langle A\chi|\psi\rangle^*$)

Lecture 1/27/2010 Note: scanned version of chalkboard notes (more condensed version of text) at end

VI) Transformation Operators

A) Introduction

We had mentioned **diagonalization:** in our discussion of degenerate spaces: the idea was we said there is we could always find a set of basis vectors $|\phi_n\rangle$ for which the following would be true for A

$$\textbf{diagonalization:} \quad \langle\phi m|A|\phi n\rangle = \langle\phi n|A|\phi m\rangle\, \delta mn \quad \text{(no implied sum)}$$

This is just a completely literal statement of the fact that the matrix representation of A in this basis is diagonal

Remembering from **linear algebra** that the way we **diagonalize matrices** means that we

$$\textbf{find} \text{ the matrix } \mathbf{S} \text{ such that we can form the matrix } \quad D = S^{-1}AS \quad \text{where } S^{-1}S = 1.$$

We also remember **if** it's a **symmetric** matrix **(**if $\mathbf{A^T = A)}$ then $\mathbf{S^{-1} = S^T}$.

For operators we will think of the **matrix S→ operator U** called the "**transformation operator".**

Just as for symmetric matrices, **for hermitian operator B (B=B†)**, the operator $U_B = U$ which diagonalizes it, will always have the property that $U^\dagger = U^{-1}$ or

$$UU^\dagger = U^\dagger U = \mathbb{I}$$

Note that such an operator is **NOT hermitian** in general (but could be in specific cases? e.g. Proj operators?).

We will remember for matrices, that this is essentially **equivalent** to finding the eigenvectors of the matrix, since S is made up of the eigenvectors. (ev1|ev2|…) On the operator side, if we have a mathematical form for the operator, and we have a complete set of basis states, it is the problem of finding the expansion of each eigenstate in the old basis:

$$|b_n\rangle = \sum c_m\, |a_m\rangle$$

that is, finding the coefficients c_m ***for every n*** $([c_m]_n)$: this will be the ingredients for forming U (in fact we will find, $\mathbf{U_{mn}}$ **(matrix rep)** $\mathbf{= [c_m]_n}$.) Note however simply expanding |an> in the |b> basis is **not** *what is* ***performed*** when we ***apply*** U to a ket: ie it's not what we use U for. For that all we need is to insert closure = 1 → $|b\rangle = \sum |a\rangle\langle a|b\rangle$. What we **DO** with U is very different as we shall see.

You might be thinking at this point: **don't we already have the answer for what U is: $[c_m]_n = \langle a_m|b_n\rangle$? The answer is actually true –however you've missed the point:** to calculate $\langle a_m|b_n\rangle$, we need **first need the expansion parameters $[c_m]_n$!** So then we can actually calculate <am|bn> = as

$$\langle a_m|(\sum c_k\, |a_k\rangle$$

–only in this form do we know how to "remove" the innerproducts.

B) Tempting Confusions in Sakurai:

1) Above: Bra/ket expressions so simple (e.g. Bra/ket form given for U) one thinks there is magic ket way of avoiding Linear Algebra work w/ Matrices to find eigenvectors. There isn't, you must still do the same Linear Algebra operations

2) simple expansion in different bases (passive rotations) which is performed with the completeness operator (ie 1) with b) the action of U (which is to actively rotate vectors).

C) Action: What does this transformation operator actually DO?:

Let's think deeper about this. **We said ALL operators are transformations.** How are these "**transformation transformations"** different?

When we diagonalize a matrix what are we actually doing? Answer: we have a set of basis vectors $\hat{a}_n$, that are unit vectors .e.g $\hat{a}_2 = (0,1,0 \dots)$ and we have some matrix B which has eigenvectors $\vec{b}_n$ which are NOT unit vectors, but when we diagonalize B, we "switch places" between the b's and a's: now the b's become the unit vectors, and the a's will actually no longer be unit vectors in the new "diagonal" space. In fact explicitly this is what the S matrix does: $a = S^{-1}b$. Thus this is also exactly what U does in our bra-ket formalism:

Answer: It provides a 1 to 1 mapping of one orthonormal basis state to another--- FOR ALL BASIS STATES!!!!

Thus if we want to change from basis a to b: $|b1\rangle = U|a1\rangle$ and $|b2\rangle = U|a2\rangle$ ….

Compare this to $|b1\rangle = \sum |a_m\rangle$ etc.. it is very different. One we are just writing $|b1\rangle$ as an expansion in $|a\rangle$'s-- in the other **changing the a's to b's**

Now think of our pneumonic or the FORMAL matrix reps: these are vectors. We already said we can always think of different orthonormal bases spanning the same space. If each of these bases are an eigenbasis of two operator. From completeness we know we can always expand a vector in any of these basis.

$$|\alpha\rangle = \sum |b\rangle\langle b|\alpha\rangle = \sum |c\rangle\langle c|\alpha\rangle$$

Remembering how our SG example of Sx "basis" could be thought of as a rotated version of the Sz basis, should be clear that it is like a rotation of a coordinate frame.

Geometrically it is clear that this is just a **"change in coordinate frame"** (from orthonormal1 to orthonormal2). That is, it's like a **passive rotation**: vector stays the same, coordinates (coefficients of the basis states) change.

The **"transformation transformation"** we're talking about is like an **active rotation:** you take the vector (e.g. |a1>) and rotate IT into one of the |bn's> (which we might as well label |b1>)—COORDINATES stay the same!!! with respect to the new coordinate basis.

The U's are indeed like "rotation matrices"—many parallels --rotation matrices always have det = 1→like being unitary. Think of the Stern Gerlach: But NOTE that we are NOT talking about the SG rotation in REAL space of 90 degrees, (which we said was meaningless for the SG), but rather the rotation in |ket> space of 45 degrees. Abstract rotations in our abstract vector space. And there are more than just rotations in ket space that can be unitary. Real rotations in 3-D space will be discussed in Chapter 3.

C.1) Transformation Operations Within the SAME Basis

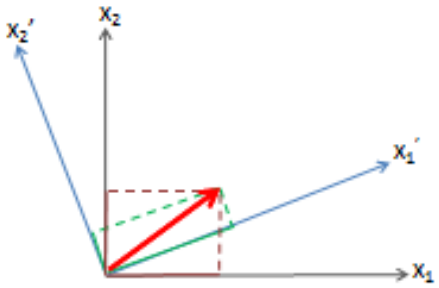
Transformation operators do not ***have*** to connect 1 eigenbasis to a ***different*** eigenbasis. One can also imagine a transformation operator that maps each basis ket into a different basis ket IN THE SAME BASIS.

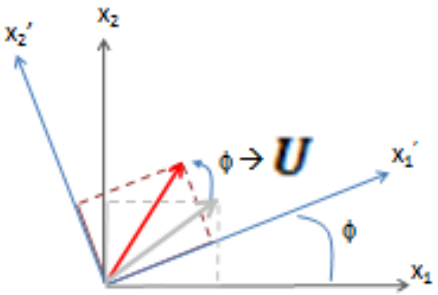
$$|a2\rangle = U|a1\rangle,\ |a4\rangle = U|a2\rangle, \text{ etc…}$$

These are necessarily all rotations of 90 degrees. Such transformations are usually related to some symmetry.

Other than our first application of unitary operators, most later applications will be of this type.

Transformations: Visual





$\vec{a} = 2.5\hat{x}_1 + 2\hat{x}_2 = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}$

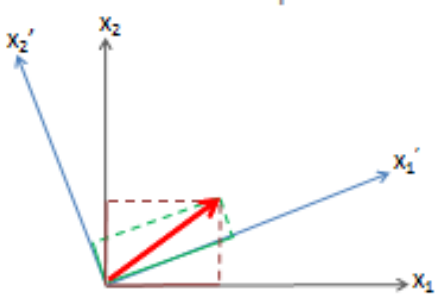
$= 3\hat{x}_1' + 1\hat{x}_2' = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

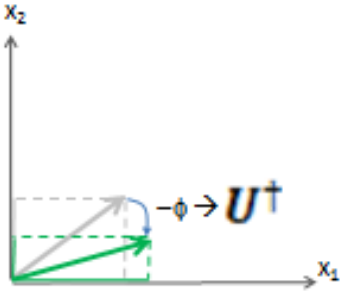
$\vec{a}' = 2.5\hat{x}_1' + 2\hat{x}_2' = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}$

$$\vec{a}' = U\vec{a} \neq \vec{a} \text{ !!!!}$$

Transformations: Neat Relation

Easy to confuse when thinking of Sak. MR equation: New = U† Old





$\vec{a} = 2.5\hat{x}_1 + 2\hat{x}_2 = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}$

$= 3\hat{x}'_1 + 1\hat{x}'_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\vec{a}'' = 3\hat{x}_1 + 1\hat{x}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\vec{a}'' = U^\dagger \vec{a} \neq \vec{a}$!!!!!

$\vec{a} = 2.5\hat{x}_1 + 2\hat{x}_2 = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}$

$= 3\hat{x}'_1 + 1\hat{x}'_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\vec{a}' = 2.5\hat{x}'_1 + 2\hat{x}'_2 = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}$

$\vec{a}' = U\vec{a} \neq \vec{a}$

$\vec{a}'' = U^\dagger \vec{a} \neq \vec{a}$

$\vec{a}'' = 3\hat{x}_1 + 1\hat{x}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$U^\dagger$

D) Explicit forms for U

It is easy to see that we can make an easy representation of U in our ket-bra notation as

$U=\sum_m |b_m\rangle\langle a_m|$

where $|b_m\rangle$ is the basis one may have the expansion of some vector $|\alpha\rangle$ in, and $|a_m\rangle$ is the new basis. (obviously we could replace $|b\rangle$ by any other orthonormal basis that spans the same space).

Also as with the matrix representation, such an explicit representation for a unitary **operator only applies to discrete kets**. (discrete summation)

It is very easy to construct $U^\dagger$ and prove that $UU^\dagger = 1$.

You see again, although U has the above simple form, to find the matrix elements of U in either basis e.g. in $|a\rangle \rightarrow \langle a_n|U|a_m\rangle$ it will be necessary to know the expansion of $|b\rangle$ in the $|a\rangle$ basis, so it is perhaps good to think of the above simple form of U as really

$$U=\sum_m \left(\sum_n [c_m]_n \, |a_n\rangle\right)\langle a_m|$$

or something similar with only $|b_m\rangle\langle b_n|$ outer products in it. Thus notice despite the original nice compact form of U, we still haven't magically gotten away from this problem of having to find the group of numbers$[c_m]_n$ to specify it.

Not to be confused with the completeness operator as described above:

$$1 = \sum_m |a_m\rangle\langle a_m| = \sum_m |b_m\rangle\langle b_m|$$

E) Matrix Representations and U (see above diagram)

Hopefully I've convinced you that U acting on a ket will never just give us the expansion in another basis. Neither will $U^\dagger$. However you will notice that Sakurai seems to concentrate on something that sounds similar: that

> **one can get the MATRIX REPRESENTATION of an arbitrary state $|\alpha\rangle$ in the NEW BASIS by multiplying the MATRIX REPRESENTATION of $U^\dagger$ in the OLD BASIS by the MATRIX REPRESENTATION of $|\alpha\rangle$ in the OLD BASIS. --(proper statement)**

For example on page 38 he writes eq. (1.5.11): **(New) = $U^\dagger$ (Old)**

It is of extreme importance that you do NOT interpret this as

|new basis ket> = $U^\dagger$ |old basis ket> (NO!!!!!)

Such an interpretation isn't at all true. In fact, we stated that $|b\rangle = U|a\rangle$ and $|b\rangle$'s where the "new basis ket's". So at best this relation would be "backwards" if interpreted this way. It can in fact only be interpreted as my "proper statement" above. This is perhaps the clearest case that can distinguish our "pneumonic" from the real matrix representations—it is something that only makes sense for the FORMAL matrix representations of the kets, not for the kets themselves.

As long as this is clear, then I will finally state that, yes, there is something else we can ***use*** U for in **MATRIX REPRESENTATIONS**, if we already have U's own matrix representation defined, that is, all its matrix elements are calculated. It is just what we started out saying about how the ingredients of U → U_{mn} were related to the expansion of |b> in the basis |a>.

So we can think of the "other use" as the inverse relation of this: if have the matrix element numbers U_{mn} already, we actually have a convenient formula that Sakurai shows, for the projections of an arbitrary state |α> onto the new basis states <b|α> in terms of the old projections, <a|α>

$$\mathbf{\langle b_n|\alpha\rangle = \sum_n U^{\dagger}_{nm}\langle a_m|\alpha\rangle}$$

Notice once more: it is nothing like U operating on |α>. The $U^{\dagger}_{mn}$'s are indeed ***pure numbers*** – in fact since matrix $A^{\dagger} = A^{T*}$ we can actually write these numbers more explicitly as $(U_{mn})^{*}$ (note the ordering of the indices switched →>transpose). All we have really done here is taken our original expansion, bra-ed it to the right, and then taken the c.c. of both sides!

As it is just the uses of the numbers Unm that Sakurai concentrates on, it should now be clear that the in the sub section where he discusses how to find these numbers using the standard methods from linear algebra, the point is not ***HOW*** one does it (since we should already know this from linear algebra), but rather ***THAT*** one needs to do it, despite the convenient ket-bra sum explicit form that was already given.

F) Unitary Equivalent Observables

A and the operator **UAU† are** said to be **unitary equivalent observables/operators**. It does NOT mean they are equal!!! If you are trying to diagonalize A then UAU† will be diagonal but there will be other cases where will want to consider unitary equivalent observables/operators O where UOU† (nor O) are diagonal.

IV Transformation Operators

- Book - Dense! Very Important

Easy to miss some points

→ Read lots of notes.

Intro Diagonalization. (of A)

Find $|\varphi_n\rangle$ such that

$\langle\varphi_m|A|\varphi_n\rangle = \delta_{mn}\langle\varphi_n|A|\varphi_n\rangle$

LA: $D = S \cdot A S^{-1}$

$= S^{*} A S^{T}$ if A hermitian

$S^T S = S^{-1} S = 1$

QM:

$S \Rightarrow U$

Sak: Simple ket/bra form of U

How to find S_z in LA

How " " U in QM/Fam

Same work! Same procedure

Must use LA!

Confusion

1) Sik: Simple form of U no LA? There is!

2) Confusing Expansion in diff basis w/ Action of U:

1) In LA: $S = (\hat{e}_1 | \hat{e}_2 | \ldots)$ $\hat{e} \equiv$ eigen

In QM: to diagonalize S given basis

$\Rightarrow$ Find expansion: $|b_n\rangle = \sum_m a_{nm} |e_m\rangle$

(same as LA if $|e_m\rangle = a_{nm}$)

Simple expression: $a_{nm} = \langle b_n | e_m \rangle \Rightarrow$ U?

ingredients.

Yes, but need $(a_{nm})_n$ for all n!

must use LA

Action of U:

(Translation Transformation special)

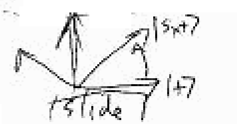
1-to-1 mapping:

"old" basis $|a\rangle$ to new basis $|b\rangle$

$|b\rangle = U|a\rangle$: $|b_1\rangle = U|a_1\rangle$, $|b_2\rangle = U|a_2\rangle$ ...

Same U for all n? Like rotation in ket space!

e.g. $|S_{x+}\rangle = U|+\rangle$



- "Active" Rotation!

= Not Passive Rot = Expansion in (new basis)

see plot

NOTE! Map: "new" / "old" can be same.

e.g. $U^{aga} \Rightarrow |a_2\rangle = U|a_1\rangle$, $|a_3\rangle = U|a_2\rangle$

what angle (in ket space?)?

This class is must like this

D) Bra/Ket Representations:

Explicit Form of U (active)

$U^{bra} = \sum_m |b_m\rangle\langle a_m|$

- Easy to form $U^\dagger$ & prove: $UU^\dagger = \underline{1}$

Deceiving: Remember

$= \sum_m \left(\sum_n |a_n\rangle\right)\langle a_m|$

$\Rightarrow$ MR of U.

Contrast to Expansion = $\underline{1}$ (completeness) (passive)

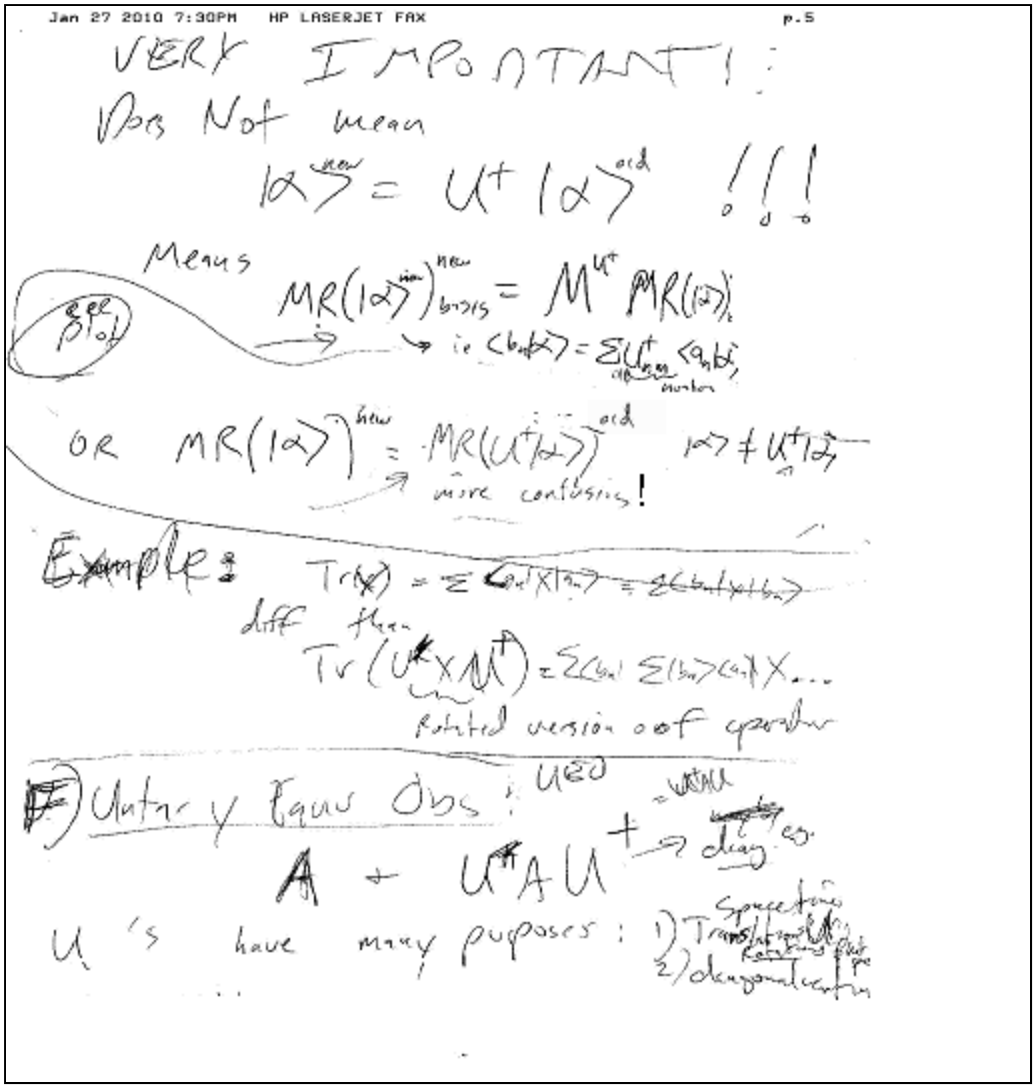
$|\alpha\rangle = \sum |b_n\rangle\langle b_n|\alpha\rangle = \sum |a_n\rangle\langle a_n|\alpha\rangle$

$= a_n|b_n\rangle$

E) M.R. + U (Sak)

1) $U \Rightarrow U_{mn}$ from L.A./M.R.

2) Sak: (New) $= U^\dagger$ (old)



Lecture 1/29/2010: 1) Pset will be posted Monday—will include Sak 1.28 c. 2) Upcoming expanded reading assignment by next Friday including Sak section 3.9 and 3.4 up to p. 181 (see website). 3) Midterm:

Sakurai shows the following which seems **not very useful ever in this course**: but just to further clarify its meaning, so as to convince you of this:

if A|a> = a|a>

UA|a> =U a|a>

and since $UU^\dagger = 1$

$$UAU^\dagger\ (U|a\rangle) = a\,(U|a\rangle).$$

(**U|a>** also **eigenket** of **UAU†**)

U could have been be any such transformation operator to any basis, but let's indeed choose to transform to **|bn> w/ operator B**. Thus **suppose U|a> = |b>** so we have

$$UAU^{\dagger}\,|b\rangle = a|b\rangle.$$

Thus **|b> is an eigenstate also of UAU†.** |b>'s were eigenkets of the operator B originally.

Sakurai then **states** that **UAU† = B in "many cases of physical interest."** What are we to make of this statement? That **usually the statement is true?** I think it's better to say **usually not**.

The above statement that |b> 's are eigenkets of the operator UAU† is nothing more than the following**. Remember:** how we already said we could write **any such** Hermitian **B** operator with eigenvalues bn as

$$B = \sum_n b_n\,|b_n\rangle\langle b_n| \;?$$

Well by the **same logic we can make arbitrary diagnonal operators X with the same form**

$$X = \sum_n x_n|b_n\rangle\langle b_n|$$

These **operators X will have the |bn>'s as eigenvectors and eigenvalues x_n**. (think of the pneumonic representation—the |bn>'s are just unit vectors (1,0,0..) and X is any diagonal matrix).

Thus in effect **that's all we've done here by finding the diagonal form UAU†**

$$UAU^{\dagger} = \sum_n a_n\,|b_n\rangle\langle b_n|.$$

Two points that are demonstrated here: 1) it is clear that **only if** the **an's equal the bn's** will **B= UAU†** . (eigenvalues same). There seem to be many such operators that don't have the same eigenvalues (any other observable!) and thus diagonal operators we can imagine that AREN'T equal to B (any X which has different xn's) this is why my statement would be that they are **usually not.**

G) Diagonalization

If we want to USE diagonalization to solve a problem, the general procedure is as follows.

General Procedure:

1) Find U (using LA)

2) Rotate States/Operators with U (actually states go w/ U† , basis vectors w/ U)

3) Do Calc's etc

4) For many things (e.g. modified states) MUST ROTATE BACK (U†/U)a

Ie it's good to think of these active rotations as being applied temporarily in the case of diagonalization. For other uses of U this won't be the case.

G.1) Points to remember when diagonalizing:

If we wish to do any **calculations <u>related to B</u>, unless we want to also form UBU† which will NOT be diagonal, UAU† is NOT a stand-in for A!!!!** in the **space of |b>** --

E.g. Question 1): if we want to take the product AB in the basis b, can we just multiply the two diagonal operators , B and UAU† ? NO—in the space of |b>, A ≠ UAU† If we wish to take the product

AB → (non-diagonal) (diagonal) = nondiagonal

=UAU† U B U† → (diagonal)(non-diagonal)

=U(AB)U† → non diagonal

Question 2): If I specify for you the state of the system as |α> (say I give you its expansion in some basis |a>) and you want to work in the diagonal basis , does the transform of the state, U|α> still represent the same physical state? In some sense but remember:

This is a rotated state: →NOT the same state.

In fact you should form U|α> (as long as you always form UOU† for every operator O) and indeed while doing your calculations it does *represent* the same state. But when the calculations are done you need to rotate back. Thus it is really like a temporary rotation, which is always waiting to be rotated back.

Concrete example: If system is in state |α> =|Sz+> and one wishes to diagonalize into the basis where operator Sx is diagonal (|Sx±>), then the U we want is such that U|Sz+> = |Sx+> - U|α> is obviously NOT the state we are actually in . So when doing calculations it may be helpful to form U|Sz+>, but when the calculation is done, if we still wish to to describe the same state we have to transform it back. Thus all we can do further that is helpful in that case, is to expand |α> = |Sz+> in the |Sx> basis:

$$\mathbf{|Sz+\rangle = U_{+-}\,|Sx+\rangle + U_{-+}\,|Sx-\rangle.}$$ where $\mathbf{U_{\pm\mp} \sim \{\pm 1/\sqrt{2}\}}$ **ie numbers.**

In effect all this means is if you transform a state during a calculation, any results of the calculation must be transformed back into the original basis.

For diagonalization: I said "you do not want to form the state U|α > and expect it to represent the physical state". For diagonalization: you may form U|α > and indeed do calculations with all operators

transformed as UOU† --in this sense you CAN ***in the diagonal space only*** think of U|α> representing your state.

e.g. expectation values of **O** in new space will be

$$\mathbf{\langle \alpha | U\,(\,U^{\dagger} O U)\;U^{\dagger} | \alpha \rangle} \;\rightarrow \text{ invariant of U “rotations”}$$

However if not invariant quantity like the expectation value (ie in general things aren’t) -- you must **transform back** when done w/ calc..

E.g. calculation finds state changes **in diagonal representation** from U|α> → |β’> (define as = U|β>)

Real new state is NOT now U†|β> but must be transformed back U |β ‘> = |β>

Just as original state is still|α> not U|α>

This is what is meant by diagonalization is passive rotation →always transform back (ie we never changed the state—just exanded it in different basis)

Add example here: 3-D? 2 pset examples. Better examples for time dependence…

IV Position And Momentum In the Formalism

Introduction

Sakurai sections 1.6-7 tells us how to fit **wave mechanics** into our **bra-ket formalism.** The key point is that the **basis we will use** for the position operator **|x>** will be our type of continuous kets.

One thing we **never discussed:** was **summing up probabilities for multiple eigenvalues**. We said our measurement probability for discrete or continuous kets would be

$$P(a_n) = |\langle a_n|\alpha\rangle|^2$$

First though consider that by definition the probability of getting one of a **group of an values (3,4,5)** would just be the sum of all probabities.

$$P(a_n) = |\langle a_n|\alpha\rangle|^2 \rightarrow P(\text{an one of } \{a3, a4, a5\}) = \sum_{n=3 \text{ to } 5} |\langle a_n|\alpha\rangle|^2$$

Actually this statement we need to **clarify slightly for continuous kets |ξ>:.**

For our continous kets, just as we initially defined them, to make linear combinations of these we need to integrate over our basis labels: e.g. for **some random |ϕ >** (like |α>)

$$|\phi\rangle = \int c(\xi)\, d\xi\, |\xi\rangle$$

where obviously by completeness **c(ξ) = <ξ|ϕ >.** In this expression we never specified what the **integration limits ?** were**:** they could be over **any range of ξ in fact, even** $\pm\infty$ (∞ ***e. g.*** **completeness).** Suppose |ξ > is an eigenvector of some operator ξ s : then it should be **easy to guess** that when trying to find the probability given a state |ϕ >that ξ will be is in some range ξ 1 - ξ 2 when measured, we just integrate the over that range.

$$P(\xi 1 < \xi < \xi 2) = \int_{\xi 1}^{\xi 2} |c(\xi)|^2 d\xi = \int_{\xi 1}^{\xi 2} |\langle \phi|\xi\rangle|^2 d\xi$$

A) Position Operators and Eigenkets

For position : we will formally define the

position operator in 1-D : x

and

→**its eigenkets:** as **continuous kets |x>.**

Thus **all of what we described as ξ will apply for x: Reminder: all the definitions/rules apply except with an integral sign, in place of summation such as completeness.**

A word about completeness: In fact **we postulate that |x> forms a complete basis**, which certainly **makes sense:** considering that **we allow all x values** from +/- infinity.

However one might wonder as per our statements regarding the fact that as continuous kets |x> has an infinite norm and in fact we said that the ket |x> by itself is not a proper member of a genuine Hilbert space, whether this is **"technically true"**. (infinite norm→not proper Hilbert space members?) -

The answer is **yes:** the eigen-differentials we discussed that where the actual Hilbert space member, must satisfy a completeness relations, and we will assume that ***this*** completeness relation can be rewritten ∫ |x><x| dx

Note that since the operator **x should have only real eigenvalues**, then by definition we **must force it to be hermitian. x = x†**

Following our expression for probability of continuous kets, having in mind the probability for the eigenvalue x to be found in some range (x1, x2)—we know the expression will be

$$P(x1 < x < x2) = \int_{x1}^{x2} |c(x')|^2 dx'$$

were **|c(x)|2 = |<x|α>|² if** the system is **in** the **state |α>.**

Looking back to our **wave mech review** we can see that it appears that the projection of |α> on to the ket |x> must be the **wave function ψ(x)** from wave mechanics!!

<x|α> = ψ α(x)

If we like, we can **extend the concept to 3 D:**, defining such **operators for each space dimension, x, y, z.** These will have the **same properties** as for x (**e.g. hermitian**)

We will say these three operators are all **inter-compatible so they commute [x,y] = [y,z] =[z,x] = 0**.

Thus they will have **simultaneous eigenkets**, **|Pos>**

x|Pos> = x_{Pos}|Pos>, y |Pos> = y_{pos} |Pos>, z...

which would naturally be **infinitely degenerate!** for each value of 1 dimension (in the other 2 dimensions for any value for the 3rd. **(** Now one will note our discussions of degeneracies so far were only involving discrete kets. We **won't discuss resolving continuous ket degeneracies**. **)**

Thus we will **label the 3-D ket |x, y, z> or better $|\vec{x}>$.**

The 3-D wave function will then turn out to be the→ **projection of the state |α>** onto this 3-D ket.

$$\langle \vec{x} | \alpha \rangle = \psi_{\alpha}(\vec{x})$$

Because we will obviously want the probabibility of particle to be located in some 3-D range {x1, x2: y1, y2: z1,z2} as a triple integral

$$\int_{x1}^{x2}\int_{y1}^{y2}\int_{z1}^{z2} dz'dy'dx' \,|\langle \vec{x}|\alpha\rangle|^2$$

It is **important (!)** to realize that the **1-D x eigenket |x>** is **very different than** the eigenket **|x,y,z>** even though both e.g. satisfy the eigenvalue equation.

$$x|x'\rangle = x'|x'\rangle \quad \& \quad x|x',y',z'\rangle = x'|x',y',z'\rangle$$

Matrix Elements of Position Operators, and Functions thereof:

Remembering our **norm for continuous kets**: **$\langle x|x'\rangle = \delta(x-x')$** it is obvious that the

Matrix element of x: $\langle x|x|x'\rangle = x'\,\delta(x-x')$

which say means that it **is "diagonal" just like if $\langle n|A|m\rangle = A_{nm}\delta_{mn}$.**

Similarly **any operator which is a function f(x)** of **only** the operator **x (**and **no other operator)**

has a matrix element (in the position (==|x>) basis !)

$$\langle x|f(x)|x'\rangle = f(x')\,\delta(x-x')$$

With this and **in order to review many points at once**, consider the **general overlap integral expression** for our expression of the projection of one ket state onto another (the **inner product**):

In our **wave mechanics review** we said this projection of state α onto state β was given by the integral

$$\int dx'\, \psi_{\alpha}^*(x')\ \psi_{\beta}(x')$$

Now it should be clear that in ket-bra notation, this is exactly what is implied by the expression $\langle\alpha|\beta\rangle$. So suppose $|\beta\rangle$

$$\langle\alpha|\beta\rangle \text{ where } |\beta\rangle = f(x)|\beta\rangle \;=\; ????? \;=\; \int dx'\, \psi_{\alpha}^*(x')\, f(x')\, \psi_{\beta}(x')$$

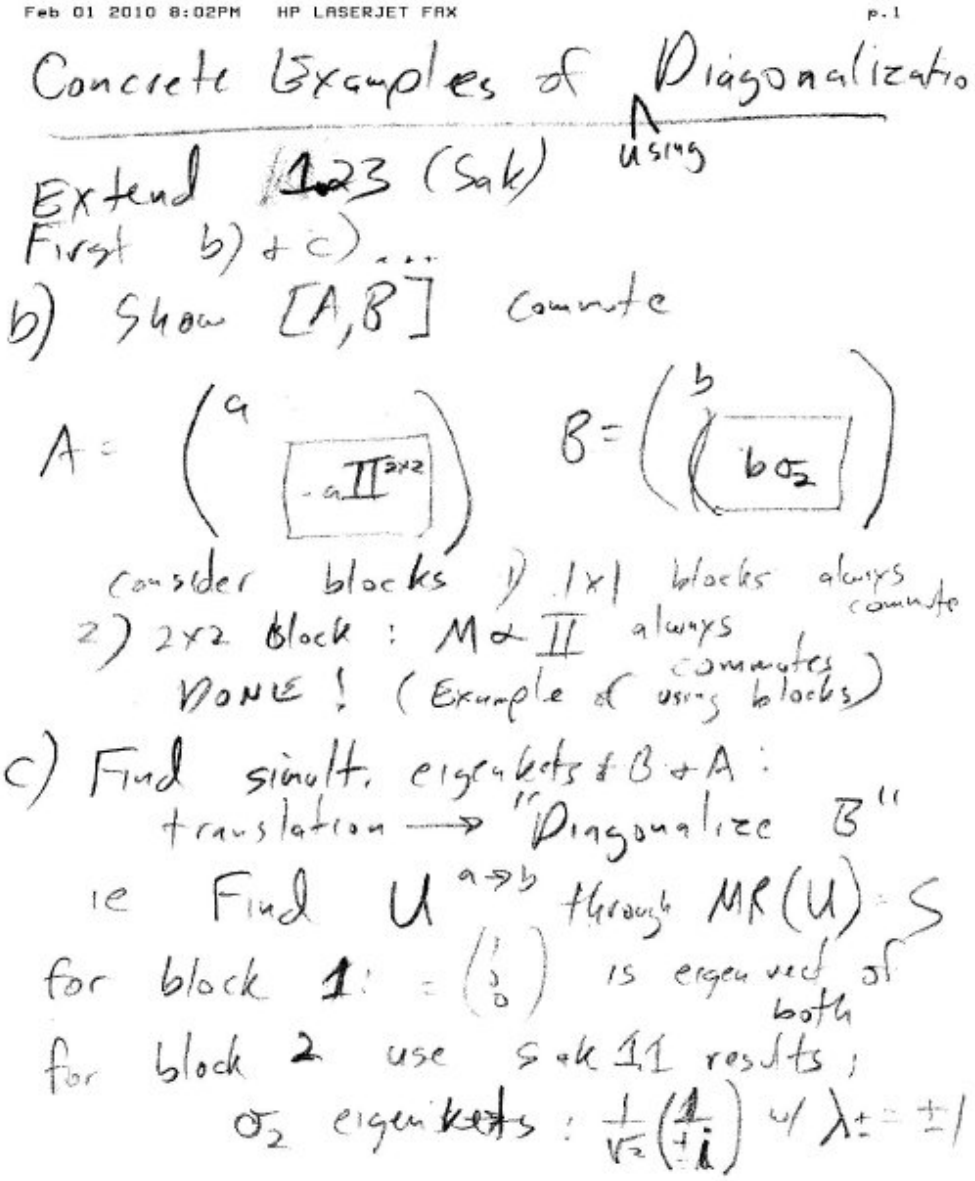
We already described this integral in the wave mech review and described it as what $\langle\alpha|\beta\rangle$ implies:

Q: Why? $(\langle\alpha\,|f(x)|\beta\rangle = \langle\alpha|\,(\int f(x)\,|x\rangle\langle x|\beta\rangle)$

Question: what is the expectation value of x in a position eigenket?

Lecture 2/1/2010:

Concrete Uses of Diagonalization Here is an example of the steps I outlined previously for *using* diagonalization (ie transformation operators etc…)



So like S, $M_R(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{pmatrix} \doteq U$

$\equiv M^R$

$U^+ \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 1 & i \end{pmatrix}$

Good Exercise: Check $U^+ B U \Rightarrow D^{diag}$

Now Expanded Problems Using Diagonalization

#1) Change Problem slightly so that

$B = \begin{pmatrix} b_1 & & \\ & b\sigma_2 & \end{pmatrix}$ $b_1 = 5$, $b_2 = b_3 = 2$

If state of system $|\alpha\rangle = \frac{1}{\sqrt{2}}(|b_1\rangle + |b_2\rangle)$,

Find $\langle B \rangle$.

Remember Steps From Notes

1) Find U
2) Rotate States ($U^+|\alpha\rangle$, $U|\alpha'\rangle = |b\rangle$)
3) Do Calcs, etc.
4) Rotate back if necc → Expectation Vals not needed

This case we're given "rotated" states ie $|\alpha\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Step 2 done for us for states

only need rotated operator

$B \Rightarrow U^+ B U$ (e.g using U's)

$$= \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -b \end{pmatrix} = \begin{pmatrix} 5 & & \\ & 2 & \\ & & -2 \end{pmatrix}$$

So $\langle \alpha | B | \alpha \rangle$

$$= \langle \alpha \underbrace{U}_{1} \underbrace{(U^+ B U)}_{1} \underbrace{U^+}_{1} | \alpha \rangle$$

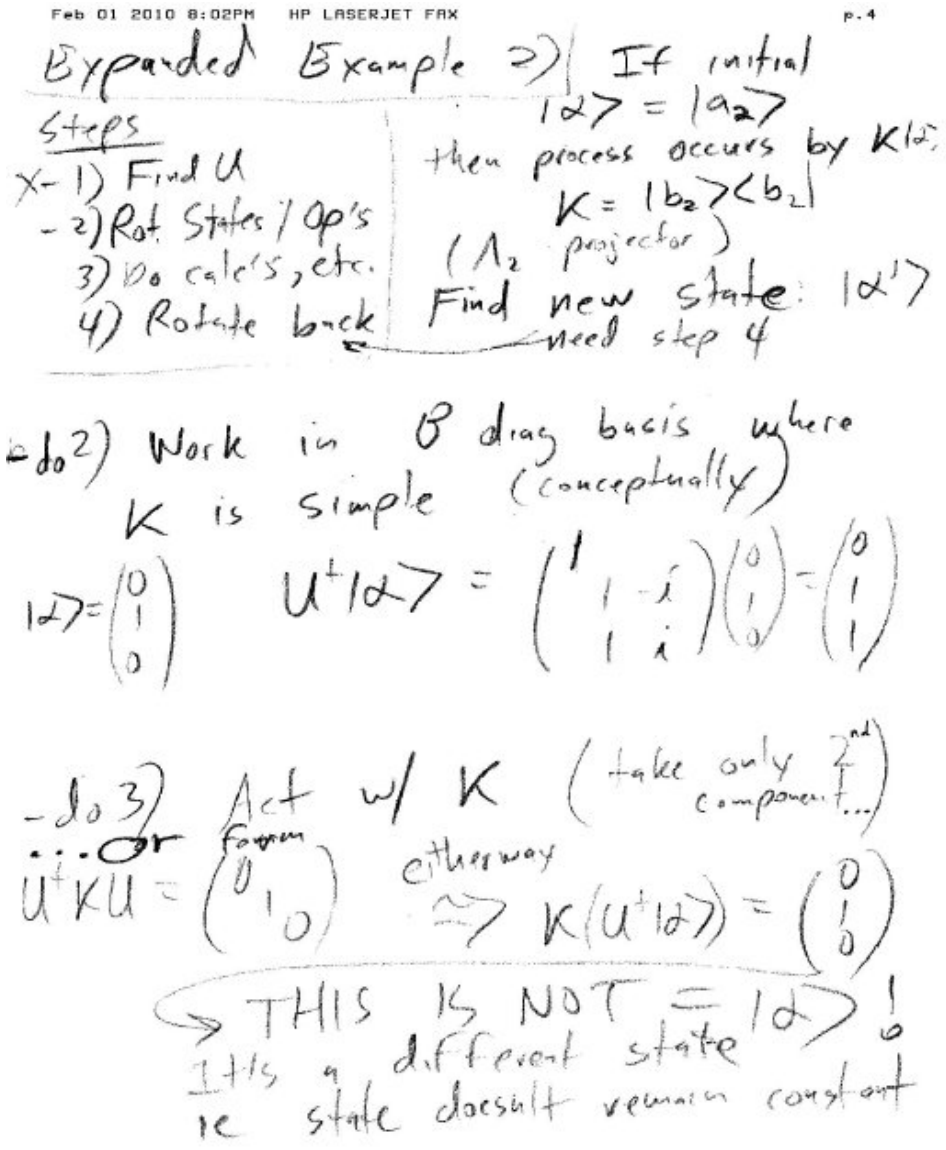
$$= \frac{1}{2}\left[(1\ 1\ 0) \begin{pmatrix} 5 & & \\ & 2 & \\ & & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] = \frac{7}{2}$$

Same result as if we never consider A: + just said equal prob's of 5 + 2 on avg $= (5+2)/2 = \langle B \rangle$

But needed diag form of B.

Realize diag steps "underneath"

In this second example imagine we have a physical process that is represented by the operator K which projects into the eigendirection of $|b_2\rangle$ This could be like as in a measurement of a third observable that resolves the degeneracy btw 2 and 3.



To finally answer Q, we must rotate back

$$U(|\alpha\rangle) \doteq M \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

New State:

$$|\alpha'\rangle \underset{(\text{prop. to})}{\propto} 1\,|a_2\rangle + i\,|a_3\rangle$$

Now has some $|a_3\rangle$ in it

$$\text{Prob}\,|\alpha'\rangle(a_3) = \text{Prob}\,|\alpha'\rangle(a_2)$$

Also concerning our discussions about unitary operators and diagonalization: remember we said Sak. introduces the explict $U=\sum_m |b_m\rangle\langle a_m|$ form of U emphasizing its nice property of it being very easy to see why $UU^\dagger = 1$.

Another easy way to explicitly create a unitary operator (which will work for both discrete or continuous ket spaces) is to consider the following form of an operator

$$\exp[\pm i A]$$

where A is Hermitian. exp A is shorthand for the taylor expansion of exp(x)

$$\exp A = 1 + A + A^2/2 + \ldots$$

And it is easy to see that these "exponentiated operators" obey our normal expectations for exponentiated numbers. e.g. since $\exp(-A) = 1 - A + A^2/2 + \ldots$

$$\exp(-A)\exp A = (1 + A - A + A^2 - A*A + \ldots) = 1$$

It is easy to see therefore that $(\exp[\pm i A])^\dagger = \exp[\mp iA]$ for as long as A is hermitian, we only need to take the complex conjugate of i. And thus we have the required $U^\dagger U = 1$.

For problem 1.28 you want to do an expansion like this, and simply apply the following commutation relation $[x,p_x] = i\hbar$ which we will prove early next week. Purely algebraic problem you can already do using these simple algebraic relations. (another hint—start with finding what $[x,p_x] = i\hbar$ implies for $[x,p_x^n]$)

B) The Translation and Momentum Operators

In Sakurai 1.6 we see the **first of several instances of introducing important operators through** the role they play in **very fundamental Transformation Operators** $\mathscr{U}$ that we discussed in the last section. **How fundamental? These Transformation Operators will always correspond to some symmetry of space-time itself!**

In this case we will introduce the **momentum operator** →through the fundamental transformation of **Space translation**:

Comments: Notice that in choosing our **definition of base kets |x>** we considered any **no certain situation**, like in **the case of spin, where we had the results of the SG experiment in mind**. **Thus** as is implied by their simple labels they should somehow be **dependent only on** the **properties of space itself, not** e.g. of **any particular energy configuration** or Hamiltonian.

As we can **move around in space**, we expect there to be a →**unitary transf. operator** that can take us **from** one location's ket **$|x_1\rangle$ to another $|x_2\rangle$**

In ket space this is a **rotation of (Q how many? 90) degrees** in 1 s**pecial direction**. In terms of **the real space** we can consider this a →**translation**:

if $x_2 - x_1 = a$ (some constant)

then we will call such a **translation transformation operator** T_{12} so that we have

$$|x2\rangle = |x1+a\rangle = T_{12}\,|x1\rangle$$

Suppose we **consider** a **T** which does the exact **same translation** operation **for any value of x: T ↔ T(a)**, not just |x1> → |x2>.

If this is the case**, besides** automatically needing this to be a Unitary operator (as all such transformation operators must)

$$T^\dagger T = 1$$

we expect (require!) this **T to follow** all our **intuitions about translations** in space:

1. Successive translations: $T(a)T(b)|x\rangle = T(a+b)|x\rangle$

2.) Inverse same as opposite direction translation $T(-a) = T^{-1}(a) = T^\dagger(a)$

3) If the translation $a \rightarrow 0$ it should become the identity operator $\lim_{a\rightarrow 0} T(a) = 1$

Especially because of 1 we should immediately **think of exponentiation** : $z^a z^b = z^{a+b}$

And **indeed the other 2 requirements along conveniently with the unitary requirement** all at once with an operator of the form we discussed last time:

$T(a) = \exp(\pm ika)$ where k could be **ANY** hermitian operator!

We will choose $\exp(-ika)$ **-- reasons will be obvious**

Lecture 2/2/2010

Although any operator k will satisfy the necessary properties (1,2,3 and Unitarity) we want of the translation operator**, something else special will be required of** how the operator **k** acts on the position kets $|x\rangle$ in order **to** actually **accomplish** the **real ket translation** we want:

$$|x1+a\rangle = T(a)|x1\rangle$$

All that's required : for this however is that k must obey the following commutation relation with the x operator:

$$[x,k] = i$$

That this commutation **relation works** is **proven in** your **homework** (we will **prove later that $k \propto p$.**

That this commutation **relation works** is **proven in** your **homework** (we will **prove later that k ∝ p),** but let's mention how **Sak proves it**. He considers **infinitesimal translations T(ϵ)**

Remembering "dx'" is just a number! I prefer to use the more usual convention for such a small number→**ϵ**

see that exp(ikϵ) in that case can just be reduced to

$$T(\epsilon) \approx 1-ik\epsilon$$

since we **can ignore any term of order in epsilon of ϵ^2** or higher in comparison to ϵ itself. This is extremely convenient—any time we see a term of ϵ 2 or higher we can simply cancel it immediately.

Further, we see that our **posited commutation relation [x,k] = i will imply** that

$$\mathbf{[x,k] = i \rightarrow [x,T(\epsilon)] = \epsilon}$$

Q why?

Using these we **apply T(ϵ) to |x'>** and **operate** on it with the **position operator**

$$\mathbf{x\,(T(\epsilon)\,|x'>) = (T(\epsilon)x + [x,T(\epsilon)])|x'>}$$

$$\mathbf{\text{imagine if } [x,T(\epsilon)] = \epsilon}$$

$$\mathbf{= (T(\epsilon)x + \epsilon|x'>}$$

but to this we can add any term $\propto \epsilon^2$ without penalty→ do so, in the form of $ik\epsilon^2|x'>$

$$\mathbf{\simeq x'T(\epsilon) + \epsilon + ik\epsilon^2\,|x'> = (x'+\epsilon)\,|x'> + i\epsilon\, k\,|x'>}$$

$$\mathbf{= x'T(\epsilon) + \epsilon(1+ik\epsilon)|x'>}$$

$$\mathbf{= (x'+\epsilon)\,T(\epsilon)|x'>}$$

Notice that it is such a manipulation that justifies line Sakurai 1.6.24:

$$\mathbf{[T(\epsilon),x]\,|x'> = \epsilon\,|x+\epsilon> \simeq \epsilon\,|x>}$$

Not some kind of Taylor expansion of |x+ϵ> : One cannot Taylor expand one basis state in terms of others!!!

Q: Why? (operators OK, not orthogonal kets**)**

So we have what we set out to get—

x(**T(ϵ)|x'>) = (x'+ϵ) T(ϵ)|x'>** that is, **T(ϵ)|x'>** is an eigenket of the operator x, with eigenvalue **(x'+ϵ).**

Thus it must be what we call the ket $|x'+\epsilon>$ **or at least proportional to it. (ie $T(\epsilon)|x> = \text{const}^{x}|x+\epsilon>$) Apparently following Sakurai we will choose this constant to be 1 (remember we already said these kets have INFINITE NORM anyways). And thus**

$$T(\epsilon)|x'> = |x'+\epsilon>$$

SUCH A CONCLUSION WILL BE USED MANY TIMES IN THIS COURSE—GET USED TO USING IT! : if we have $B(A|b_1>) = b_2(A|b_1>)$ we will always say $A|b_1>$ must $= c|b_2>$ where c is some numerical constant.

Our translation operator does what we wanted. So you see it is just a **simple algebraic relation** that allows for this.

Why k is proportional to the momentum

It **turns out** that we can **identify k as being proportional to the momentum of the particle.** To see how this comes about, it is instructive to observe its matrix element

Look at the wave function of the state

$|\alpha'> = T(\epsilon)\,|\alpha> \rightarrow <x'|\alpha'> = \psi_{\alpha'}(x) = <x'|\,T(\epsilon)\,|\alpha>$

$$<x'|\,T(\epsilon)\,|\alpha> = <x'|\,T(\epsilon)\,|\alpha>$$

(2-side Parallel)

LHS: By our properties of $T(\epsilon) = T^{\dagger}(-\epsilon)$

$$= [\,<x'|T^{\dagger}(-\epsilon)\,]\;|\alpha> = <x-\epsilon|\alpha> = \psi_{\alpha}(x-\epsilon)$$

Taylor expansion in ϵ

$$= \psi_{\alpha}(x') - \frac{d}{dx'}\psi_{\alpha}(x')\,\epsilon + \dots$$

For RHS:

$$T(\epsilon) \approx 1-ik\epsilon \rightarrow T(\epsilon) = \exp(-ik\epsilon) = 1-ik\epsilon+1/2\,(ik\epsilon)^2+\dots$$

So we have $<x'|\,T(\epsilon)|\alpha>$

$$= <x'|\,=\,1-ik\epsilon+1/2\,(ik\epsilon)^2+\dots|\alpha>$$

$$= \langle x'|\alpha\rangle - i\langle x'|k|\alpha\rangle\,\epsilon + \dots$$

Since this equation is actually true for all ϵ (Taylor series exp(x+a) converges for all a!), then like our wave mech review, we can set term by term the RHS = LHS which yields

$$\langle x'|k|\alpha\rangle = -i\frac{d}{dx'}\psi_\alpha(x') \left(= -i\frac{d}{dx'}\langle x'|\alpha\rangle\right)$$

This is a simple relation that **IN WORDS says NOT that** k **<u>is</u>** -id/dx', but **rather** that its **action of k changes the state ket such that** the

new projection is d/dx' (old projection)

This is perhaps a fancy way of saying we do NOT say k ≠ -id/dx' in above expression, because <x'| k ≠ k <x'|

So we know what **k by itself does to a ket**, but **to make the full identification with momentum** we **consider again** the **general overlap expression:** where

we bra- the k'ed state (think of |ψ > = k|α>) **with another arbitrary state <β|**

$$\langle\beta|k|\alpha\rangle = \int dx' \langle\beta|x'\rangle\langle x'|k|\alpha\rangle = \int dx' \psi^*_\beta(x') \left(-i\frac{d}{dx'}\right)\psi_\alpha(x')$$

$$= \frac{1}{\hbar}\int dx' \psi^*_\beta(x') \left(-i\hbar\frac{d}{dx'}\right)\psi_\alpha(x')$$

Thus we see **from our comparison to wave mechanics relation (→)** that it is obvious that **(-iℏ d/dx')** is what we called the **momentum operator in wave mechanics.**

Note this is the wave mechanics relation you will want to use for problem 1.21.

Thus we have that

action of $\hbar^x$k operator (k_{ket}) in formalism → action of p ($\hat{p}_{wave\ mech}$) in Wave mechanics

Thus by**" equality of action"** we can identify

$\hbar$ k = p ≡ $\hat{p}_{ket}$(meaning ket operator p)

Warning again: NOT that $\hat{p}_{ket}$ = $\hat{p}_{wave\ mech}$ ≡ (-iℏ d/dx') rather just that <u>projection</u> of p operation is same.

By this logic we can see better why the constant $\hbar$ must appear as the proportionality constant in p ∝ k. It is from all of the considerations of wave mechanics, primarily the original "quantum" statement of

De Broglie: $2\pi/\lambda = p/\hbar = k$

We also see that k itself should just be the wave number operator.

My main point is Sakurai first just STATES that k = p/$\hbar$ (and gives a reason that I do not agree with—see below--why it makes sense from the point of view of classical mechanics) then shows that the above action of p ($\propto$ action of wave mech d/dx) is a simple result. My point is that we can borrow all of De-Broglie's and other wave mechanics arguments for why the $\hbar$ should appear at all, if we first show that there is this operator k that has this action, and then "match up" the action with momentum from those arguments.

Three dimensional momentum/commutation

Above was all for 1-D.

We can define a corresponding momentum operator for each of our 3-D position operators

$p_y \rightarrow y$, $p_z \rightarrow z$

pi $\rightarrow$ xi (index notation)

Because translations in perpendicular directions commute, we expect that the

pi's also all inter commute like the xi's.

[xi,xj] = [pi, pj] = 0 and also that

We already specified the commutation relation for each xi,pi - pair, but for the same reason, that translations in perp directions commute, we also expect that

$$[x_i, p_j] = i\hbar\,\delta_{ij}$$

Digression:

Definition 1: since all [pi,pj]'s inter-commute, and these are the called **'generators'** of the group of translation transformations in 3-D, we say that group is an "Abelian" group.

Definition 2: See in group theory when you have a group of transformations which can generally be written exp(iA) then A is called the generator of the that group.

More generally, (though not really accurate to do so) "Abelian" usually implies that any sort of commutation relation holds, while "non-Abelian" will imply some sort of non-commutation.

QCD, the quantum field theory of the strong interaction is a "non-Abelian theory"

Lecture 2/2/09

Review

reading assignment Sak 2.3 up to time dev of the oscillator.

Question : True/False in the bra ket formalism is the operator p is -ih d/dx?

For example, in Sak. 1.29 the relation: Sak 1.29 [p,F(x)] = dF/dx or [x,G(p)]

→ you may remember from wave mech: [introduce "test function" $p_{\text{wave mech}}$ = ihd/dx]

The action of p => ihd/dx NOT pket = ihd/dx

continuint from last time we said:

[xi,xj] = [pi, pj] = 0

We already specified the commutation relation for each xi,pi - pair, but for the same reason, that translations in perp directions commute, we also expect that

$[x_i, p_j] = i\hbar\, \delta_{ij}$

Relations to Classical Mechanics /Commutators

Sakurai tries to **justify k = p/$\hbar$** as a **"trick of language"**

generator <-- > generating function

The **language makes sense** especially in this case for because somehow in this case **p/$\hbar$ is "responsible" for the translation action.**

But I do not think Sakurai's analogy to the classical mechanics **"Generating Function"** is very intuitive. See my comment in the optional reading. Just because they share the same root word "generate " they are not very alike.

Contrived: means **not** something that would happens **natural:** but **artificially prepared.**

Comment: Often authors try in this way to point out **similarities between Quantum and Classical mechanics** as if to **"it had to be that way"** or "one can actually **derive" Quantum Mechanics from Classical Mechanics" (?). But this is not really the case.**

It wasn't by observing these classical expressions and then magically forming the "quantum" equivalent and they saw it worked. No it is the other way around. We had to construct the quantum theory and all it's weird conceptual implications first, with only very loose relations to classical mechanical concepts. (for example one loose relation that was used is to "insert" $\hbar$'s in a way that in the limit of $\hbar$

→ 0 we recover classical behavior -- already mentioned in the review). But in general, this kind "searching through" classical relations for analogies came afterwards.

Point: Often (~50%?) analogies to classical: "artificial"/not rigorous/not useful

Exceptions:

1) Usually though, such analogies are at least **useful for remembering** one (e.g. the classical) if you already remember the other (e.g. the quantum).

2) HOWEVER: One relation to classical mechanics that is worth mentioning is that there is indeed **a similarity** btw/**of** the **classical mechanical "commutator"** defined as

$[A,B]_{classical}$ = dA/dq dB/dp – dB/dq dA/dp

Which indeed plays many analogous roles to our what we will use our commutators for in Quantum (and we will see more).

Thus obviously the classical commutator is indeed

$$[xi,pj]_{classical} = \delta ij$$

The **reason** that the **classical commutator** will seem to **have so many similarities** is exactly because its algebraic properties--it has the **→same algebra as** the **commutator.**

3-D Momentum Operator: Gradient ---------

Another nice feature of the exponential form of our translation operator is it handles the 3-D translations automatically as a dot-product.

Review: Thus if we want to make the vector translation in the direction $\vec{\epsilon}$.

$$\vec{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$$

we need/want the product of three orthogonal translation operators

$$e^{-\frac{ip_1\epsilon_1}{h}} e^{-\frac{ip_2\epsilon_2}{h}} e^{-\frac{ip_1\epsilon_1}{h}}$$

because we can think of 3-D translation as 3 successive translations in each directions—further we must be able to rearrange the order of the translations freely since that is how real translations work--now we see why we must have [pi,pj] = 0.

But this can be nicely written as

$$= \exp(-i\, \vec{p} \cdot \vec{\epsilon}/\hbar)$$

We can go through the same proof as before for each component separately ($T(\vec{\epsilon}) \simeq 1 + i\,\vec{p}\cdot\vec{\epsilon}/\hbar$)

and prove that the matrix element $\langle \vec{x}' | p_j | \alpha \rangle = \partial_i \langle \vec{x}' | \alpha \rangle$.

Thus now three operators implied by $\langle x | \vec{p} | \alpha \rangle = i\hbar \nabla \langle \vec{x} | \alpha \rangle$

Things follow straight forwardly in 3-D: e.g. Problem 1.29 3-D: point of this problem: 3-D.

Went over in class:

The Momentum Basis

Now that **we've introduced the p operator**, and derived a way to represent **→its effects on a position basis kets. (Wave Mech)**

For the **formalism,** it is more appropriate to discuss the momentum operator as we have every other operator so far: in terms of **its own eigenkets**. Indeed we can write one

$$\text{Dynamical variable } p \longleftrightarrow \text{operator } p.\ \text{Basis } p|p'\rangle = p'|p'\rangle$$

it is a **continuous ket also**

$$\text{Vector } |\alpha\rangle = \int dp'\, |p'\rangle\langle p'|\alpha\rangle \quad ;|\langle p'|\alpha\rangle|^2 dp' = \text{probability } p \in [p', p'+dp']$$

$$\text{Wave function in momentum space } \Phi_\alpha(p') = \langle p'|\alpha\rangle$$

since <p|α> is expansion parameter it automatically implies probability.

Take projection on to |x> basis:

$$\langle x'|\alpha\rangle = \int dp' \underbrace{\langle x'|p'\rangle}_{*} \langle p'|\alpha\rangle$$

Two ways to view :

1) wave function of p operator eigenket (actually "eigenfunction" –defined later…)
2) * interpreted as transformation "matrix" from $|x'\rangle$-basis to $|p'\rangle$-basis

…our transformation matrix element**, remember:** was **Umn = <am|U|an> = <am|bn>**

"Continuous Eigenvalue Analysis:" (find matrix element numbers: <x|p>)

Since $\langle x'|p|p'\rangle = p'\langle x'|p'\rangle$ and, using eq. (3), the LHS can be written $-i\hbar\frac{d}{dx'}\langle x'|p'\rangle$ and one obtains the very simple first order differential equation

$$-i\hbar\tfrac{d}{dx'}\langle x'|p'\rangle = p'\langle x'|p'\rangle$$

with the solution $\langle x'|p'\rangle = N\exp(ip'x'/\hbar)$, where $N = 1/\sqrt{2\pi\hbar}$ from the normalisation condition. This is the plane wave obtained without solving a second order Schrödinger equation!

Our expansion can be written:

$$\psi_\alpha(x') = N\int dp'\, e^{ip'x'/\hbar}\,\Phi_\alpha(p')$$

which is the Fourier expansion of $\psi_\alpha(x')$ in eigenfunctions $N\,e^{ip'x'/\hbar}$ to p. The wave function of $|\alpha\rangle$ in p-space is

$$\Phi_\alpha(p') = \langle p'|\alpha\rangle = \int dx'\,\langle p'|x'\rangle\langle x'|\alpha\rangle = N\int dx'\, e^{-ip'x'/\hbar}\psi_\alpha(x')$$

i.e. the Fourier transform is contained in the Dirac formalism.

========== end of lecture 2/2/09

Lecture 2/3/09

Midterm: Mon 2/16 Not sure if in class or in evening – maybe start early in class?

The Hamiltonian Operator, Eigenfunctions, and Discrete Eigenfunctions

In our formalism with the introduced **Wave functions** already, where do **eigenfunctions** themselves generally fit in**?** Remember these were a **(key part of wave mechanics.)**

In general if we have another operator like p, and we take the wave function ***of an eigenket of that operator***, we will call this a special kind of wave function: an eigenfunction

For p:

$$\underbrace{\langle x'|p'\rangle}_{\bullet}$$

This is the eigenfunction of |p'>.

However, we remember that **usually in wave mechanics we have a discrete** set of eigenfunction solutions. How does this fit **in** our **formalism**?

The main point comes from the beginning of **Sak .1.7.** We assume that in our bra ket formalism that the same space the contin. kets span (the **cont. ket space) can** also **be spanned by** a set of **discrete kets:**

Discrete set is complete. (Note: Remember discussion of |ξ> (|x>,|p>) not being members of the Hilbert space themselves, only their eigendifferentials are—[∝ ∫ |x>dx] (see Messiah)—it is that space [the eigendifferential space] that we are talking about, that is also spanned by the Discrete set).

Thus we can expand a single basis ket |x> as some **sum over some other discrete observable A**

$$|x\rangle = \sum |a\rangle\langle a|x\rangle$$

Most important example-- Define Hamiltonian operator

$$H = p2/2m + V(x)$$

(same form as classical mech energy -- same as our wave mechanics)

But now however note that these operators can also be interpreted as our ***ket*** **operators p and x.**

Ie we do not imply h2 d/dx2 + V(x).

Thus **if H eigenvals are discrete, eigenkets/eigenfunctions of H are discrete**

Concrete example: Particle in the box:

With our knowledge of the **wave mech**anics solution (pset #2 : Sak 1.21), we know that

H has a complete basis of eigenkets $|h_n\rangle$.

where **hn ≡ En = $\hbar$ 2 n2/(2m a2).**

Thus we can expand any arbitrary state $|\alpha\rangle$ in terms of this basis

$$= \sum |h_n\rangle\langle h_n|\alpha\rangle$$

By taking the projection of this expansion on the position basis ket $|x\rangle$

We see that this is just an expansion of $\psi_\alpha = \sum c_n u_n(x)$

where **c_n is still just the number $\langle h_n|\alpha\rangle$** and u(x) is what we call the **eigenfunction $u_n(x) = \langle x|hn\rangle$.**

Q: How did we know that there was this discrete set of eigenvalues /states? EXPERIMENT: the **same as for** the **SG experiment!**

With **wave mechanics** we **can derive** that **this** discrete basis of kets exists and know what the eigenvalues are

thus → from this standpoint, **wave mech**anics is actually generally **more powerful than** our **formalism**. (It however **requires advanced calculus**.**)**

However remember formalism already includes Wave mechanics through the relation.

$$\langle x|p|\alpha\rangle = ihd/dx \langle x|\alpha\rangle$$

Also: it may **not be impossible to derive** these eigenstates **strictly within the formalism** however, meaning without having to resort to wave mechanics. → **strictly algebraically based only on commutator algebra.** We can do this in some cases. (In all cases? Who knows? We do have an lot of freedom in how we may define the Hamiltonian with ket operators.

For the case of the simple harmonic oscillator in this weeks pset, you will see that we **can derive what the eigenvalues** are (and thus define the kets) **purely algebraically** using the operators themselves, just assuming only that they do have some ket. **(ie using ket's formalism only)**

Simple Harmonic Oscillator (SHO)

Most important problem in QM. Why?

1) **One of the only practical problems that is exactly solvable.**

2) because the annihilation/creation operators and algebra provide the basis for Quantum Field Theory.

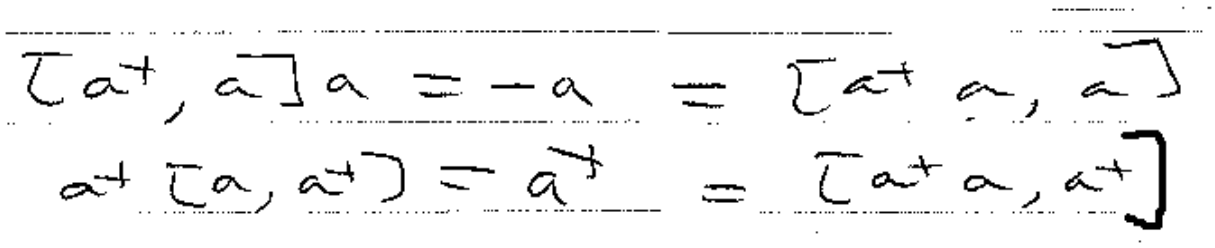
Start by first discussing these op's:

Suppose two operators a, $a^\dagger$ who have the following commutation algebra.

$[a, a^\dagger] = 1$

(hey not very different from [x,p] = ih → and same as classical commutator!)

Then immediately we have



which is to say if we define an Operator

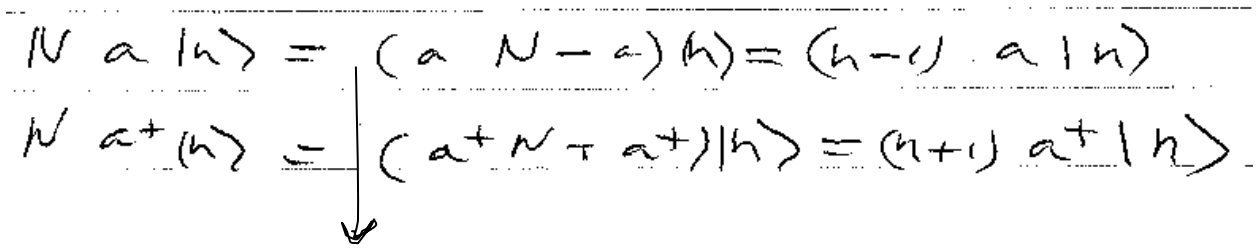
$$N = a^\dagger a$$

that has eigenbasis states $|n_n\rangle$ such that

$$N|n_n\rangle = n_n|n_n\rangle$$

(Let's just call it n, the same way we call $A|a\rangle = a|a\rangle$)

Then playing same game as w/ translation operator, we have the following interesting relation



aN+[N,a]

PURELY derived because of the commutator algebra:

Which of course IMPLIES:

$a|n\rangle$ is eigenket w/ eigenvalue n-1

$a^\dagger|n\rangle$ is eigenket w/ eigenvalue n+1

So if we start with any eigenstate $|n\rangle$ we can always "generate" a new state $|n-1\rangle$

$$a|n\rangle = C_n|n-1\rangle$$

This process has to stop because

$$n = \langle n|a^\dagger a|n\rangle = \langle an|an\rangle \geq 0$$

ALWAYS for all n (we require this to be so—because we want only positive definite normalizations)

This is only possible if there exists a zero eigenvalue

$$N|0\rangle = 0$$

remains zero after further lowering

$$N|0\rangle = a^\dagger a|0\rangle = 0$$

$$\Rightarrow a|0\rangle = 0$$

a called "annihilation" operator. it "kills" $|0\rangle$ and always decrements n

$a^\dagger$ called "creation" operator . it cannot "resurrect" a killed state ($a^n|0\rangle$) --"creation" bad term?--but it always increments n

Actually the reason for these terms is because we think of each increment of energy N as a excitation above some base state $|0\rangle$. It is also part of general quantum thinking (e.g. including wave mech) to think of any discrete excitation as a ***particle.*** In this case the particles would be called ***phonons*** of the SHO.

In Quantum Field Theory (QFT) the situation is generalized for ALL REAL fields, e.g. electromagnetic or particles. In place of our momentum states in this class $|p\rangle$, we construct new basis states that carry an "number of particles" for a given momentum→$|P\rangle = |p\rangle|n\rangle$. E.g. for a propagating free electromagnetic field, n is the number of "photons" with momentum $|p\rangle$. In exactly

the same way, n could be the number of electrons if the state describes electrons—and conversely electrons are thought of as just excitations in a some global "electron field".

Now we've established that **N has eigenvalues that are**

- chosen as integers

-and as **only positive**

At least most conveniently—otherwise we need new postulates…

Why is this **relevant?**

Of course we **know the answer already for the SHO.** It's because our **H**amiltonian for the **SHO** is can be written **in terms of the N** operator **only—ie with no other "non-trivial" operator.** Done by **forming linear combinations of x,p** that satisfy the above requirements for **a,a†**

But **before** we show **that…**

DIGRESSION:

Neat: Note that we **don't** even **need** to consider any **H**amiltonian **at all** really.

The above actually works for ANY linear combination of x, p that fulfill this algebraic requirement.

Q: what's the most trivial example, thinking of [x,p] = ih ? → [x,p/ih] = [x,k/i] = 1.

(using a' so as not to confuse w/ SHO a) Thus if we can simply define a' = x, a'† = p/ih, N' = xp /iℏ

Means instead of our continuous bases, |x>,|p> we can always define a discrete basis of the operator xp /iℏ

What is this operator ? I don't know. It almost looks like a reasonable observable (pos*mom), but it probably isn't. In fact it almost looks like angular momentum is $L = x \times p = \epsilon_{ijk} x_i p_j$ but it's $\propto x_i p_i$

Who cares! point is we always can define at least one operator who has a discrete eigenbasis spanning the same Hilbert space as our |x>, |p> basis

END DIGRESSION

OK so now back to the SHO. (FINALLY!)

First define it: what is SHO?

Answer: a system with the following Hamiltonian:

$$H = p^2/2m + V(x) = p^2/2m + m\omega^2/2$$

(ie V(x) = x2 remember ω = √(k/m) so that Force = dV/dx = -kx)

Immediately we see **a2-b2 → think of perfect square (a+b)(a-b)** (remember a2+b2 = a2 - (ib)2)

Problem set: define an a,a† in terms of x, p so that we can write H above in the form

$$H = K(a^\dagger a + aa^\dagger) = K(2N + 1)$$

So **H commutes with N** obviously**—> same eigenbasis**

And K works out such that energy eigenvalues are $H_n = E_n = \hbar\omega\,(n+1/2)$ Ground state--lowest E = hw/2

H is an observable quantity in this case obviously!

Lecture 2/4/09

Reading Assignment Sakurai 2.1.

Reminder Midterm : 2/16 Info/times/format TBD.

-Yesterday introduced discussed Discrete Bases that span same space as |x>,|p> bases. (note there is some technicality regarding which space we're talking about since |x>, |p> we said themselves weren' t part of the space).

-Introduced Hamiltonian

-Two examples of 1-D Wave Mech problems framed in terms of formalism—both 1-D box/SHO—discrete basis of Hamiltonian operator is more convenient to work with.

Using the Position/Momentum Basis in the Formalism

So we see that the first , "most important" application of using the p, x operators avoided using the |x>, |p> basis at all! and preferred switching to a discrete basis.

Generally true for all Bound State problems in Wave Mech.

For what are the position / momentum basis states useful ?

Answer 1) Approximation Schemes

The momentum basis will be most useful for approximation schemes especially particle scattering applications/approximations. Exactly solvable results not usually encountered in real world. Approximation schemes are necessary.

Notice two things about the Hamiltonian: $H = p^2/2m + V(x) = T + V(x)$

1) Obviously with V(x) = 0 The momentum basis is also the basis of the Hamiltonian.

2) We should usually be able to approximate potentials V(x) as some sort of series sum

Most obvious such expansion → V(x) = V1(x) + V2 (x)+… like Taylor expansion where other later terms are relatively smaller and smaller, so we can drop them.

Separable Potentials

But we can also do expansions of a kind that is more convenient towards using a particular set of basis states:

Following the idea of making a general operator out of ket-bra outer products (e.g. we said in it's own eigenbasis an operator A has the form $\sum a_n |a_n\rangle\langle a_n|$) it can be useful to approximate using arbitrary potentials in the form

$$V = |g\rangle\lambda_{gf}\langle f|$$

Where ,g,f can be some state, which can of course be written as linear combinations of basis states, possibly momentum kets $|p\rangle$ or position kets $|x\rangle$ and λ is some number.

Potentials are called **separable** if they can be written as a combination of such terms (e.g. most commonly a discrete sum)

$$V_{sep} = \sum_{kn} |k\rangle\lambda_{kn}\langle n|$$

I think that word is used because it acts on each combination of basis states "separately."

A convenient choice for these could be the momentum basis states

$$\int dpdp' \quad |p\rangle\lambda_{pp}\langle p'|$$

Why ? Because then the entire Hamiltonian including the p2/2m operator (sometimes called T) is entirely composed of operators which act directly on $|p\rangle$ basis kets !!!

YESTERDAY NUCLEAR SEMINAR: EXACTLY ONE EXAMPLE!!!

DIGRESSION: Why momentum kets instead of position?

Greater question: Perfect symmetry between p, x? PSET: The Position Operator in the $|p\rangle$ Basis—also fulfills derivative relation

$\langle p|x|\alpha\rangle \propto d/dp (\langle p|\alpha\rangle) = d\phi(p)/dp$

So everything looks very symmetric.

Answer: **no** generally because of free Hamiltonian "preference" for p operator. p is likely the usually the preferred basis (in lieu of exact Hamiltonian basis).

Actually for separable potentials, it is actually usually angular momentum basis states.

end DIGRESSION

Perturbation Theory:

More generally there is perturbation theory which we will not discuss this quarter. But we discussed it during the Wave Mech review (please go back and take a look)—its generalization in the bra/ket formalism should not be hard to imagine.

For high energy scattering problems, it is most convenient to treat the "base Hamiltonian" as

$$H_0 = p^2/2m = T$$

and because by assumption the T = KE is so high, treat the entire potential V(x) as a perturbation. Thus again the base states you want to start w/ for H_0 are obviously the $|p\rangle$ states.

The main difference between Perturbation Theory though is that PT is a full expansion of the STATES of the full Hamiltonian, in terms of the states of the base H, not just an approximate expansion of the potential.

Answer 2) "Artificial" Construction of States

Another reason $|p\rangle$, $|x\rangle$ basis useful is it's easy to construct arbitrary states using them—through the wave functions themselves. Sometimes we may not know or care what the Hamiltonian is, but we have a better idea how to describe the state in terms of the $|x\rangle$ or $|p\rangle$ states.

Examples:

-Suppose we know a particle is localized to some area

- Suppose we know its momentum

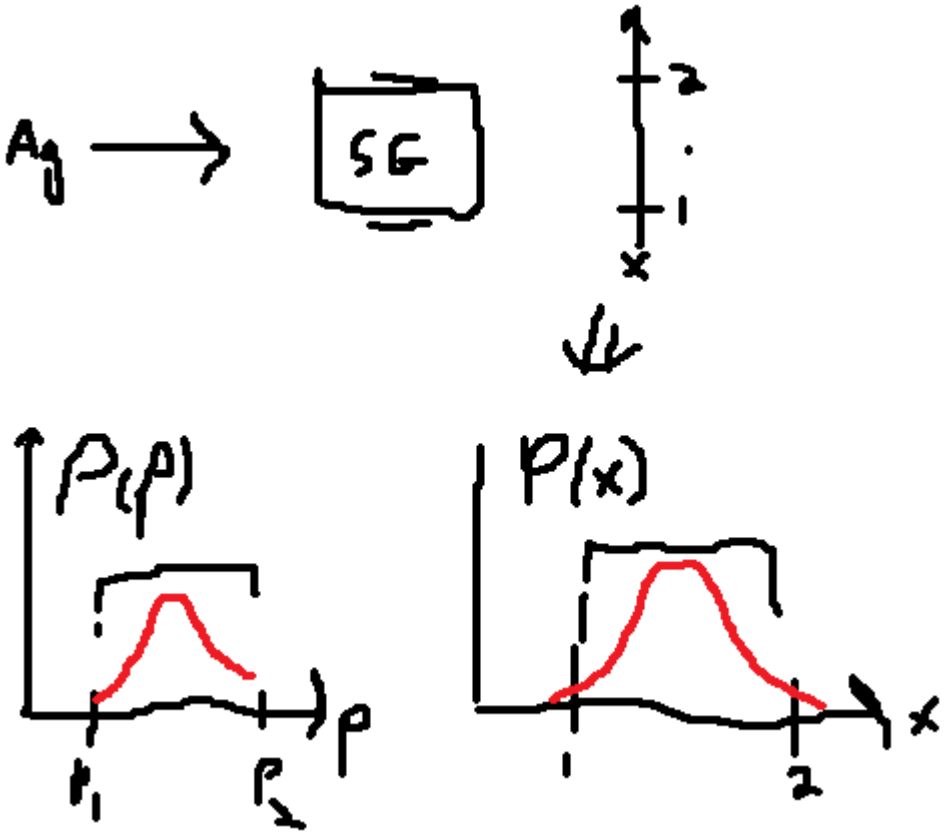
This is especially useful for the question of initial state.

Remember question: What is the state of the particle before it hits the SG device? Spin states may not be determined, but there is also a momentum/position state for the atom: this we do have some idea how it should look.

A) it has some fixed momentum. (this has to be tuned right to "see" the splitting due to spin) within some uncertainty range

B) it has some fixed position (changing w/ time) (this is what allows us to say it is going through the SG device)—within some uncertainty range.

This is demonstrated in this plot:



DIGRESSSION:

CENTRAL LIMIT THEOREM (CLM): (classical thinking, not just QM) for some derived quantity (Observable) F(a,b,c,…), (F depends on a,b,c…) no matter what the probability distributions of a, b, c, inputs are, the probability distribution of F will always tend towards a Gaussian.

POINT: All uncertainties/experimental errors tend to be Gaussian because they are usually due to some many microscopic inputs.

end DIGRESSION

Thus following the example set by the CLT we may reasonably assume it's also true for x,p in SG. Thus the following construction works very nicely to describe this situation:

Gaussian Wave Packets

So just CONJECTURE that the state initial space state for the SG looks like this:

Ψ α(x)= 

=definite momentum eigenstate exp(ikx) "smeared" around x = 0. (by this choice we have put our x origin @ 0)

Called a "PACKET" (group) of x states. It has a Gaussian form. ($\propto \exp(-x^2)$)

Has the right form for P(x) then because

$$P(x) = \psi^* \psi \langle \alpha | x \rangle \langle x | \alpha \rangle \propto \exp(-x^2/d^2)$$

We would choose d according to the box size. So we are PICKING the form of the WAVE FN first! This is what I mean by "artificial". Not any operator or eigenket that the state might be related to. Could $|\alpha\rangle$ be the eigenstate of some other observable? Who knows! Is it a Discrete sum over eigenstates of some other observable? Who knows!

Actually we will show that the SHO ground state has this form.

Now the NEAT property of this state is that we already said

$$\langle x'|\alpha\rangle = \int dp' \, \underbrace{\langle x'|p'\rangle} \, \langle p'|\alpha\rangle$$

$$\psi_\alpha(x') = N \int dp' \, e^{ip'x'/\hbar} \, \Phi_\alpha(p')$$

which is the Fourier expansion of $\psi_\alpha(x')$ in eigenfunctions $N\, e^{ip'x'/\hbar}$ to p. The wave function of $|\alpha\rangle$ in p-space is

$$\Phi_\alpha(p') = \langle p'|\alpha\rangle = \int dx' \, \langle p'|x'\rangle \langle x'|\alpha\rangle = N \int dx' \, e^{-ip'x'/\hbar} \psi_\alpha(x')$$

i.e. the Fourier transform is contained in the Dirac formalism.

DIGRESSION: Remembering Properties of the Fourier Transform (FT): The FT of a Gaussian form, is another Gaussian form.

FT(Gaus(x)) = Gaus(p)

FT(Gaus(p)) = Gaus(x)

end DIGRESSION

Thus for our artificial state $|\alpha\rangle$ It's momentum wave function must also be a Gaussian in p:

Form is given in Sak. eq 1.7.42:

$$\langle p'|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[-\frac{(p'-\hbar k)^2 d^2}{2\hbar^2}\right]$$

Obviously this is a state with mean value p = "hk" but with some uncertainty around it.

Why? because we can just read the value off From "Wikipedia: Gaussian distribution" Gaussian of x has form:

$$\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where $\sigma_x^2 = \langle \Delta x^2\rangle$ and $\mu = \langle x\rangle$

Thus matching symbols , by inspection, $\langle p\rangle = \hbar k$

Our probability distribution $P(p) = \langle\alpha|p\rangle\langle p|\alpha\rangle \propto \exp(-2(p-\hbar k)^2 d^2/2\hbar^2)$ so

$\langle \Delta p^2\rangle = \hbar^2/2d^2$

NOTE THAT SAKURAI DERIVES THESE USING THE DERIVATIVE FORM OF p in sect. 1.7 → ie w/ wave mechanics. YOU SHOULD BE COMFORTABLE DOING THAT TO… (YOU MAY WANT TO PRACTICE). E.g. $\langle x\rangle = 0$ is easy to see: because $\exp(-x^2)* x$ odd

Anyway, the conjectured form of $|\alpha\rangle$ fulfills what we want expected for our SG experiment. And actually even predicts something: based on the input requirement for the distribution of x we see what the P(p) of p must look like!

Note about Uncertainty Relation

While we're add it let's read off uncertainty value for $\langle \Delta x2\rangle$

Ψ α(x)= $$\langle x|\alpha\rangle = \frac{1}{\sqrt[4]{\pi}\sqrt{d}}\, e^{ikx - \frac{x^2}{2d^2}}$$

$P(x) = \psi^*\psi\langle\alpha|x\rangle\langle x|\alpha\rangle \propto \exp(-x^2/d^2)$

So it must be just $\sigma = d^2/2$.