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RINGS IN WHICH EVERY RIGHT IDEAL IS QUASI-INJECTIVE

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It is well known that if every right ideal of a ring R is injective, then R is semi simple Artinian. The object of this paper is to initiate the study of a class of rings in which each right ideal is quasi-injective. Such rings will be called q-rings. It is shown by an example that a q-ring need not be even semi prime. A number of important properties of q-rings are obtained.

Throughout this paper, unless otherwise stated, we assume that every ring has unity $1 \neq 0$. If M is a right R-module, then $\hat{M}$ will denote the injective hull of M. For any positive integer n, R_n will denote the ring of all $n \times n$ matrices over the ring R. R^{Δ}, $J(R)$ and $B(R)$ will denote the right singular ideal, the Jacobson radical and the prime radical respectively. A ring R is said to be a right duo ring if every right ideal of R is two-sided. Left duo rings are defined symmetrically. By a duo ring we mean a ring which is both right and left duo ring.

It is shown that $R_n (n > 1)$ is a q-ring if and only if R is semi-simple Artinian. Some of the main results are: (i) a prime q-ring is simple Artinian, (ii) a semi-prime q-ring is a direct sum of two rings S and T, where S is a complete direct sum of simple Artinian rings, and T is a semi-prime q-ring with zero socle, and (iii) a semi-prime q-ring is a direct sum of two rings A and B, where A is a right self injective duo ring, and B is semi-simple artinian.

2. Let R be a right self injective ring. If B is any right ideal of R, then $\hat{B} = eR$ for some idempotent e of R. Let $K = \mathrm{Hom}_R(\hat{B}, \hat{B})$. Then $K \cong eRe$. In fact every element in K can be realized by the left multiplication of some element of eRe. By Johnson and Wong ([3], Theorem 1.1) B is a quasi injective as a right R-module if and only if $KB = B$. Hence B is quasi injective if and only if $B = KB = (eRe)B = (eR)(eB) = \hat{B}B$. Hence every two-sided ideal in a right self injective ring is quasi-injective. So, the following is immediate.

2.1. Every commutative self injective ring is a q-ring.

Now, we give an example of a q-ring which is not semi-prime.

EXAMPLE 2.2. Let Z be the ring of integers. Set $R = Z/(4)$. It is trivial that R is a q-ring. But R is not semi-prime, since its only proper ideal is nilpotent.

In fact, $Z/(n)$ is a q-ring for every integer $n > 1$, since it is self injective (cf. Levy [5]). Also we remark that $Z/(n)$ has nonzero nilpotent ideals if n is not square free.

Next we prove

THEOREM 2.3. *The following are equivalent*

(1) *R is a q-ring*

(2) *R is right self injective, and every right ideal of R is of the form eI, e is an idempotent in R, I is a two sided ideal in R.*

(3) *R is right self-injective, and every large right ideal of R is two sided.*

Proof. Assume (1). Therefore R is right self injective. Let B be any right ideal of R. Then $\hat{B} = eR$ for some idempotent e. Since B is quasi injective $B = \hat{B}B = eRB = eI$, where $I = RB$, the smallest two-sided ideal of R containing B. Hence (1) implies (2).

Assume (2). Let A be a large right ideal of R. Then $A = eI$, $e^2 = e$, I is a two sided ideal. Since $A \cap (1-e)R = 0$, $(1-e)R = 0$. This implies that $e = 1$. Hence $A = I$, proving (3).

Now assume (3). Let B be a right ideal of R. If K is a complement of B, then $B \oplus K$ is large in R. By assumption $B \oplus K$ is a two-sided ideal in R, hence quasi-injective. This implies B is a quasi-injective, completing the proof.

THEOREM 2.4. *Let $n > 1$ be an integer. Then R_n is a q-ring if and only if R is semi-simple Artinian.*

Proof. Suppose that R is not semi-simple Artinian. By Lambek ([4], Proposition 2, p. 61), there exists a large right ideal B of R such that $B \neq R$. Let e_{ij}, $1 \leqq i, j \leqq n$ be the matrix units of R_n and let $E = \{\sum a_{ij}e_{ij}: a_{1j} \in B, 1 \leqq j \leqq n \text{ and } a_{ij} \in R, 1 \leqq i, j \leqq n\}$. It is clear that E is a right ideal in R_n. But E is not two-sided, for $e_{nn} \in E$ and $e_{1n}e_{nn} = e_{1n} \notin E$. Now, we prove that E is a large right ideal in R_n. Let $0 \neq x = \sum_{i,j=1}^{n} b_{ij}e_{ij}$. If $b_{1j} = 0$, $1 \leqq j \leqq n$, then $x \in E$. So, let $b_{1k} \neq 0$ for some k. Since B is large in R, there exists $a \in R$ such that $0 \neq b_{1k}a \in B$. Then,

$$x(ae_{kk}) = (\textstyle\sum_{i,j=1}^{n} b_{ij}e_{ij})(ae_{kk}) = \sum_{i=1}^{n} b_{ik}ae_{ik} \in E.$$

Hence, $0 \neq x(ae_{kk}) \in E$. Therefore E is a large right ideal in R_n which is not two-sided, and by Theorem 2.3, R_n is not a q-ring. This proves "only if" part. Other part is obvious.

We are now ready to show the existence of right self injective rings which are not q-rings.

EXAMPLE 2.5. Let R be a right self injective ring which is not semi-simple (we can take $R = Z/(4)$). Let $n > 1$ be an integer. By Utumi ([6], Th. 8.3) R_n is right self injective. But R_n is not a q-ring, by the above theorem.

Next we prove

THEOREM 2.6. *A simple ring is a q-ring if and only if it is Artinian.*

Proof. Let R be a simple q-ring. Let B be a large right ideal in R. Then B is two-sided, and hence $B = R$. This proves that R does not contain any proper large right ideal. Hence R is Artinian. The converse is trivial.

Now, we give an example of a right self injective simple ring which is not a q-ring.

EXAMPLE 2.7. Let S be a noncommutative integral domain which is not a right Öre domain (cf. Goldie [1]). Let $R = \hat{S}$. Then R is a right self injective simple regular ring which is not Artinian. By the above theorem R is not a q-ring.

LEMMA 2.8. *Let R be a q-ring. Then B(R) is essential in J(R) as a right R-module.*

Proof. Since R is self injective, $J(R) = R^J$, by Utumi ([6], Lemma 4.1). Let $0 \neq x \in J(R)$. There exist a large right ideal E of R such that $xE = 0$. Then $xE \subset P$ for every prime ideal P of R. Since R is a q-ring, E is two-sided. This implies that either $x \in P$ or $E \subset P$.

Let $\{P_i\}_{i \in I}$ be the set of all prime ideals of R such that $x \in P_i$ for every $i \in I$, and $\{P_j\}_{j \in J}$ be the set of all prime ideals of R such that $x \in P_j$ for every $j \in J$. Let $X = \bigcap_{i \in I} P_i$, and $Y = \bigcap_{j \in J} P_j$. $X \neq 0$, since $0 \neq x \in X$. On the other hand, $E \subset P_j$ for every $j \in J$. Thus $E \subset Y$, which implies that Y is large in R. Therefore $B(R) = X \cap Y \neq (0)$. Moreover, there exists $a \in R$ such that $0 \neq xa \in Y$. This implies that $0 \neq xa \in X \cap Y = B(R)$, completing the proof.

Hence, we have the following

THEOREM 2.9. *A q-ring is regular if and only if it is semi-prime.*

Proof. The result follows by the above lemma, and Utumi ([6], Corollary 4.2).

THEOREM 2.10. *Let V be a vector space over a division ring D, and let $R = \mathrm{Hom}_D(V, V)$. Then R is a q-ring if and only if V is of finite dimension over D.*

Proof. The "if" part is obvious. Conversely, suppose that V is of infinite dimension over D. Let $X = \{x_1, x_2, \cdots\}$ be a denumerable set of linearly independent elements of V. X can be extended to a basis $X \cup Y$ of V. Let F be the ideal in R consisting of all elements of finite rank. Let $\sigma \in R$ be defined by $\sigma(x_{2i}) = x_{2i}$, $\sigma(x_{2i-1}) = 0$ for every i, and $\sigma(y) = 0$ for every $y \in Y$. Let $E = \sigma R + F$. Then $F \subset E$. Since F is a two-sided ideal in R, F is large. Therefore E is a large right ideal in R. We proceed to prove that E is not two-sided. Let $\lambda_1, \lambda_2 \in R$ be defined by: $\lambda_1(x_i) = x_{2i}$ for every i, and $\lambda_1(y) = 0$ for every $y \in Y$, $\lambda_2(x_{2i}) = x_i$, $\lambda_2(x_{2i-1}) = 0$, for every i, and $\lambda_2(y) = 0$ for every $y \in Y$. Let $\lambda = \lambda_2\sigma\lambda_1$. Then $\lambda(x_i) = x_i$ for every i. Hence $X \subset \lambda(V)$. We assert that $\lambda \notin E$; for otherwise, let $\lambda = \sigma r + f$, $r \in R$, $f \in F$. Then $X \subset \lambda(V) = (\sigma r + f)(V) \subset \sigma(V) + f(V)$. But since f is of finite rank, there exists an integer n such that $x_{2n-1} \notin f(V)$. Also, by definition of σ, $x_{2n-1} \notin \sigma(V)$. Hence $x_{2n-1} \notin \sigma(V) + f(V)$, which is a contradiction. Thus $\lambda \notin E$, as desired. However $\lambda \in R\sigma R \subset RE$. Hence E is not a two-sided ideal. Therefore, by Theorem 2.3, R is not a q-ring. This completes the proof.

We remark that the above theorem is also a consequence of Theorem 2.4.

The right (left) socle of a ring R is defined to be the sum of all minimal right (left) ideals of R. It is well known that in a semi-prime ring R, the right and left socles of R coincide, and we denote any of them by soc R.

LEMMA 2.11. *A semi-prime q-ring R with zero socle is strongly regular.*

Proof. Let M be a maximal right ideal in R. Either M is a direct summand of R or M is large in R. If M is a direct summand of R, then its complement is a minimal right ideal. This implies that soc $R \neq 0$, a contradiction. Therefore, every maximal right ideal is large, hence two-sided. By Lemma 2.8, $J(R) = 0$. Thus R is isomorphic to a subdirect sum of division rings, which implies that R has no nonzero nilpotent elements. Since R is regular, by Theorem 2.9, R is strongly regular.

LEMMA 2.12. *A prime q-ring has nonzero socle.*

Proof. Let R be a prime q-ring. If possible, let $\operatorname{soc} R = 0$. By the above lemma, R is strongly regular. Hence R is a division ring, and $\operatorname{soc} R = R$ contradicting our assumption. Therefore $\operatorname{soc} R \neq (0)$.

THEOREM 2.13. *A prime ring R is a q-ring if and only if R is simple Artinian.*

Proof. By Theorem 2.9, and the above lemma, R is a prime regular ring with nonzero socle. Hence, by Johnson ([2], Th. 3.1), $\hat{R} = \operatorname{Hom}_D(V, V)$, where V is some vector space over a division ring D. But then $R = \operatorname{Hom}_D(V, V)$, since R is right self injective. By Theorem 2.10, V has finite dimension over D. Let $(V: D) = n$. Then $R \cong D_n$, completing the proof.

LEMMA 2.14. *Let $\{R_\alpha\}_{\alpha \in I}$ be a finite set of rings. Then the direct sum $\sum_{\alpha \in I} \oplus R_\alpha$ is a q-ring if and only if each R_α is a q-ring.*

The proof is obvious.

That Lemma 2.14 is not true for an infinite number of rings is shown by the following example which is due to Storrer.

EXAMPLE 2.15. Let R be a 2×2-matrix ring over a field F. Let $\{R_\alpha\}_{\alpha \in I}$ be an infinite family of copies of R and let $S = \pi R_\alpha, \alpha \in I$. Let E be the right ideal of S consisting of those elements $[x_\alpha]$ of S such that all but finite x_α's are matrices with first row zero. Since $R_\alpha \subset E$ for all $\alpha \in I$, E is a large right ideal of S. To show that E is not two-sided, consider $[x_\alpha] \in E$ where $x_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for all $\alpha \in I$. Let $[y_\alpha] \in S$ be such that $y_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all $\alpha \in I$. Then $[y_\alpha][x_\alpha] = [z_\alpha]$, where $z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. But then $[z_\alpha] \notin E$, and E is not two-sided. Hence, by Theorem 2.3, S is not a q-ring.

Example 2.15 also suggests the following.

THEOREM 2.16. *Let $\{R_\alpha\}_{\alpha \in I}$ be a family of simple Artinian rings and let R be their complete direct sum. Then R is a q-ring if and only if all R_α's excepting a finite number of them are division rings.*

The above theorem shows, in particular, that a regular q-ring may not be Artinian.

LEMMA 2.17. *Let R be a semi-prime q-ring such that* $\operatorname{soc} R$ *is* *large in R. Then R is a complete direct sum of simple Artinian rings.*

Proof. Since $\operatorname{soc} R$ is large, every nonzero right ideal of R contains a minimal right ideal. Also R is regular, by Theorem 2.9. Hence by Johnson ([2], Th. 3.1), R is a complete direct sum of rings R_i, where each R_i is the ring of all linear transformations of some vector space V_i over a division ring D_i. But then by Lemma 2.14 and Theorem 2.10, each R_i is a simple Artinian ring. This completes the proof.

In the following two theorems we assume that every ring has a unity element which may be equal to zero.

THEOREM 2.18. *Let R be a semi-prime q-ring. Then* $R = S \oplus T$, *where S is a complete direct sum of simple Artinian rings and T is a semi-prime q-ring with zero socle.*

Proof. Let $F = \operatorname{soc} R$. Since $R^J = 0$, $\hat{F} = \{x \in R : xE \subset F$ for some large right ideal E of $R\}$. Then it is immediate that $\hat{F}$ is a two-sided ideal in R. Since R is self injective, $\hat{F} = eR$ for some idempotent e. Then e is central, since R is regular. Let $S = eR$ and $T = (1 - e)R$. Hence $R = S \oplus T$. By Lemma 2.14, both S and T are q-rings. Further, it can be easily verified that (i) S is a semi-prime ring, $\operatorname{soc} S = F$, and F is large in S, and (ii) T is a semi-prime ring with zero socle. By the above lemma S is a complete direct sum of simple Artinian rings, completing the proof.

As a consequence of Lemma 2.11, Theorem 2.16 and Theorem 2.18 we have the following.

THEOREM 2.19. *A semi-prime ring R is a q-ring if and only if* $R = A \oplus B$, *where A is a right self injective duo ring and B is semi-simple Artinian.*

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