Jain, S.K López-Permouth, S.R.
Computational Algebra
Marcel Dekker
205-232 (1994)

A Survey on the Theory of Weakly-Injective Modules

S.K. JAIN, S.R. LÓPEZ-PERMOUTH* Department of Mathematics, Ohio University, Athens, OH 45701.

Abstract

We survey results on the theory of weakly-injective modules obtained during the last five years. In particular, we show that arbitrary direct sums of weakly-injective right modules are weakly-injective if and only if every cyclic right R-module has finite univorm dimension. We also discuss right weakly-semisimple rings (those rings for which every right module is weakly-injective), characterize semiprime Goldie rings as those rings for which every right ideal is weakly-injective, and completely characterize the weakly-injective abelian groups.

Dedicated to the memory of Professor Pere Menal.

Introduction

The purpose of this article is to give an introduction to the theory of weakly-injective modules. We intend to bring together the basic facts and results from several articles developed during the last five years. We intend to point out the connections between the concepts discussed here and several concepts and problems in other areas of ring theory.

The concept of weak relative-injectivity of modules was originally introduced to obtain a characterization of semiperfect rings over which each

*A portion of this paper was written during S.R. López-Permouth's visit to the Centre de Recerca Matemàtica in Bellaterra, Spain. S.R. López-Permouth wishes to acknowledge the support of the Ministry of Education and Science of Spain for that visit and to thank everyone at the Universitat Autònoma de Barcelona for their kind hospitality.

cyclic right module is embeddable as an essential submodule of a projective module (CEP rings). In analogy to a characterization of quasi-Frobenius rings, a ring R is right CEP if and only if R is right artinian and each indecomposable projective right R-module is weakly R-injective (see [15] and [17]). An R-module M is called weakly-injective if for each finitely generated R-module N, and for each R-homomorphism $\varphi : N \to E(M)$ (where $E(M)$ is the injective hull of M), there exists a submodule X of $E(M)$ such that $\varphi(N) \subset X \simeq M$. Chapter 1 of this paper contains some of the basic results on this subject.

Chapter 2 studies, among others, the questions: For what rings is it the case that each weakly-injective module is injective and when are the direct summands of weakly-injective modules again weakly-injective? In analogy to the Matlis-Papp theorem on noetherian rings, rings over which direct sums of weakly-injective modules are weakly-injective are also characterized. There are precisely those rings over which each cyclic module has finite uniform dimension.

Chapter 3 considers rings over which every right module is weakly-injective. Such rings are called right weakly-semisimple. Various characterizations of right weakly-semisimple rings are given. In particular, right weakly-semisimple rings have the property that quasi-injective right modules are injective (i.e., they are right QI-rings). Chapter 4 addresses the question: What abelian groups are weakly-injective as Z-modules?

Chapter 1

Basic Results

Throughout this paper all rings have unity and all modules are right and unital unless otherwise stated. Let R be a ring and let M and N be R-modules. Let $E(M)$ denote the injective hull of M. Recall that M is injective relative to N or simply N-injective if for each homomorphism $\varphi : N \to E(M)$, $\varphi(N) \subset M$. This motivates the following definition.

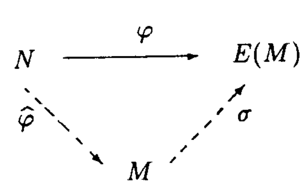
1.1 DEFINITION. Weak Relative-Injectivity

A module M is called weakly-injective relative to the module N or weakly N-injective if for each homomorphism $\varphi : N \to E(M)$, $\varphi(N) \subset X \simeq M$, for some submodule X of $E(M)$. A ring R is right weakly N-injective if the right module R is weakly N-injective.

Notice that the ring of integers Z is weakly Z^n-injective for all $n \in Z^+$, but Z is not Z-injective. Indeed, any commutative integral domain R which is not a field (or even a semiprime Goldie ring R which is not semisimple artinian) is weakly R^n-injective for all $n \in Z^+$, but not self-injective. Also, notice that if a module M is weakly $E(M)$-injective, then M must be injective.

The definition of weak relative-injectivity may be given in terms of commutative diagrams as follows.

1.2 LEMMA. *M is weakly N-injective iff for each homomorphism $\varphi : N \to E(M)$ there exists a monomorphism $\sigma : M \to E(M)$ and a homomorphism $\widehat{\sigma} : N \to M$ such that the diagram*



commutes, that is $\varphi = \sigma\widehat{\varphi}$.

1.3 LEMMA. *Let M and N be R-modules. Then the following statements are equivalent:*

(i) *M is weakly N-injective;*

(ii) *M is weakly N/K-injective for all $K \subset N$; and*

(iii) *For every submodule K of N, and for every monomorphism $h : N/K \to E(M)$ there exist monomorphisms $\sigma : M \to E(M)$ and $\widehat{h} : N/K \to M$ such that $h = \sigma\widehat{h}$.*

PROOF. The proof is straightforward from the definition of weak-injectivity.

The following remarks illustrate the meaning of weak relative-injectivity in specific cases.

1.4 REMARK. An R-module M is weakly R^n–injective iff for all $x_1, \ldots, x_n \in E(M)$ there exists a submodule X of $E(M)$ such that $x_i \in X \simeq M$, $i = i, \ldots, n$.

PROOF. Since every homomorphism $\varphi : R^n \to E(M)$ is determined by choosing arbitrary elements $x_1, \ldots, x_n \in E(M)$, the proof follows from the definition of weak R^n-injectivity.

1.5 REMARK. A ring R is weakly R^n-injective if and only if for all $x_1, x_2, \ldots, x_n \in E(R)$ there exists an element $b \in E(R)$ such that $r \cdot \mathrm{ann}_R(b) = 0$ and $x_1, x_2, \ldots x_n \in bR$.

PROOF. As in the previous remark, if R is weakly R^n-injective, $x_1, x_2, \ldots, x_n$ must be contained in a submodule X of $E(R)$ which is isomorphic to R (via $\varphi : R \to X$, say). Let $b = \varphi(1)$. Then $x_1, x_2, \ldots, x_n \in bR$ and $r \cdot \mathrm{ann}_R(b) = 0$.

Weak relative injectivity is closed under finite direct sums (Lemma 1.6) and under essential extensions, but the direct summands of a weakly injective module need not be weakly injective (see Example 1.11) (vii)).

1.6 LEMMA.

(i) *If L and M are weakly N-injective modules, then $L \oplus M$ is weakly N-injective. In fact, if R is right noetherian, then arbitrary direct sums of weakly N-injective modules are weakly N-injective.*

(ii) *If M is weakly N-injective and L is an essential extension of M, then L is weakly N-injective.*

PROOF. Straightforward.

1.7 DEFINITION. Weakly-injective Module

An R-module M is called weakly-injective if M is weakly N-injective for every finitely generated module N. Equivalently, M is weakly-injective if M is weakly R^n-injective for all $n > 0$ (see Lemma 1.3*). (Note the restriction* *of being finitely generated on N, since if M were weakly N-injective for all N then M would be injective, by considering $N = E(M)$). A ring R is right weakly injective if the right module R is weakly-injective.*

1.8 LEMMA. *A cyclic R-module is weakly-injective if and only if it is weakly R^2-injective.*

PROOF. One implication is trivial. Suppose that M is weakly R^2-injective and let us proceed by induction. Let M be weakly R^{n-1}-injective and let $x_1, x_2, \ldots, x_{n-1} \in E(M)$. By the inductive hypothesis, there exists $x\,R \subseteq E(M)$ such that $x_1, \ldots, x_{n-1} \in x\,R$. By the weak R^2-injectivity of M, there exists $X \simeq M$ such that $x, x_n \in X$. Hence $x_1, x_2, \ldots, x_n \in X$ concluding our proof.

1.9 LEMMA. *Let U be M-injective and M be weakly R-injective, then U is $E(M)$-injective.*

PROOF. If U is not $E(M)$-injective, then by Zorn's lemma there exists a submodule A of $E(M)$ and a homomorphism $f : A \to U$ which cannot be extended to any $f' : B \to U$ with B a submodule of $E(M)$ containing A properly. Let $b \in E(M)/A$. Notice that $A \subset' E(M)$; so $C = bR \cap A \neq 0$. Let $f_1 : C \to U$ be the restriction of f to C. As M is weakly R-injective, bR embeds in M. Therefore, U is bR-injective and f_1 extends to $g : bR \to U$. Define $f' : A + bR \to U$ by $f'(a+br) = f(a) + g(br)$, whenever $a \in A$, $r \in R$. Since f' extends f, this is a contradiction. Therefore, U is $E(M)$-injective.

1.10 PROPOSITION. *A weakly R-injective module M which is also quasi-injective must be injective.*

PROOF. Take $U = M$ in Lemma 1.9.

1.11 EXAMPLES.

(i) A ring R is quasi-Frobenius if and only if it is right artinian and right weakly-injective.

(ii) A domain R is right weakly R-injective if and only if it is right Ore and right weakly-injective if and only if it is two-sided Ore.

(iii) Every semiprime right and left noetherian ring is right and left weakly-injective.

(iv) A ring R is right self-injective if and only if it is right continuous and right weakly-injective.

(v) A semiprime right weakly-injective right Goldie ring is left Goldie.

(vi) A von Neumann regular right weakly-injective ring is right self-injective. A right weakly R-injective regular ring need not be self-injective.

(vii) Let R be a commutative ring and E_R be an injective module whose submodules are linearly ordered. Then, for all $M \subset E$, $N = M \oplus E$ is weakly R-injective. In particular, $Z_2 \oplus Z_{2^\infty}$ is weakly Z-injective. (Note, however, Z_2 is not weakly Z-injective.)

PROOF.

(i) One implication is trivial. Let R be right artinian and weakly-injective. Let $x \in E(R)$. By weak-injectivity there exists $X \subseteq E(R)$, $X \simeq R$, such that $1, x \in X$. Then $R \subseteq X$ and X is right artinian and isomorphic to R. Thus $R = X$ and, therefore, $x \in R$. So $R = E(R)$ is quasi-Frobenius.

(ii) Let R be a domain. The injective hull of R is its Utumi ring of quotients Q. Assume that R is right Ore, then Q is the classical right ring of quotients of R, a division ring. It is obvious that R is right weakly R-injective. On the other hand, if R is right weakly R-injective, for every $q \in Q$ there exists q' in Q with $r \cdot \text{ann}(q') = 0$ and $r \in R$ such that $q = q'r$. This implies that Q is again a domain and, therefore, being self-injective, a division ring. Thus, R is right Ore. If R is two-sided Ore, then Q is both a right and a left ring of quotients for R. Let $q_1, q_2 \in Q$, then there exists $r_1, r_2, s \in R$ such that $q_1 = s^{-1}r_1$ $q_2 = s^{-1}r_2$. Since $r \cdot \text{ann}(s^{-1}) = 0$, $q_1, q_2 \in s^{-1}R$. Using Remark 1.5 we conclude that R is right weakly-injective. The converse follows from (v).

(iii) Let R be a right and left noetherian semiprime ring. Then $E(R_R) = E({}_RR) = r \cdot Q_{c\ell}(R) = \ell \cdot Q_{c\ell}(R)$ where $r \cdot Q_{c\ell}(R)$ and $\ell \cdot Q_{c\ell}(R)$ respectively, denote right and left classical ring of quotients of R. Let $q_1, q_2 \in Q$. There exist $r_1, r_2 \in R$, $s \in R - \{0\}$ such that $q_1 = s^{-1}r_1$, $q_2 = s^{-1}r_2$. By Remark 1.5 R is right weakly-injective. Similarly, R is left weakly-injective.

(iv) While one implication is trivial, the converse requires only condition C_2 in the definition of a continuous module [13], namely *a submodule which is isomorphic to a summand must be summand*. Let R be weakly-injective satisfying the above condition. Let $x \in E(R)$. There exists $X \subseteq E(R)$ which is isomorphic to R and contains 1 and x. Since $R \simeq X$, X is also continuous (hence has C_2). Also, $R \subset X$. So by continuity of X, R is a direct summand of X. However, $R \subset X \subset E(R)$ implies $R \subset' X$. Therefore, $R = X$ and so $x \in R$ for all $x \in E(R)$, proving R is self-injective.

(v) Let R be a semiprime right Goldie ring. Then the injective hull of R is its complete ring of right quotients Q. Assume R is weakly-injective and let $q \in Q$. There exists $q' \in Q$ such that $r \cdot \mathrm{ann}(q') = 0$ with $1, q \in q'R$. It follows that $q' = r^{-1}$ for some $r \in R$ and there exists $s \in R$ such that $q = r^{-1}s$. Therefore, Q is a left ring of quotients for R and hence R is left Goldie.

(vi) Every von Neumann regular ring satisfies condition C_2 in the definition of continuity. So, the first result follows from the proof of (iv). An alternative proof follows from the characterization of weakly-injective nonsingular rings (see Corollary 1.15). F be a field and K be a proper subfield. Let $S = \prod_{i=1}^{\infty} F_i$, $F_i = F$ and $R = \{(x_i) \in S|$ all but finitely many $x_i \in K\}$. Then R is von Neumann regular, weakly R-injective but R is not self-injective.

(vii) Clearly, $E(N) = E \oplus E$. Let $q = (a,b) \in E(N)$, $a \in E$, $b \in E$. Now either $aR \subset bR$ or $bR \subset aR$. Without loss of generality, let $bR \subset aR$. Hence $b = ax$ for some $x \in R$. Thus we have $q = (a,b) = (a, ax) \in \{(c, cx) : c \in E\} = Y \simeq E$. Choose $X = Y \oplus \{(0,c) : c \in M\} \simeq E \oplus M$. Therefore, $N = M \oplus E$ is weakly R-injective.

Example 1.11 (ii) illustrates the fact that being weakly R-injective does not imply being weakly R^2-injective (and so there is no analog to the Baer-criterion for weak-injectivity).

1.12 PROPOSITION. *Every nonsingular module over a noetherian prime ring is weakly-injective.*

PROOF. Over a noetherian prime ring R, every torsionfree right module contains an essential submodule which is a direct sum of uniform submodules. Since weakly-injective modules over noetherian rings are closed under arbitrary direct sums and under essential extensions, it suffices to show that every uniform nonsingular right R-module is weakly-injective. Let U be a uniform nonsingular right R-module and let V be a finitely generated submodule of $E(U)$. Since R is prime and noetherian, it follows that V is isomorphic to a right ideal of R and that, therefore, it embeds in U via a monomorphism φ, say. By the injectivity and indecomposability of $E(U)$, φ extends to an automorphism $\widehat{\varphi} : E(U) \to E(U)$. Let σ be the restriction of $\widehat{\varphi}^{-1}$ to U, then σ is a monomorphism satisfying that $V \subseteq \sigma(U)$, proving our claim.

The result in Proposition 1.12 also holds for nonsingular modules over semiprime Goldie rings [22].

1.13 COROLLARY. *For any module A over a noetherian prime ring R, A is weakly-injective if and only if its singular submodule $Z(A)$ is weakly-injective.*

PROOF. The injective hull of A may be written as $E(A) = E(Z(A)) \oplus K$, where $Z(A)$ is the torsion submodule of A and K is some nonsingular submodule of $E(A)$. If A is weakly-injective and N is a finitely generated submodule of $E(Z(A))$, then there exists, $X \simeq A$ such that $N \subseteq X \subseteq E(A)$. But N is singular, hence $N \subseteq Z(X) \subseteq Z(E(A)) \subseteq E(Z(A))$. Also $Z(X) \simeq Z(A)$, proving our claim.

Recall that for a right nonsingular ring R the maximal right ring of quotients, $r \cdot Q(R)$, of R is a regular right self-injective ring which coincides with $E(R_R)$ as a right R-module.

1.14 PROPOSITION. *For a right nonsingular ring R, the following statements are equivalent:*

(i) *R is a right weakly-injective ring; and*

(ii) *For all $q_1, q_2 \in Q$, there exist $c \in R$ such that $q_1, q_2 \in c^{-1}R$. In particular, Q is a classical left ring of quotients of R.*

PROOF. (i) $\Rightarrow$ (ii). Now $Q = E(R_R)$ is a regular right self-injective ring. Let $1, q_1, q_2 \in Q$. By Remark 1.5 there exists $b \in Q$ such that $r \cdot \mathrm{ann}_R(b) = 0$ and $1 \in bR$, $q_1 \in bR$ and $q_2 \in bR$. Since $r \cdot \mathrm{ann}_R(b) = 0$, b has a left inverse, say c, in Q. Also $1 \in bR$ implies b has a right inverse in R. Thus $q_i \in c^{-1}R$, where $c \in R$, $i = 1, 2$. To prove that Q is a classical left ring of quotients, we need to show in addition that every regular element in R is invertible in Q. We first note that ${}_RR \subset' {}_RQ$. Next let $x \in R$ be a regular element. Then $r \cdot \mathrm{ann}_Q(x) = \ell \cdot \mathrm{ann}_Q(x) = 0$ since $R_R \subset' Q_R$ and ${}_RR \subset' {}_RQ$. Hence x is invertible in Q.

(ii) $\Rightarrow$ (i). Obvious.

Proposition 1.14 yields a new proof for the result in Example 1.11 (vi).

1.15 COROLLARY. *If R is a von-Neumann regular ring, then R is a right self-injective ring if and only if R is a right weakly-injective ring.*

PROOF. Straightforward from Proposition 1.14.

We show now that for a right nonsingular ring right and left weak-injectivity implies the coincidence of the classical ring of quotients with the maximal ring of quotients.

1.16 THEOREM. *Let R be a right nonsingular ring. Then the following statements are equivalent:*

(i) *R is right and left weakly-injective;*

(ii) $E({}_RR) = \ell \cdot Q(R) = r \cdot Q_{c\ell}(R) = \ell \cdot Q_{c\ell}(R) = r \cdot Q(R) = E(R_R)$.

PROOF. (i) $\Rightarrow$ (ii). By Proposition 1.14 we have $Q = E(R_R) = \ell \cdot Q_{c\ell}(R)$. Therefore, considering Q as a left R-module, we have ${}_RR \subset' {}_RQ$. Since Q is von-Neumann regular, $Z({}_RR) = 0$. Therefore, applying Proposition 1.14 to the left weakly-injective left module R, we get $r \cdot Q_{c\ell}(R) = \ell \cdot Q(R) = E({}_RR)$. Since both classical right and left quotient rings exist, they must coincide. Hence

$$E({}_RR) = \ell \cdot Q(R) = r \cdot Q_{c\ell}(R) = l \cdot Q_{c\ell}(R) = r \cdot Q(R) = E(R_R).$$

(ii) $\Rightarrow$ (i). This follows by the definition of weak-injectivity and Remark 1.5.

Chapter 2

Direct Sums of Weakly Injective Modules

We start by pointing out that when we say that M is weakly N-injective, we are saying more than *every quotient of N which is embeddable in $E(M)$ is embeddable in M*. The following lemma sheds light on the distinction.

2.1 LEMMA. *Given two right modules M and N, M is weakly N-injective if and only if for every submodule Q of N and for every monomorphism $\sigma : N/Q \to E(M)$:*

(i) *there exists a monomorphism $\sigma' : N/Q \to M$ and*

(ii) *for every complement K of $\sigma'(N/Q)$ in M there exists $K' \subset E(M)$ such that $K' \cap \sigma(N/Q) = 0$ and $K' \simeq K$.*

PROOF. Let $\sigma : N/Q \to E(M)$ be a monomorphism. By Lemma 1.3 (iii), there exist monomorphisms $\alpha : M \to E(M)$ and $\sigma' : N/Q \to M$ such that $\sigma = \alpha\sigma'$. Thus (i) holds. Let K be a complement of $\sigma'(N/Q)$ in M; then $K' = \alpha(K)$ is isomorphic to K and independent from $\sigma(N/Q)$ proving that (ii) is also necessary. Conversely, let us assume that (i) and (ii) hold and let $\sigma : N/Q \to E(M)$ be a monomorphism. By (i) there exists a monomorphism $\sigma' : N/Q \to M$. Let K be a complement of $\sigma'(N/Q)$ in M. Using (ii), we get a monomorphism $\alpha : \sigma'(N/Q) \oplus K \to E(M)$. Since $\sigma'(N/Q) \oplus K \subset' M$, we may extend α to a monomorphism $\beta : M \to E(M)$. It is straightforward that $\beta\sigma' = \sigma$. Using Lemma 1.3 (iii), M is weakly N-injective.

2.2 DEFINITION. Relative Tightness.

An R-module M is called tight relative to the R-module N if whenever a quotient N/K of N is embeddable in $E(M)$, N/K is also embeddable in M (possibly under a different embedding).

2.3 REMARK. For a uniform module M, M is weakly N-injective if and only if M is N-tight.

PROOF. Obvious from Lemma 2.1 and the definition of relative-tightness.

2.4 DEFINITION. Tight module.

We call an R-module M tight if M is N-tight for every finitely generated R-module N.

In this chapter we study rings over which arbitrary direct sums of (weakly-) injective modules are weakly-injective. We show that this condition is equivalent to the requirement that direct sums of tight modules be tight. These rings are precisely those rings over which all cyclic modules have finite uniform dimension.

We will consider when tight modules are weakly-injective, when weakly-injective modules are injective and when weakly-injective modules are closed under direct summands.

2.5 Definition. QFD rings.

A ring R is said to be right q.f.d. *ring if every cyclic right R-module has finite uniform (Goldie) dimension. (This is equivalent to the requirement that every cyclic or every finitely generated R-module has a finitely generated socle which may possibly be zero.)*

The class of right q.f.d. rings contains all rings with right Krull dimension. So, in particular, every right noetherian ring is right q.f.d.

The following theorem and its corollary extend the main result proved in [1]. Here, we are able to replace weak-injectivity by tightness in every condition of the main theorem of [1], both on the summands as well as on the sums.

2.6 Theorem. *For a right R, the following conditions are equivalent:*

(i) *R is a right* q.f.d. *ring;*

(ii) *every direct sum of injective right R-modules is weakly-injective;*

(iii) *every direct sum of injective right R-modules is tight;*

(iv) *every direct sum of tight R-modules is tight;*

(v) *every direct sum of weakly-injective right R-modules is tight;*

(vi) *every direct sum of weakly-injective right R-modules is R-tight; and*

(vii) *every direct sum of indecomposable injective right R-modules is R-tight.*

Proof. Let us start by showing that (i) $\Rightarrow$ (ii). Consider $M = \bigoplus_{i\in\Lambda} E_i$ where, for every $i \in \Lambda$, E_i is an injective right R-module. Let N be a finitely generated submodule of $E(M)$. By the hypothesis, N contains as an essential submodule a direct sum of uniform submodules $U_1\oplus\cdots\oplus U_k$. Since $M \subset' E(M)$, there exist $0 \neq q_i \in U_i \cap M$. So, $\bigoplus_{i=1}^{k} q_i R$ is contained in a finite direct sum $E_{i_1} \oplus \cdots \oplus E_{i_t}$ where, for $j = 1, \ldots, t$, $i_j \in \Lambda$. This implies that $E_{i_1} \oplus \cdots \oplus E_{i_t}$ contains an injective hull E of $\bigoplus_{i=1}^{k} q_i R$. Since E is injective and contained in M, we may write $M = E \oplus K$, for some submodule K of M. On the other hand, let $E(N)$ be an injective hull of N inside $E(M)$. Then $E(N) = \bigoplus_{i=1}^{k} E(U_i) = \bigoplus_{i=1}^{k} E(q_i R) \simeq E$. Since $\bigoplus_{i=1}^{k} q_i R \subset' E(N)$, it follows that $E(N) \cap K = 0$. So, let $X = E(N) \oplus K \simeq E \oplus K = M$. Then $N \subset X$, proving our claim. Clearly, (ii) $\Rightarrow$ (iii). We proceed to prove (iii) $\Rightarrow$ (iv). Consider now a direct sum $\bigoplus_{i\in\Lambda} M_i$ where, for each $i \in \Lambda$, M_i is tight. Let N be a finitely generated submodule of $E\left[\bigoplus_{i\in\Lambda} M_i\right] = E\left[\bigoplus_{i\in\Lambda} E_i\right]$ where, for each $i \in \Lambda$, $E_i = E(M_i)$. By the hypothesis, N is embeddable in $\bigoplus_{i\in\Lambda} E_i$ via a monomorphism φ, say. Now, $\varphi(N)$ is, therefore, contained in a finite direct sum $E_{i_1} \oplus \cdots \oplus E_{i_t}$ where, for $j = 1, \cdots, t, i_j \in \Lambda$. Now, $M_{i_1} \oplus \cdots \oplus M_{i_t}$, being a finite direct sum of tight modules, is tight. So $N \simeq \varphi(N) \subset E_{i_t} = E(M_{i_1} \oplus \cdots \oplus M_{i_t})$ is embeddable in $M_{i_1} \oplus \cdots \oplus M_{i_t}$ and hence in $\bigoplus_{i\in\Lambda} M_i$, proving our claim. The implications (iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (vii) are obvious. We conclude by proving that (vii) implies (i). We shall do this by proving that every cyclic right R-module has a finitely generated socle. Let M be a cyclic right R-module. If $\mathrm{Soc}\,(M) = 0$, we are done. On the other hand, if $\mathrm{Soc}\,(M) \neq 0$ let K be a complement of $\mathrm{Soc}\,(M)$ in M. Then $\mathrm{Soc}\,(M)$ embeds as an essential submodule of the quotient of M by K (again a cyclic right R-module). So, we may assume, without loss of generality, that M has an essential socle. Let $\mathrm{Soc}\,(M) = \bigoplus_{i\in A} S_i$, where, for each $i \in A$, S_i is simple. Then $E(M) = E\left[\bigoplus_{i\in A} E(S_i)\right]$. Now, since $\bigoplus_{i\in\Lambda} E(S_i)$ is R-tight and M is a cyclic submodule of its injective hull, it follows that M embeds in $\bigoplus_{i\in A} E(S_i)$. Consequently, M embeds in a submodule $L = \bigoplus_{i\in B} E(S_i)$ where B is a finite subset of A. Thus, since L has finitely generated socle, M does also, concluding our proof.

Several other equivalent conditions are given in the following corollary.

2.7 COROLLARY. *A ring R is a right* q.f.d. *ring if and only if any one of the following conditions hold:*

(a) *every direct sum of weakly-injective right modules is weakly-injective;*

(b) *every direct sum of weakly-injective right modules is weakly R-injective;*

(c) *every direct sum of indecomposable injective right modules is weakly R-injective.*

PROOF. Obviously, (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (c) implies condition (vii) of the previous theorem and hence R is a right q.f.d. ring whenever (c) holds. So, it is only left to show that every right q.f.d. ring R satisfies property (a). Consider the module $M = \bigoplus_{i\in\Lambda} M_i$, a direct sum of weakly-injective modules M_i, $i \in \Lambda$. Let N be a finitely generated submodule of $E(M)$. By condition (ii) of the previous theorem, we know that the direct sum of injectives $\bigoplus_{i\in\Lambda} E(M_i)$ is weakly-injective. Also,

$$M \subset' \bigoplus_{i\in\Lambda} E(M_i) \subset' E(M).$$

Hence, there exists a submodule $Y \subset E(M)$ such that $N \subset Y$ and $Y \simeq \bigoplus_{i\in\Lambda} E(M_i)$. Write $Y = \bigoplus_{i\in\Lambda} E(Y_i)$ such that $Y_i \simeq M_i$ for all $i \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma \subset \Lambda$ such that $N \subset \bigoplus_{i\in\Gamma} E(Y_i) = E\left[\bigoplus_{i\in\Gamma} Y_i\right]$. Since the Y_i's are weakly-injective, the finite sum $\bigoplus_{i\in\Gamma} Y_i$ is weakly-injective and, therefore, there exists $X_1 \simeq \bigoplus_{i\in\Gamma} M_i$ such that $N \subset X_1 \subset E\left[\bigoplus_{i\in\Gamma} Y_i\right]$. But then $N \subset X_1 \oplus \bigoplus_{i\in\Lambda-\Gamma} Y_i = X \simeq M$, proving our claim.

We shall consider next when tight modules are weakly-injective. Proposition 2.2 in [16], Theorem 3.1 in [21] and Proposition 3.8 in [22] deal with results regarding when for certain modules being tight implies being weakly-injective. Our next theorem has each of these results as a corollary.

2.8 THEOREM. *Let $\mathcal{A}$ be a class of modules which is closed under submodules and under injective hulls. Let N be a finitely generated module. If every cyclic module in $\mathcal{A}$ has finite Goldie dimension, then for every $M \in \mathcal{A}$, M is N-tight if and only if M is weakly N-injective.*

PROOF. Arguing as in the proof of the proposition in [8], we see that, under these hypotheses, every finitely generated module in $\mathcal{A}$ has finite Goldie dimension. It also follows, using Zorn's Lemma, that every module in $\mathcal{A}$

contains, as an essential submodule, a direct sum of uniform submodules. Let M be an N-tight module in $\mathcal{A}$ and let $\varphi : N \to E(M)$. Since $\varphi(N)$ is a finitely generated module in $\mathcal{A}$, we conclude that $\varphi(N)$ has finite Goldie dimension and thus its imjective hull is a sum of indecompsable injectives, say $E(\varphi(N)) = E_1 \oplus \cdots \oplus E_n$. For $i = 1, \cdots, n$. Let $W_i = E_i \cap \varphi(N)$, then $E_i = E(W_i)$. Since M is N-tight, there exists a map $\psi : N \to M$ such that $\psi(N)$ is isomorphic to $\varphi(N)$. Let $\theta : \varphi(N) \to \psi(N)$ be an isomorphism. Let K be a complement of $\psi(N)$ in M and let $\bigoplus_{i \in I} U_i \subset' K$, where for all $i \in I$, U_i is uniform. Similarly, let $\widehat{K}$ be a complement of $\varphi(N)$ in $E(M)$ and let $\bigoplus_{j \in J} V_j \subset' \widehat{K}$, a direct sum of uniform submodules. It follows that

$$E_1 \oplus \ldots \oplus E_n \oplus \bigoplus_{j \in J} E(V_j) \subset' E(M)$$

and

$$E(\theta(W_1)) \oplus \cdots \oplus E(\theta(W_n)) \oplus \bigoplus_{i \in I} E(U_i) \subset' E(M).$$

Hence, by Corollary 4.2 in [25],

$$E_1 \oplus \cdots \oplus E_n \oplus \bigoplus_{i \in J} E(V_j) \simeq E(\theta(W_1)) \oplus \cdots \oplus E(\theta(W_n)) \oplus \bigoplus_{i \in I} E(U_i).$$

Using the Azumaya-Krull-Schmidt theorem, one gets that there exists an isomorphism

$$\eta : \bigoplus_{i \in I} E(U_i) \to \bigoplus_{j \in J} E(V_j).$$

Denote η' the restriction of η to $\bigoplus_{i \in I} U_i$. Consider then the one-to-one map

$$\sigma = \theta^{-1} \oplus \eta : \psi(N) \oplus \bigoplus_{i \in I} (U_i) \to E(M)$$

which extends to a monomorphism

$$\widehat{\sigma} : M \to E(M) \quad (\text{since } \psi(N) \oplus \bigoplus_{i \in I} U_i \subset' M).$$

This monomorphism satisfies that

$$\varphi(N) = \theta^{-1}(\psi(N)) = \sigma(\psi(N)) = \widehat{\sigma}(\psi(N)) \subseteq \widehat{\sigma}(M),$$

as desired.

2.9 REMARK. As mentioned before, Proposition 2.2 in [16] follows from Theorem 2.8 (let $\mathcal{A}$ consist of all submodules of $E(M)$). The same is true for Proposition 3.8 in [22], by letting $\mathcal{A}$ consist of all nonsingular modules over the semiprime Goldie ring R. Because of its importance, we shall state Theorem 3.1 from [21] below.

2.10 THEOREM. *Every tight module over a right* q.f.d. *ring R (in particular over a right noetherian ring R) is weakly-injective.*

PROOF. Let $\mathcal{A}$ be the class of all right R-modules and use Theorem 2.8.

It is open whether the converse of Theorem 2.10 holds. Theorem 3.2 in [21] seems to point in the direction of a proof for the converse and suggested the next example.

2.11 EXAMPLES. Let R be the ring of endomorphisms of an infinite dimensional vector space V over a field F. Then there exists a tight R-module M which is not weakly-injective.

PROOF. Let $M = \mathrm{Soc}\,(R_R) \oplus R$. As is well known, R is right self-injective and $\mathrm{Soc}\,(R_R) \subset' R_R$. Also, there exists a simple right R-module S and an infinite cardinal $\mathcal{N}$ such that $\mathrm{Soc}\,(R_R) \simeq S^{(\mathcal{N})}$. So, $E(\mathrm{Soc}\,(R_R)) = R$ and $E(M) = E(\mathrm{Soc}\,(R_R)) \oplus R \simeq E(\mathrm{Soc}\,(R_R)) \oplus \mathrm{Soc}\,(R_R)) \simeq E(\mathrm{Soc}\,(R_R)) = R$. Obviously, since $E(M)$ is isomorphic to a submodule of M, M is tight. On the other hand, M is not even weakly R-injective. Since $E(M) = R \oplus R \simeq R$, $E(M)$ is a cyclic module. If M were weakly R-injective, we would have $X \simeq M$, such that $E(M) \subseteq X \subseteq E(M)$ and then M would be injective, a contradiction.

We will now consider direct summands of weakly-injective modules.

As pointed out before, weakly-injective modules are not necessarily closed under direct summands. For example, for any field F, the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is weakly-injective but $S = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ as an R-module is not weakly-injective. The following two propositions illustrate to what extreme this condition fails.

2.12 PROPOSITION. *Every completely reducible module over an arbitrary ring R is a direct summand of a weakly-injective R-module.*

PROOF. Let M be a completely reducible right R-module. Let us write $M = \bigoplus_{i \in I} [S_i]$, where $[S_i]$ represents the homogeneous component of M corresponding to the simple submodule $S_i \subset M$. It follows that for every

$i \in I$ there exists a cardinal $\aleph_i$ such that $[S_i] \simeq S_i^{(\aleph_i)}$. Let N be an infinite cardinal greater than both the cardinality of R and the number of summands of M. In particular, for every $i \in I$, $N > \aleph_i$. Notice that for every finitely generated right R-module N, if $\bigoplus_{\alpha\in\Gamma} U_\alpha$ is an internal direct sum of nonzero submodules of N then the cardinality of Γ is less than $\aleph$. Let $V = M \oplus E(M^{(\aleph)})$. We claim that V is weakly-injective. Notice, first of all, that $E(V) \simeq E(M^{(\aleph)})$ and $\operatorname{Soc}(V) = \operatorname{Soc} E(V) \simeq \bigoplus_{i\in I}[S_i]^{(\aleph)} \simeq \bigoplus_{i\in I}(S_i^{(\aleph_i)})^{(\aleph)} \simeq \bigoplus_{i\in I} S_i^{(\aleph)}$. Let N be a finitely generated submodule of $E(V)$. Then the number of simple summands in any decomposition of $\operatorname{Soc} N$ is less than $\aleph$. Let us say that $\operatorname{Soc}(N) = \bigoplus_{i\in I}[[S_i]]$, where $[[S_i]]$ denotes the (possibly zero) homogeneous component of $\operatorname{Soc}(N)$ corresponding to S_i. Since for every $i \in I$ the number of simple summands in $[[S_i]]$ is less than $\aleph$, we conclude that the homogeneous component of $\operatorname{Soc} E(V)$ corresponding to S_i equals $[[S_i]] \oplus K_i$, for some $K_i \simeq (S_i)^{(\aleph)}$. Hence, we get $\operatorname{Soc} V = \operatorname{Soc} N \oplus T$, for some $T \simeq \operatorname{Soc} V$. Therefore, $E(\operatorname{Soc} V) = E(V) = E(N) \oplus E(T)$, and $E(T) \simeq E(V)$. Let Y be a submodule of $E(T)$ isomorphic to V and define $X = E(N) \oplus Y$. Then $X \simeq E(N) \oplus M \oplus E(M^{(\aleph)}) = M \oplus E(N \oplus M^{(\aleph)}) \simeq M \oplus E\left(\bigoplus_{i\in I}[[S_i]] \oplus \bigoplus_{i\in I}(S_i)^{(\aleph)}\right) \simeq M \oplus E\left(\bigoplus_{i\in I}(S_i)^{(\aleph)}\right) \simeq M \oplus E(M^{(\aleph)}) = V$. Since $N \subset X$, this concludes our proof.

2.13 COROLLARY. *Over a right semi-artinian ring R, every right R-module is a summand of a weakly-injective right module.*

PROOF. This follows from the previous proposition since weak-injectivity is preserved by essential extensions.

2.14 PROPOSITION. *Over arbitrary rings, every module is a summand of a tight module. If R is a right* q.f.d. *ring, every right R-module is a summand of a weakly-injective right module.*

PROOF. Let M be a right module over the right q.f.d. ring R and let $\aleph$ be any infinite cardinal. Consider the module $N = M \oplus E(M^{(\aleph)})$. Since $E(N)$ is isomorphic to a submodule of N, N is tight. In light of Theorem 2.4, if R is a right q.f.d. ring, then N is weakly-injective.

In light of the previous results, one has to wonder if, any ring R and for any R-module M, M is a summand of a weakly-injective module. This is an open problem.

Propositions 2.12 and 2.14 are instrumental in characterizing those rings for which weakly-injective modules are closed under direct summands and those rings for which the only weakly-injective modules are the injective ones.

2.15 THEOREM. *Let R be a ring. Then the following are true:*

(i) *Direct summands of weakly-injective (tight) right R-modules are weakly-injective (tight) if and only if every R-module is weakly-injective.*

(ii) *Every weakly-injective (tight) right module is injective if and only if R is semisimple-artinian.*

PROOF. If weakly-injective (tight) right R-modules were closed under direct summands, Proposition 2.12 implies that every completely reducible right R-module would be weakly-injective (tight) and thus injective. This implies that R is right noetherian [17]. Then, by Proposition 2.14 and the hypothesis, R is right weakly-semisimple. One can argue in the same way to prove that if every weakly-injective module is injective then every right R-module is injective and hence R is semisimple artinian.

Chapter 3

Weakly-Semisimple Rings

In this chapter we consider those rings over which all right modules are weakly-injective. We start by considering the rings for which all right ideals are weakly-injective. A theorem of López-Permouth, Rizvi and Yousif characterizes these rings as being precisely the semiprime Goldie rings. On the other hand, rings over which each right module is weakly-injective are precisely the right noetherian rings over which all finitely generated uniform right modules are compressible. These rings have the property that all quasi-injective right modules are injective.

3.1 LEMMA. *Let R be a ring such that each right ideal is weakly R-injective. Then R is nonsingular and semiprime.*

PROOF. Let $Z(R)$ denote the right singular ideal. Write $E(Z(R)) \oplus K = E(R)$ and $1 = a + b$ where $a \in E(Z(R))$, $b \in K$. Since $Z(R)$ is weakly R-injective, aR is embeddable in $Z(R)$ and so there exists an essential right ideal I such that $aI = 0$. But then $bI = I$ and hence $I \subset K$ is nonsingular. Since $Z(I) = I \cap Z(R)$ and $Z(I) = 0$, we obtain $Z(R) = 0$ because I is essential.

Let Q be the maximal right quotient ring of R. We know that Q is regular and right self-injective. Let N be an ideal in R such that $N^2 = 0$, we shall prove $N = 0$. Now, $E(N) = eQ$, $e = e^2$. Thus by the weak R-injectivity of N, there exists $X \subset Q$ such that $e \in X \simeq N$. This yields that $eN = 0$. Since $eK \subset N$ for some essential right ideal K, it follows that $eK = 0$. Hence $e = 0$ because Q is nonsingular. This proves that R is semiprime.

3.2 LEMMA. *Let R be a ring such that each right ideal is weakly R-injective then R has finite uniform (Goldie) dimension.*

PROOF. Let $A = \bigoplus_{i \in I} U_i$ be a direct sum of nonzero right ideals, and suppose A is essential in R. We will show that I must be a finite set. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \lambda\ } & R \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\widehat{\varphi}} & \\ & E(A) & \end{array}$$

where λ and φ are inclusion maps while $\widehat{\varphi}$ is the (necessarily one-to-one) map granted by the injectivity of $E(A)$. Since A is weakly R-injective, there exists a submodule $X \subseteq E(A)$ such that $\widehat{\varphi}(R) \subseteq X \simeq A$. We may then write $X = \bigoplus_{i \in I} X_i$, where $X_i \simeq U_i$. But then $\widehat{\varphi}(1) \in \bigoplus_{i \in J} X_i$, where J is a finite subset of I. Consequently $\widehat{\varphi}(1)R = \widehat{\varphi}(R) \subseteq \bigoplus_{i \in J} X_i$. On the other hand,

$$A = \varphi(A) = \widehat{\varphi}\lambda(A) \subseteq \widehat{\varphi}(R) \subseteq E(A).$$

So $\widehat{\varphi}(R)$ is essential in $E(A)$. Since for every $i \notin J$, $X_i \cap \widehat{\varphi}(R) = 0$, we conclude that $X_i = 0$, a contradiction. So, $I = J$ is finite.

3.3 PROPOSITION. *Let R be a ring in which each right ideal is weakly R-injective. Then R is a semiprime right Goldie ring. Furthermore, if R is weakly-injective, then R is also left Goldie.*

PROOF. By Lemmas 3.1 and 3.2 R is semiprime nonsingular and has finite uniform dimension. So the right quotient ring Q must be semisimple artinian. This proves R is a right Goldie ring. If R were also weakly-injective, then, by Example 1.11 (v), R is also left Goldie.

Proposition 3.3 may be proved by assuming only that each essential right ideal is weakly R-injective (see [22]).

3.4 DEFINITION. *A uniform module M is said to be compressible if for all nonzero $N \subseteq M$ there exists a monomorphism from M into N. In general, a module M is compressible if for every essential submodule N of M, M embeds in N.*

3.5 LEMMA. *Given an indecomposable injective right module E, the following statement are equivalent:*

(i) *every submodule of E is weakly-injective;*

(ii) *every cyclic submodule of E is weakly-injective; and*

(iii) *every finitely generated submodule of E is compressible.*

PROOF. Clearly (i) implies (ii). Assume (ii) and let M be a finitely generated submodule of E and N a non-zero submodule of M. Let $x \in N$, $x \neq 0$. Since xR is weakly-injective, M is embeddable in xR and hence in N. Let us now assume (iii). Let N be an arbitrary non-zero submodule of E. Suppose $x \in N$, $x \neq 0$. If $x_1, x_2, \ldots, x_n \in E(N) = E = E(xR)$ by the compressibility of $M = \sum_{i=1}^{n} x_1 R$, there exists an embedding σ of M into $xR \cap M \subseteq N$. Using Remark 2.3, we conclude that N is weakly-injective.

A more general version of this result for an arbitrary injective module E appears in [22](Proposition 3.7)

3.6 PROPOSITION. *Let R be a semiprime right and left Goldie ring. Then each right ideal is weakly-injective.*

PROOF. Since each right ideal I contains a finite direct sum of uniform right ideals which is essential in I, it is sufficient to prove that each uniform right ideal is weakly-injective. So let U be a uniform right ideal. Since R is semiprime and Goldie, every finitely generated nonsingular module is compressible ([16], Theorem 22.15). So, by Lemma 3.5, every submodule of $E(U)$ is weakly-injective. In particular, U is weakly-injective, as claimed.

3.7 REMARK. Let R be a semiprime right Goldie ring. Then R as a right R-module need not be weakly R-injective. For example, consider a right Ore-domain K which is not a left Ore-domain. Let $R = M_2(K)$ be the 2×2 matrix ring over K. Then R is a semiprime right Goldie ring but R is not weakly R-injective as follows from the next lemma.

3.8 LEMMA. *If K is a right Ore-domain and $R = M_2(K)$ is right weakly R-injective, then K is also a left Ore-domain.*

PROOF. Let a and b be nonzero elements in K. Since K is a right Ore domain, $E(K_K) = Q$ is a division ring. Consider the element $q = \begin{bmatrix} a^{-1} & b^{-1} \\ 0 & 0 \end{bmatrix}$ $\in E(R_R) = M_2(Q)$. Given that R is right weakly R-injective, there exists an element $y \in M_2(Q)$ such that $r \cdot \mathrm{ann}_R(y) = 0$ and $q \in yR$. But then y is invertible. Let $y^{-1} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$. Certainly not every entry in the first column of y^{-1} can be zero. Without loss of generality, assume $q_{11} \neq 0$. We then get that $q_{11}a^{-1} = \gamma_1$ and $q_{11}b^{-1} = \gamma_2$ belong to K. So $q_{11} = \gamma_1 a = \gamma_2 b$ and, therefore, $K a \cap K b \neq 0$. We conclude that K is left Ore.

We now consider rings over which each module is weakly-injective. Since rings over which each module is injective are semisimple artinian, we define.

3.9 DEFINITION. Weakly-Semisimple Ring

A ring R is called right weakly-semisimple if each right R-module is weakly-injective.

3.10 LEMMA. *A right weakly-semisimple ring is right QI.*

PROOF. This follows obviously from Proposition 1.10.

3.11 LEMMA. *A right* q.f.d. *ring R over which every cyclic uniform right R-module is weakly-injective must be right weakly-semisimple.*

PROOF. Since every right module over a right q.f.d. ring contains an essential submodule which is a direct sum of cyclic uniform submodules, the result follows from Corollary 2.7.

3.12 THEOREM. *The following conditions on a ring R are equivalent:*

(i) *R is right weakly-semisimple;*

(ii) *every finitely generated right R-module is weakly-injective and R is right noetherian;*

(iii) *every cyclic right R-module is weakly R^2-injective and R right noetherian;*

(iv) *every uniform cyclic right R-module is weakly R^2-injective and R is right noetherian; and*

(v) *every finitely generated uniform right R-module is compressible and R is right noetherian.*

PROOF. From Lemma 3.10 and the fact that every right QI-ring is right noetherian, it follows that (i) implies (ii). Clearly (ii) implies (iii) and (iii) implies (iv). The implication (iv) $\Rightarrow$ (i) is Lemma 3.11. The equivalence of (iv) and (v) follows from Lemma 3.5.

3.13 REMARK. Theorem 3.12 is specially interesting when compared to a result by Boyle which characterizes right QI-rings as being those right noetherian rings for which every uniform cyclic is strongly prime [6] in the following sense. A module M is strongly prime if it satisfies one of the equivalent conditions:

(i) M is contained in every quasi-injective submodule of its injective hull;

(ii) for all $x, y \in M$, there exist $r_1, r_2, \ldots, r_n \in R$ such that $r \cdot \mathrm{ann}(x) = r \cdot \mathrm{ann}(yr_1, yr_2, \ldots, yr_n)$.

It can be shown easily that every compressible module is strongly prime.

In relation to Theorem 3.12 and Remark 3.13, it would be interesting to characterize right noetherian rings over which every cyclic right R-module is weakly R-injective and right noetherian rings over which every cyclic uniform right R-module is compressible. Clearly, in statements (ii) through

(v) of Theorem 3.12 one could replace right noetherian by right q.f.d. and the theorem would remain valid. Could the requirement of right q.f.d. be removed from (or even weakened in) any of these statements? While this does not seem likely, we do not have a counterexample.

3.14 REMARK. Every right QI-ring is a right noetherian right V-ring. Also, it is well known [5] that for a (two-sided) noetherian hereditary ring R, the following conditions are equivalent:

(i) R is a right V-ring;

(ii) R is a left V-ring;

(iii) R is a right QI-ring; and

(iv) R is a left QI-ring.

Our next theorem extends this list of equivalent statements to include right and left weakly-semisimple.

3.15 THEOREM. *Let R be a hereditary noetherian ring. Then the following are equivalent:*

(i) *R is a right QI-ring;*

(ii) *R is a right weakly-semisimple ring; and*

(iii) *R is a left weakly-semisimple ring.*

PROOF. It suffices to show that (i) implies (ii). Without losing generality, we may assume that R is simple [20]. Reasoning as in [5], we may also assume that R is a right (and left) Ore-domain. Let M be a finitely generated uniform right R-module. Since every finitely generated right module over a hereditary noetherian prime ring is a direct sum of a projective and a torsion module [11], M is either projective or torsion. If M is projective, then, being uniform, M must be embeddable in R. It follows, because R is a domain, that M must contain (essentially, since M is uniform) a submodule isomorphic to R. Since R is a noetherian domain, R is weakly-injective and so M is weakly-injective because R is essentially embeddable in M (Lemma 1.6). If M is torsion, then M is artinian [11]. Hence, since simple modules are injective, $M = \mathrm{Soc}\, M$ is injective. The result now follows from Theorem 3.12.

3.16 COROLLARY. *For a hereditary noetherian ring R, the following conditions are equivalent:*

(i) *R is (right) weakly-semisimple*

(ii) *every (right) R-module is weakly R-injective;*

(iii) *domains of injectivity by right R-modules are closed under injective hulls; and*

(iv) *R is (right) QI.*

PROOF. (i) implies (ii) is trivial. Lemma 1.9 yields that (ii) implies (iii). Assuming (iii), a quasi-injective module M is $E(M)$-injective thus injective. This gives (iii) implies (iv). By Theorem 3.15, (iv) is equivalent to (i).

We conclude this chapter with a remark related to the early part of the chapter, specifically to Remark 3.7. Y. Zhou [26] has characterized semiprime one-sided Goldie rings in terms of the concept of "strong compressibility". A module M is said to be strongly compressible if for every essential submodule N o f M there exists $X \subseteq E(M)$ such that $M \subseteq X \simeq N$. The connection between strong compressibility and weak-injectivity is given in the following proposition (c.f. Proposition 3.7 of [22]).

3.17 PROPOSITION. *Let E be an injective module then every submodule of E is weakly-injective if and only if every finitely generated submodule of E is strongly compressible.*

PROOF. See [26], Proposition 9.

Chapter 4

Weakly-Injective Abelian Groups

The purpose of this chapter is to characterize weakly-injective abelian groups. A finitely generated abelian group is weakly-injective if and only if it is torsion free. Indeed, every torsionfree abelian group is weakly-injective (Proposition 1.12). A p-torsion group is weakly-injective if and only if for all $n > 0$, $\bigoplus_{i=1}^{n} Z_{p^i}$ is embeddable in A. A weakly-injective abelian group A may always be decomposed as $A = B \oplus C$ where B is torsion, divisible (injective) with finite-dimensional primary components and C satisfies the property: if Z_p embeds in C, then for all $n > 0$, $\bigoplus_{i=1}^{n} Z_{p^i}$ embeds in C.

In this chapter, note that the concepts of weak-injectivity and tightness coincide (Theorem 2.10).

We restate a special case of Corollary 1.13 in

4.1 Lemma. *An abelian group A is weakly-injective if and only if the torsion subgroup T is weakly-injective.*

Thus in order to characterize weakly-injective abelian groups, it suffices to consider torsion ones.

4.2 Lemma. *A torsion abelian group is weakly-injective if and only if its p-subgroups are weakly-injective.*

Proof. The proof is left as an exercise.

4.3 Lemma. *If a torsion abelian group A has finite uniform dimension and is weakly-injective, then it must be injective.*

Proof. By hypothesis there exists a finite direct sum $\sum_{i=1}^{n} S_i$ of simple groups S_i such that $\oplus \sum S_i$ is essential in A. Now $E(S_i) = Z_{p_i^\infty}$ for some prime p_i. Let $0 \subset a_{11}Z \subset a_{22}Z \subset \cdots \subset Z_{p_i^\infty}$ be the unique composition series of $E(S_i) = Z_{p_i^\infty}$. Then for every $m > 0$, $\operatorname{Soc}^m E(A) = a_{1m}Z \oplus \cdots \oplus a_{nm}Z$. It is clear that $E(A) = \bigcup_{m=1}^{\infty} \operatorname{Soc}^m E(A)$. So in order to prove that A is injective, it suffices to prove that $\operatorname{Soc}^m E(A) = \operatorname{Soc}^m A$. Since A is weakly-injective, for every $m \in Z^+$ there exists an embedding $\varphi : \operatorname{Soc}^m E(A) \to A$. We will first prove by induction that for every embedding $\varphi : \operatorname{Soc}^m E(A) \to A$, $\varphi(\operatorname{Soc}^m E(A)) = \operatorname{Soc}^m A = \operatorname{Soc}^m E(A)$. The result is clear if $m = 1$. Suppose it is true for $m = j - 1$ and assume that $\varphi : \operatorname{Soc}^j E(A) \to A$ is an embedding. By the inductive hypothesis, the restriction of φ to $\operatorname{Soc}^{j-1} E(A)$ is an isomorphism onto $\operatorname{Soc}^{j-1} A = \operatorname{Soc}^{j-1} E(A)$. Then

$$\frac{\operatorname{Soc}^j E(A)}{\operatorname{Soc}^{j-1} E(A)} \simeq \frac{\varphi(\operatorname{Soc}^j E(A))}{\varphi(\operatorname{Soc}^{j-1} E(A))}$$

$$= \frac{\varphi(\operatorname{Soc}^j E(A))}{\operatorname{Soc}^{j-1} E(A)} \subset \operatorname{Soc}\left[\frac{A}{\operatorname{Soc}^{j-1} E(A)}\right] \subset \operatorname{Soc}\left[\frac{E(A)}{\operatorname{Soc}^{j-1} E(A)}\right]. \quad (1)$$

From the first inequality in (1), $\varphi(\operatorname{Soc}^j E(A)) \subset \operatorname{Soc}^j A$. Also by (1), the Goldie dimension of $\dfrac{\operatorname{Soc}^j A}{\operatorname{Soc}^{j-1} A}$ is at least n, since $\dfrac{\operatorname{Soc}^j E(A)}{\operatorname{Soc}^{j-1} E(A)} = \displaystyle\sum_{i=1}^{n} \frac{a_{ij} R}{a_{i,j-1} R}$, a direct sum of n simples. On the other hand, since $A \subset' E(A)$, the Goldie dimension of $\dfrac{\operatorname{Soc}^j A}{\operatorname{Soc}^{j-1} A}$ is less than or equal to the Goldie dimension of $\dfrac{\operatorname{Soc}^j A}{\operatorname{Soc}^{j-1} E(A)}$, which equals n. So, using (1) once again, we get that $\dfrac{\varphi(\operatorname{Soc}^j E(A))}{\operatorname{Soc}^{j-1} E(A)} = \dfrac{\operatorname{Soc}^j A}{\operatorname{Soc}^{j-1} A}$ and hence $\varphi(\operatorname{Soc}^j E(A)) = \operatorname{Soc}^j A = \operatorname{Soc}^j E(A)$, as desired. This concludes our induction.

A characterization of weakly-injective torsion modules with infinite Goldie dimension is given in the next lemma.

4.4 LEMMA. *Let A be a p-torsion abelian group and of infinite Goldie dimension. The following statements are equivalent:*

(i) *A is weakly-injective;*

(ii) *$\oplus \sum_{i=1}^{n} Z_{p^i}$ is embeddable in A for all $n > 0$.*

PROOF. Follows from definition of weak-injectivity.

4.5 REMARK. It can also be shown that a p-torsion abelian group A is weakly-injective if and only if the external direct sum $\oplus \sum_i B_i$ of all proper subgroups of Z_{p^∞} is embeddable in A.

4.6 THEOREM. *For an abelian group A, the following statements are equivalent:*

(i) *A is weakly-injective;*

(ii) *there is a decomposition* $A = B \oplus C$ *such that* (i) *B is torsion, injective and has finite dimensional primary components,* (ii) *C satisfies that if* Z_p *embeds in C then for all* $n, \oplus \sum_{i=1}^{n} Z_{p^i}$ *embeds in C, and* (iii) *B and C have no isomorphic simple subgroups;*

(iii) *there is a decomposition* $A = B \oplus C$ *such that B is injective and C satisfies that if* Z_p *embeds in C then* $\oplus \sum_{i=1}^{n} Z_{p^i}$ *embeds in C for all* $n > 0$.

PROOF. If A is weakly-injective, so is the torsion subgroup T (Lemma 4.1) and also so are the primary components of T (Lemma 4.2). Let B be the (direct) sum of all the primary components of T with finite Goldie dimension. By Lemma 4.3, each such primary component is injective and, therefore, so is B. It follows that we may write $A = B \oplus C$, where C is chosen so that it contains the primary components of T not already contained in B. If $S = Z_p$ and a monomorphism φ embeds S in C, then S actually embeds in the primary component N (say) of T corresponding to $\varphi(S)$. By the weak-injectivity of N and in light of Lemma 4.4, we conclude that $\oplus \sum_{i=1}^{n} A_{p^i}$ embeds in N for all $n > 0$ and consequently in C, as claimed. The decomposition $A = B \oplus C$ satisfies conditions (i), (ii) and (iii) in (2) and, therefore, we conclude that (1) implies (2). Obviously (2) implies (3). The conditions in (3) imply that torsion subgroup T' of C is weakly-injective (by Lemma 4.3). Therefore, by Lemma 4.1, C is weakly-injective and hence A, being the sum of two weakly-injective, is weakly-injective. Thus (3) implies (1).

4.7 REMARK. The results in this chapter can be extended to modules over Dedeking prime rings or even to modules over bounded hereditary noetherian prime rings [16].

References

[1] A.H. Al-Huzali, S.K. Jain and S.R. López-Permouth, *Rings whose cyclics have finite Goldie dimension,* J. Alg. **153**, 37–40 (1992).

[2] A.H. Al-Huzali, S.K. Jain and S.R. López-Permouth, *Weakly-injective rings and modules,* Osaka J. Math. **29**, 75–87 (1992).

[3] A.H. Al Huzali, S.K. Jain and S.R. López-Permouth, *On the weak relative-injectivity of rings and modules,* Noncommutative Ring Theory, Lectures Notes in Math. **1448** Springer-Verlag, New York (1980), 93–98.

[4] F.W. Anderson and K.R. Fuller, RINGS AND CATEGORIES OF MODULES, Springer-Verlag, New York, Heidelberg, Berlin (1974).

[5] A.K. Boyle, *Hereditary QI-rings,* Trans. Amer. Math. Soc. **58**(1974), 115–120.

[6] A.K. Boyle, *Injectives containing no proper quasi-injective submodules,* Comm. in Alg. **4**(1976), 775–785.

[7] A.K. Boyle and K.R. Goodearl, *Rings over which certain modules are injective,* Pacific J. Math. **58**(1975), 43–53.

[8] V.P. Camillo, *Modules whose quotients have finite Goldie dimension,* Pacific J. Math. **69**(1977), 337–338.

[9] A.W. Chatters and C.R. Hajarnavis, RINGS WITH CHAIN CONDITIONS, Pitman Publishing Inc., Boston, London, Melbourne (1980).

[10] J. Cozzens and C. Faith, SIMPLE NOETHERIAN RINGS, Cambridge University Press, Cambridge, London, New York (1975).

[11] D. Eisenbud and J.C. Robson, *Modules over Dedeking prime rings,* J. Algebra **16**(1970), 67–85.

[12] C. Faith, ALGEBRA: RINGS, MODULES AND CATEGORIES II, Springer-Verlag, New York, Berlin, Heidelberg (1976).

[13] C. Faith, *On hereditary rings and Boyle's conjecture,* Arch. der Math. **27**(1976), 113–119.

[14] J.S. Golan and S.R. López-Permouth, *QI-filters and tight modules,* Comm. Alg. **19**(8) (1991), 2217–2229.

[15] S.K. Jain and S.R. López-Permouth, *Rings whose cyclics are essentially embeddable in projective modules,* J. Algebra **128**(1990), 257–269.

[16] S.K. Jain and S.R. López-Permouth, *Weakly-injective modules over hereditary noetherian prime rings,* to appear in J. Australian Math. Soc.

[17] S.K. Jain, S.R. López-Permouth and S. Singh, *On a class of QI-rings,* Glasgow Math. J. **34** (1992) 75-81.

[18] A.V. Jategaonkar, LOCALIZATION IN NOETHERIAN RINGS, Cambridge University Press, Cambridge, London, New York, 1986.

[19] K.A. Kosler, *On hereditary and noetherian V rings,* Pacific J. Math. **103**(1982), 467–473.

[20] R.P. Kurshan, *Rings whose cyclic modules have finitely generated socle,* J. Algebra **15**(1970), 376–386.

[21] S.R. López-Permouth, *Rings characterized by their weakly-injective modules,* Glasgow Math. J. **34** (1992) 349–353.

[22] S.R. López-Permouth, S.T. Rizvi and M.F. Yousif, *Some characterizations of semiprime Goldie rings,* to appear in Glasgow Math. J.

[23] S.R. López-Permouth and S.T. Rizvi, *On certain classes of QI-rings,* Methods in module theory, Marcel Dekker, New York, Basel, Hong Kong (1992) 227–235.

[24] G.O. Michler and O.E. Villamayor, *On rings whose simple modules are injective,* J. Algebra **25**(1973), 185–201.

[25] R.B. Warfield, *Decompositions of injective modules,* Pac. Jour. Math. **31**(1) (1969), 263–276.

[26] Y. Zhou, *Strongly compressible modules and semiprime right Goldie rings,* preprint.