RINGS WITH A POLYNOMIAL CONSTRAINT

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Rings with polynomial identities and their generalizations have been studied by Amitsur [1, 2, 3, 4, 5] and Drazin [6]. In this note we have introduced a type of localization of polynomial identities called quasi-standard identities. We investigate the structure of a class of prime rings having a quasi-standard identity or a pivotal monomial. In Theorems 1 to 4 we show that a prime ring with its right singular ideal zero and possessing uniform right ideals has a classical right quotient ring if it satisfies a pivotal monomial or a quasi-standard identity. In Section 2 we prove an analogous result for integral domains.

R. E. Johnson in [11], [12] introduced the class of prime rings with zero right singular ideal and possessing uniform right ideals. Section 1 of this paper can be viewed upon as giving a sufficient condition that such rings are Goldie rings, that is, satisfy Goldies' conditions [7]. It is well known that a Goldie ring possesses the conditions introduced by Johnson, however the converse is false. For example, consider an infinite dimensional vector space V over a division ring D. Then $\mathrm{Hom}_D(V,V)$ satisfies the conditions of Johnson but not of Goldie. In view of this it would be interesting to find conditions other than those of Section 1 of this paper that will guarantee a ring satisfying Johnson's conditions also satisfy Goldies'.

1. Let R be a ring. For each x, $x_r(x_l)$ denotes the right (left) annihilator of x in R. A right ideal of R is large if it has non-zero intersection with each non-zero right ideal of R. The *right singular ideal*, R^{Δ}, of R is defined by :

$$R^{\Delta} = \{x \in R \mid x_r \text{ is a large right ideal}\}.$$

An R-module is *uniform* if each pair of non-zero sub-modules has a non-zero intersection. A right ideal is *uniform* if it is uniform as a right R-module. An R-module M is called *irreducible* if it is uniform and its singular submodule

$$M^{\Delta} = \{x \in M \mid x_r \text{ is a large right ideal of } R\}$$

is zero.

Let $x_1, x_2, \dots, x_d$ be non-commuting indeterminates; let

$$S_d(x) = \Sigma \pm x_{\delta(1)} x_{\delta(2)} \dots x_{\delta(d)},$$

where the summation is taken over all permutations of $1, 2, \dots, d$ and the sign is positive or negative according as δ is even or odd. A ring R is said to possess a *quasi-standard identity* (QSI) of degree d iff for each d-tuple $r = (r_1, r_2, \dots, r_d)$ of elements of R there is an integer $n = n(r)$ such that $[S_d(r)]^n = 0$.

Let $\pi(x) = x_1 x_2 \dots x_d$ and $P_\pi = \{x_{j_1} x_{j_2} \dots x_{j_q} \mid$ either $q > d$ or $q \leqslant d$ and for some $h \leqslant d, j_h \neq h\}$. A ring R is said to possess a *pivotal monomial* (PM) of degree d if for each d-tuple $r = (r_1, r_2, \dots, r_d)$ of elements of R there are $a_\delta \in R$, $\delta \in P_\pi$ such that

$$r_1 r_2 \dots r_d - \sum_{\delta \in P_\pi} \delta(r) a_\delta = 0, \qquad a_\delta = 0$$

except for finitely many δ.

Let V be a left vector space over a division ring D and let R be a ring of linear transformations of V. R is *weakly transitive* means there is a right order K in D and a (K, R)-submodule M of V such that M is uniform as an R-module and such that if $x_1, x_2, \dots, x_n$ is a finite D-linearly independent subset of M and if $y_1, y_2, \dots, y_n$ are arbitrary elements in M then there is an $r \in R$, $k \in K$, $k \neq 0$ and $x_i r = k y_i$, $1 \leqslant i \leqslant n$. It has been shown by Koh and Mewborn [9] that if R is a prime ring such that (1) $R^{\Delta} = 0$ and (2) the lattice of closed right ideals is atomic then R is weakly transitive. We remark that it is equivalent to the statement that a prime ring R with $R^{\Delta} = 0$ and possessing uniform right ideals is weakly transitive.

For the remainder of this section, unless otherwise stated R will denote a prime ring with $R^{\Delta} = 0$ and having uniform right ideals. Let Q be the maximal right quotient ring of R; Q is a primitive ring with a non-zero socle. Let I be a minimal right ideal of Q and let $U = I \cap R$. Then U is a uniform right ideal of R. Let

$$D = \mathrm{Hom}_R(I, I),$$

then D is a division ring (cf. [9] and [13]). Let $V = DU$ be the minimal quasi-injective extension of U [14, p. 261].

We now prove,

THEOREM 1: *Suppose R has a pivotal monomial of degree d. Then*

$$\dim_D V < d + 1.$$

PROOF: Assume $r_1, r_2, \ldots, r_{d+1}$ are linearly independent elements of U over D. By the weak transitivity of R there are $a_i \in R$ and $k_i \in K$ such that

$$r_i\, a_i = k_i\, r_{i+1} \qquad \text{for} \qquad 1 \leqslant i \leqslant d$$

and

$$r_j\, a_i = 0 \qquad \text{if} \qquad i \neq j, \qquad 1 \leqslant j \leqslant d + 1.$$

Since R has a PM of degree d, there are $c_\delta \in R$, $\delta \in P_\pi$ such that

$$a_1 a_2 \ldots a_d - \sum_{\delta \in P_\pi} \delta(a)\, c_\delta = 0.$$

Now

$$r_1 a_1 a_2 \ldots a_d = k_1 r_2 a_2 \ldots a_d = k_1 k_2 \ldots k_d r_{d+1}$$

and for another

$$\delta \in P_\pi,\ r_1\, \delta(a) = 0.$$

Thus we have

$$r_1\, [a_1 a_2 \ldots a_d - \sum_{\delta \in P_\pi} \delta(a)\, c] = k_1 k_2 \ldots k_d r_{d+1} = 0.$$

Since $(k_1 k_2 \ldots k_d) \in D$ and is non-zero, we have $r_{d+1} = 0$, which contradicts the linear independence of $r_1, r_2, \ldots, r_{d+1}$. Thus we see that $\dim_D U < d + 1$. We claim $\dim_D V = \dim_D U$. For if $x \in V$ then $x = \Sigma a_i u_i$, $a_i \in D$, $u_i \in U$. Hence any basis of V over D can be expressed in terms of D-linearly independent elements of U; consequently $\dim_D V < d + 1$.

We also have the following.

*THEOREM 2. *Suppose R has a QSI of degree d. Then*

$$dim_D\, V < \left[\frac{d}{2}\right].$$

PROOF: Let d be even, say, $2m$. Then $t = m$. Assume that $r_1, \ldots, r_{m+1}$ are *l. i.* elements of U over D. We can choose a_i in R and non-zero k_i in D such that

(1) $r_i\, a_{2i-1} = k_{2i-1}\, r_{i+1}, \quad 1 \leqslant i \leqslant m, \quad r_j\, a_{2i-1} = 0, \quad j \neq i$

(2) $r_{i+1}\, a_{2i} = k_{2i}\, r_{i+1}, \quad 1 \leqslant i \leqslant m - 1, \quad r_j\, a_{2i} = 0, j \neq i + 1.$

(3) $r_{m+1}\, a_d = k_d\, r_i, \quad r_i\, a_d = 0, \quad i \neq m + 1$

If d is odd, say, $2m + 1$, then t is also m and we can similarly choose a_i and k_i such that (1) and (3) hold and instead of (2) we have

(2′) $r_{i+1}\, a_{2i} = k_{2i}\, r_{i+1}, \quad 1 \leqslant i \leqslant m, \quad r_j\, a_{2i} = 0, \quad j \neq i + 1,$

Since R satisfies a quasi-standard identity of degree d, there is an integer $n = n(a_1, a_2, \ldots, a_d)$ such that $[S_d(a)]^n = 0$. Now

$$r_1\, a_1\, a_2 \ldots a_d = k_1\, k_2 \ldots k_d\, r_1$$

and for δ different from the identity permutation,

$$r_1\, a_\delta(1)\, a_\delta(2) \ldots a_\delta(d) = 0.$$

Thus

$$r_1[S_d(a)]^n = k_1\, k_2 \ldots k_d r_1\, [S_d(a)]^{n-1};$$

*The proof that dim $V \leqslant \left[\frac{d}{2}\right]$ is due to referee. The authors proved originally that dim $V < d$.

repeating n times we have

$$r_1[S_d(a)]^n = (k_1 k_2 \dots k_d)^n r_1 = 0.$$

Since

$$(k_1 k_2 \dots k_d)^n \neq 0,$$

we obtain $r_1 = 0$ contrary to the linear independence of the r_i. Hence $\dim_D U \leqslant \left[\frac{d}{2}\right]$ and proceeding as in Theorem 1, we obtain $\dim_D V \leqslant \left[\frac{d}{2}\right]$.

By a right Goldie ring we mean a prime ring such that

(1) every direct sum of right ideals is finite,

(2) the ascending chain condition on right anninilator ideals holds.

The following is the main theorem of this section.

THEOREM 3: *Suppose R has a PM of degree d. Then R is a right Goldie ring.*

PROOF: It is easy to check that $V = DU$ is an irreducible module. Let $x_1, x_2, \dots, x_p$ be a D-basis for V, $p \leqslant d$. Let $h \in \mathrm{Hom}_D(V, V)$. By Johnson and Wong [13, cor. 2,6] there exist $r, r' \in R$ such that

$$x_i hr = x_i r', \quad i = 1, 2, \dots, p,$$

and if $x_j h \neq 0$ for some j then r' can be chosen so that $x_j r' \neq 0$. So let $H = \mathrm{Hom}_D(V, V)$ and let $h \in H$, $h \neq 0$. Then there is a j such that $x_j h \neq 0$. Thus r, r' can be chosen as above with $r' \neq 0$. Therefore $x_i(hr - r') = 0$, $i = 1, 2, \dots, p$, yields that $hr - r' = 0$. This implies $hR \cap R \neq 0$. Thus H is a right quotient ring of R [13] and since H is a full matrix ring over a division ring, we have H is maximal right quotient ring for R. Hence it follows that R is a right Goldie ring.

In case R satisfies a QSI, we obtain the stronger result.

THEOREM 4. *If R satisfies a QSI of degree d, then R has a polymnoial identity.*

PROOF: As in Theorem 3 we can show R is a right Goldie ring. Therefore, R is of bounded index : that is the indices of its nilpotent elements are bounded. Since for each d-tuple $x = (x_1, \ldots, x_d)$ of elements of R, we have $[S_d(x)]^n = 0$ for some $n = n(x)$. We also have $[S_d(x)]^m = 0$ where m is an upper bound for the indices of the nilpotent elements.

COROLLARY: *If R has a QSI of degree d, then R is a left and right Goldie ring.*

PROOF: By Theorem 4 and Posner [8] who showed a prime *PI* ring is a left and right Goldie ring.

Let R now be a (right) primitive ring. We know that R is transitive, thus weakly transitive.

THEOREM 5: *Let R be a primitive ring with a QSI of degree d. Then R is a simple algebra finite demensional over its center.*

PROOF: Let V be a faithfully irreducible module of R, and let D be the centralizer of V. R acts transitively on V over D. By arguing in a manner similar to Theorem 2 we have $\dim_D V \leqslant \left[\frac{d}{2}\right]$. Hence R is a full matrix ring M of order $n \leqslant \left[\frac{d}{2}\right]$ over D, so QM satisfies a QSI of degree d. Since the indices of nilpotent elements of M are bounded, we have, as in Theorem 4, that M, hence R, satisfies a polynomial identity and, therefore, R is a simple algebra finite dimensional over its centre.

2. In Theorem 3 we showed that a prime ring with zero right singular ideal, possessing uniform right ideal, and a PM is a right Goldie ring. If R is an integral domain, then clearly $R^{\Delta} = 0$; however, R need not possess uniform right ideals. In this case, however, if R satisfies a PM of degree d, then the conclusion of Theorem 3 remains true.

In fact, we have,

THEOREM 6. *Let R be an integral domain with a (right) PM of degree d. Then R is a right Ore-domain.*

PROOF. Let $x_1 x_2 \ldots x_d$ be a pivotal monomial. Then for each $r_1, r_2, \ldots, r_d \in R$ there are $a_\delta \in R$ such that $r_1 r_2 \ldots r_d = \sum_{\delta \in P_\pi} \delta(r) a_\delta$.

For $\delta \in P_\pi$ we shall write δ_i for δ if the first indeterminate in δ is x_1. Suppose now that $I_1 \oplus I_2 \oplus \ldots \oplus I_d$ is a direct sum of non-zero right ideals of R and choose

$$r_i \in I_i, \qquad r_i \neq 0, \qquad i = 1, 2, \ldots, d.$$

Then we have

$$r_1 r_2 \ldots r_d = \sum_{\delta \in P_\pi} \delta(r)\, a_\delta = \sum_{i=1}^{d} \sum_{\delta_i \in P_\pi} \delta_i(r)\, a_{\delta_i}.$$

Now $\qquad r_1 r_2 \ldots r_d \in I_1,$

and for each $\qquad i = 1, 2, \ldots d, \sum_{\delta_i \in P_\pi} \delta_i(r)\, a_{\delta_i} \in I_i.$

Since the sum of the I_i is direct, we thus have

$$r_1 r_2 \ldots r_d = \sum_{\delta_1 \in P_\pi} \delta_1(r)\, a_{\delta_i}$$

and

$$\sum_{\delta_i \in P_\pi} \delta_i(r)\, a_{\delta_1} = 0 \qquad \text{for} \qquad j \neq 1.$$

Thus

$$r_1 r_2 \ldots r_d = r_1 \sum_{\tau \in P_1} \tau(r)\, a_\tau$$

where $\tau \in P_1$ iff $r_1 \tau(r) = \delta_1(r)$ for some $\delta_1 \in P_\pi$ and $a_\tau = a_{\delta_1}$.

Cancelling r_1 gives $r_2 \ldots r_d = \sum_{\tau \in P_1} \tau(r)\, a_\tau$. Now let $P_2 = \{\tau \mid \tau \in P_1$ and the initial indeterminate in τ is $x_2\}$. Since $r_2 \ldots r_d \in I_2$, we have, like above

$$r_2 \ldots r_d = \sum_{\tau \in P_2} \tau(r)\, a_\tau .$$

Since each $\tau(r)$, for $\tau \in P_2$, begins with r_2 we can cancel r_2 from both sides and obtain, eventually,

$$r_3 \ldots r_d = \sum_{\tau \in P_3} \tau(r)\, a_\tau,$$

$P_3 = \{\tau \mid \tau \in P_2$ and begins with $x_3\}$. Continuing thus we arrive at

$$r_d = \sum_{\tau \in P_d} \tau(r)\, a_\tau \qquad \text{where} \qquad \tau \in P_d \qquad \text{iff} \qquad \tau \in P_{d-1}$$

and τ begins with x_d. But since τ was obtained from some $\delta \in P_\pi$, $\delta = x_{j_1}\, x_{j_2} \ldots x_{j_q}$ with $j_d = d$ we must have $q > d$. Hence we can write $\tau(r) = r_d\, r_j\, b_j$ for some j, thus

$$r_d = \sum_{j=1}^{d} r_d\, r_j\, b_j.$$

Multiplying by r_1 on the right gives

$$r_d\, r_1 = r_d \sum_{j=1}^{d} r_j\, b_j\, r_1$$

and therefore

$$r_1 = \sum_{j=1}^{d} r_j\, b_j\, r_1.$$

Again, by the direct sum of the I_j we have

$$r_1 = r_1\, b_1\, r_1, \qquad r_j\, b_j\, r_1 = 0 \qquad \text{for} \qquad j \neq 1.$$

Thus $r_1\, b_1 = r_1\, b_1\, r_1\, b_1$. Hence $r_1 b_1$ is a non-zero idempotent and must be identity for R. But $r_1\, b_1 \in I_1$ and so $I_1 = R$ which is impossible. Thus every direct sum of right ideals has fewer than d factors. It is now easy to see that R is a right Ore-domain [(10, p. 264)].

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