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Self-injective Group Ring

By

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A beautiful result of Gentile [3], states if K is a commutative torsion-free ring then KG is right self-injective iff K is self-injective and G is finite.

The object of this note is to present its proof by using among others the well known Ikeda-Nakayama theorem for self-injective rings. The latter gives at once that if KG is right self-injective then G is locally finite. This fact simplifies the proof of Gentile's result. We also point out that the theorem holds even if K is not commutative.

Let A be any torsion-free ring and G be a group such that $R = AG$ is right self-injective.

1.1 (Ikeda and Nakayama [4], p. 16). Finitely generated left ideals of a right self-injective ring are annihilator ideals.

1.2. Let H be a finitely generated subgroup of G. Then if H is not finite the right ideal ωH generated by $1 - h$, $h \in H$ is such that its left annihilator is zero, that is $(\omega H)^l = 0$, and this implies $(\omega H)^{lr} = R$. This contradicts 1.1, since ωH is a proper right ideal. Hence G is locally finite.

1.3 (Cartan and Eilenberg, [1], Ch. VI, Ex. 10, p. 123). Let $\Phi\colon \Lambda \to \Gamma$ be a ring homomorphism and A is a left Γ-module. Then inj. $\dim_\Lambda A \leqq$ inj. $\dim_\Gamma A$, if Γ is right Λ-flat.

1.4 (Gentile [3], p. 431). If H is a subgroup of G then AG is a free right AH-module and hence by 1.3, AH is right self-injective.

1.5. Since each element in a right self-injective ring which is not a zero divisor is invertible, it follows that A which is torsion-free contains the rationals Q as a subring. Hence AG is QG-free and 1.3 yields QG is right self-injective.

1.6 (Villamayor [7], Th. 3, p. 626). QG is semi-simple.

1.7 (Utumi [6], Lemma 8, p. 19). If R is right self-injective then $R/J(R)$ is regular.

1.8 (Kaplansky [5], Lemma 1, p. 85). Each countably generated right ideal in a regular ring is projective.

1.9. If G is infinite locally finite group it must have a countable subgroup say H. Then QH is right self-injective (1.5 and 1.1), regular (1.6 and 1.7) and hence right hereditary (1.8). Since self-injective and hereditary rings are completely reducible, H must be finite, a contradiction. Hence G cannot be infinite.

1.10. AG being a free A-module implies A is also right self-injective.

1.11 (CONNELL [2], Th. 4, p. 663). Conversely if A is right self-injective and G is a finite group then AG is right self-injective.

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