Restricted Regular Rings

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A ring is called restricted regular if each of its proper homomorphic images is regular in the sense of von Neumann. Such rings will be called r-rings. Simple examples exist that an r-ring may not be regular. The object of this article is to characterize right noetherian r-rings. It has been possible to characterize them for the class of non-prime rings. The prime case remains open. However, it can be shown that if R is a prime right noetherian r-ring then each non-trivial ideal is a unique product of maximal ideals. It is also shown that if R is a right duo r-ring then either R is regular or R has exactly one non-trivial ideal. An application to group-rings is given in the last section. Throughout J will denote the Jacobson radical of a ring, and $l(J), r(J)$ respectively will denote the left, right annihilator of J in R. A ring R is called a right duo ring if each of its right ideals is two-sided. We begin with

Theorem 1. *Let R be a nonprime right noetherian ring. Then R is an r-ring if and only if*

(i) *R is semisimple artinian, or*

(ii) *R has exactly one non-trivial ideal, namely the radical J and is isomorphic to an $n \times n$ matrix ring over a local ring, or*

(iii) *R has exactly three non-trivial ideals namely J, $l(J)$, $r(J)$ and is isomorphic to $\begin{pmatrix} U & N \\ O & V \end{pmatrix}$, where U, V are simple artinian and N is irreducible $(U - V)$ bimodule.*

Proof. Suppose $J = 0$. Since R is right noetherian, each ideal is a product of prime ideals (proof similar to commutative noetherian rings ([4], p. 200)). As prime ideals are maximal, we have in particular, $0 = M_1 \ldots M_t$, where M_i are maximal ideals. Now

$$(M_1 \cap M_2 \cap \cdots \cap M_t)^t \subset M_1 M_2 \ldots M_t = 0,$$

implies

$$M_1 \cap M_2 \cap \cdots \cap M_t \subset J = 0.$$

Therefore R is a finite direct sum of simple artinian rings R/M_i. Hence R is artinian, proving (i). So let $J \neq 0$. If $J^2 \neq 0$, then R/J^2 regular implies $J = J^2$, which is not possible in a right noetherian ring. Thus $J^2 = 0$. Hence $l(J) \neq 0$, $r(J) \neq 0$. Since J is the unique minimal ideal, $l(J)$ and $r(J)$ are prime ideals and hence maximal. Also $J \neq 0$ implies R is bound to J ([1], Theorem 6), that is $l(J) \cap r(J) \subset J$. This implies $J = l(J) \cap r(J)$. In case $l(J) = r(J)$, then J is the only

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ideal of R. Hence by ([2], Theorem 1, p. 56), $R = S_n$, where S is local r-ring, proving (ii).

Suppose $l(J) \neq r(J)$. Let M be a maximal ideal of R. Then $l(J) \cdot r(J) \subset l(J) \cap r(J) \subset M$ implies $l(J) = M$ or $r(J) = M$. This shows that $l(J)$ and $r(J)$ are the only maximal ideals. It is clear that in a right noetherian r-ring, each non-zero ideal is a finite intersection of maximal ideals. Hence J, $l(J)$ and $r(J)$ are the only proper ideals in R.

We now proceed to show that R is isomorphic to $\begin{pmatrix} U & N \\ O & V \end{pmatrix}$, where U and V are simple artinian. We have $R/J \cong l(J)/J \oplus r(J)/J$, therefore $\bar{1} = \bar{e} + \bar{f}$ where $\bar{e}$, $\bar{f}$ are central orthogonal idempotents in R/J. We can assume that e and f are orthogonal idempotents in R such that $1 = e + f$. Next from the above decomposition of R/J

$$eR + J = l(J) = Re + J,$$

and

$$fR + J = r(J) = Rf + J,$$

which immediately yields

$$eJ = 0, \quad Jf = 0, \quad eae = ea \quad \text{and} \quad faf = fa \quad \text{for all } a \in R.$$

We define

$$\sigma: R \to \begin{pmatrix} fRf & fRe \\ 0 & eRe \end{pmatrix},$$

by

$$a \to \begin{pmatrix} faf & fae \\ 0 & eae \end{pmatrix}.$$

Then it is easy to check that σ is an isomorphism of R onto S where

$$S = \begin{pmatrix} fRf & fRe \\ 0 & eRe \end{pmatrix}.$$

Now $S/J(S) = fRf \oplus eRe$, which implies that fRf and eRe are semisimple artinian. But since R/J which is isomorphic to $S/J(S)$ has only two ideals, eRe and fRf are simple artinian. Finally it is clear that $N = fRe$ is irreducible $(U - V)$ bimodule where $U = fRf$ and $V = eRe$ are simple artinian. Converse is obvious when R is of type (i) or (ii). So suppose now that $R \cong S$, where

$$S = \begin{pmatrix} U & N \\ O & V \end{pmatrix}$$

and U, V are simple artinian rings and N is irreducible $(U - V)$ bimodule. Since $S/J(S)$ is regular, it is enough to show that each nonzero ideal A of S contains $J(S)$. Let $a \in A$, then

$$a = u e_{11} + x e_{12} + v e_{22}, \quad u \in U, \ v \in V, \ x \in N$$

$$u' e_{11} a v' e_{22} = u' x v' e_{12} \in A, \quad \text{for all } u' \in U, \ v' \in V.$$

Thus $U \times V e_{12} \subset A$. Since N is irreducible $(U - V)$ bimodule, $U \times V = N$. Thus $N e_{12} \subset A$. Hence $J(S) \subset A$, completing the proof of the theorem.

Assume now that R is a prime right noetherian r-ring. Let A be any nonzero ideal in R. Then $A^2 \neq 0$. Since A/A^2 is a nilpotent ideal in the regular ring R/A^2, we must have $A = A^2$. In particular R must be semi-simple. Furthermore since R/A is regular, there exist a finite set of maximal ideals M_i such that $A = M_1 \cap M_2 \cap \cdots \cap M_n$. Then $A = A^2 = (M_1 \cap M_2 \cdots \cap M_n)^2 \subset M_1 M_2 \ldots M_n \subset A$ gives that $A = M_1 M_2 \ldots M_n$. That is if A is also equal to $M_1' M_2' \ldots M_k'$, then $k = n$ and $M_i = M_{\sigma(i)}'$ where $1 \leq i \leq n$ and $\sigma(i)$ is a permutation of $1, 2, \ldots, n$, is clear. This result may be stated in

Proposition. *If R is a prime right noetherian r-ring then R is semi-simple and each non-trivial ideal is a unique product of maximal ideals.*

The next theorem characterizes right duo restricted regular rings without chain condition.

Theorem 2. *A right duo ring R is an r-ring if and only if R is strongly regular or R has exactly one proper ideal.*

Proof. First assume $J = 0$. Then $(x R)^2 \neq 0$, for any nonzero x in R. This implies $R/(x R)^2$ is regular and thus $(x R)^2 = x R$. Therefore $x \in x^2 R$, proving that R is strongly regular. Next let $J \neq 0$. Then J is the minimal ideal and thus $l(J)$, $r(J)$ are prime ideals. If $J^2 \neq 0$, then $J = J^2$ which is not possible. Thus $J^2 = 0$. Hence both $l(J)$ and $r(J)$ are not zero, and being prime ideals they must be maximal. Further R is bound to J ([1], Theorem 6), that is, $l(J) \cap r(J) \subset J$. This gives in our case $l(J) \cap r(J) = J$. Therefore, if $l(J) \neq r(J)$ then $R/J = l(J)/J \oplus r(J)/J$. In particular, $\bar{1} = \bar{e} + \bar{f}$, where $\bar{e}$ and $\bar{f}$ are orthogonal idempotents in R/J. As $J^2 = 0$, idempotents modulo J can be lifted and thus we can assume that e, f are orthogonal idempotents in R such that $1 = e + f$. But then $R = e R \oplus f R$ which yields that R is regular, a contradiction. Hence $l(J) = r(J) = J$, completing the "only if" part. Converse is trivial.

A special case of the above is

Corollary. *A commutative ring is an r-ring iff it is either strongly regular or it has exactly one proper ideal.*

2. In this section we give a characterisation of the group of a right-noetherian group-ring which is restricted regular. Let A be any ring and G a group. Let $R = AG$ be a group-ring and $w G$ the augmentation ideal.

Theorem 3. *Let R be a right noetherian group-ring which is restricted regular. Then either*

(i) *G is finite and $O(G)$ is unit in A, or*

(ii) *G is infinite group and each normal subgroup is of finite index. Further if $Z(G) \neq (e)$, then G is cyclic, or*

(iii) *G is finite simple group.*

Proof. Since $A = R/w G$, A is semi-simple artinian. First let R be non-prime. If $J = 0$, we get (i). Assume $J \neq 0$. Then the only possible non-trivial ideals

are J, $l(J)$ and $r(J)$. We assert that in group-ring R, J is the only non-trivial ideal. Consider the augmentation ideal wG. Then $wG = J$, $r(J)$, or $l(J)$. Since $J^2 = 0$, the right or the left annihilator of wG is not zero. Thus G is finite. Hence R is self-injective which implies that R is quasi-frobenious. But then $l(J) = r(J)$, proving that J is the only non-trivial ideal in R. Next, let H be any normal subgroup of G. Then $wH = J = wG$. Since w is faithful, this implies, G is a simple group. We now consider the case when R is prime. In this case we know that A is prime and G is infinite. Let H be a normal subgroup of G. Then $A\dfrac{G}{H} \cong \dfrac{AG}{wH}$ is semisimple artinian, hence G/H is finite, proving that each normal subgroup is of finite index. As a special case let the centre $Z(G) \neq (e)$, then $G/Z(G)$ finite will give that the derived group G' of G is finite ([3], 15.1.13, p. 453). This implies $G' = (e)$, since G is infinite. Hence G is abelian, and thus must be then cyclic.

Remark 1. If R is a right noetherian ring such that either (i) or (iii) holds then R can be shown to be restricted regular.

Remark 2. Examples can be given to show that there exist infinite groups each of whose normal subgroup is of finite index but the groups have trivial centres.

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