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KD_∞ IS A CS-ALGEBRA

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ABSTRACT. In this paper, it is shown that the group algebra KD_∞ is right CS if and only if $char(K) \neq 2$. Moreover, when $char(K) \neq 2$, then KD_∞ is also CS as a module over its center.

1. INTRODUCTION

Rings whose complement right ideals are direct summands are called right CS-rings. The class of CS-rings includes selfinjective rings, continuous rings etc. and have been of interest to many authors. However, there is hardly any literature on CS-group algebras. It is well known that the group algebra KG, where K is a field, is selfinjective if and only if G is a finite group. But the group algebra KG may be CS without the finiteness condition on the group G. For example if G is a torsion-free solvable-by-finite group, then KG is an Ore domain and hence is a CS-algebra. On the other hand if G is a finite group, then the group ring RG over the ring $R = M_n(\mathbb{Z})$ is not CS for any $n \geq 1$. It is, therefore, of interest to study when a given group algebra is CS. In this paper we study the group algebra $S = KD_\infty$ over a field K for its being CS or not. It is proved that S is a right CS-algebra if and only if $char(K) \neq 2$ (Theorem 3.6). It is further shown that the center $Z(S)$ of S is a Dedekind domain (Lemma 3.8) and that S is also a CS-module over $Z(S)$ (Theorem 3.9).

2. NOTATION AND PRELIMINARIES

Throughout, unless otherwise stated, K will denote a field and D_∞, the infinite dihedral group, that is the group generated by two elements a and b where a is of infinite order, b is of order 2 and $ab = ba^{-1}$. A module will always mean a right unital module. A non-zero submodule N of a module M is said to be essential in M, denoted by $N \subseteq_e M$, if, for every non-zero submodule L of M, $L \cap N \neq 0$. N is called closed in M if N has no proper essential extensions in M. A module M is said to be CS if every nonzero submodule of M is essential in a summand of M, or equivalently, if every closed submodule of M is a summand of M. CS-modules are also commonly known as extending modules ([1]). A ring R is called right CS if it is CS as a right module over itself.

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If A is an algebra over a ring R, then an element $u \in A$ is called integral over R if it satisfies a polynomial equation with coefficient in R and leading coefficient 1. A is called integral if all its elements are integral. coefficients

3. Group ring KD_∞

Throughout this section, S will denote the group algebra KD_∞, and R will denote the group algebra KA where $A = \langle a \rangle$. It is well known that KA is a PID ([5], Exercise 2, p.28). Also, by ([6], Theorem 5.1 and [3], Proposition 9, p.165), S is a prime PI-ring. We begin with some lemmas which will be useful to prove our main result. Our first lemma is well known. We state it here without proof for convenience.

Lemma 3.1 ([1], Corollary 12.8). *For a commutative domain R, the following are equivalent:*

(a) R is a Prüfer domain.

(b) $(R \oplus R)_R$ is a CS-module.

Lemma 3.2. *S is CS as a right R-module.*

Proof. Since $S = R \oplus Rb \simeq R \times R$ and R is a Prüfer domain, the result follows by Lemma 3.1. □

Lemma 3.3. *If $char(K) \neq 2$ and if U is a right ideal of S such that $S_R = U_R \oplus X_R$ for some R-submodule X of the right R-module S_R, then there exists a right ideal V of S such that $S = U \oplus V$.*

Proof. Let π_1, π_2 be the projections of S_R onto U_R and X_R respectively. Define $\gamma : S \to S$ by $\gamma(s) = \frac{1}{2}[\pi_2(s) + \pi_2(sb)b]$. Since $char(K) \neq 2$, γ is well-defined. Clearly, $\gamma(s_1 + s_2) = \gamma(s_1) + \gamma(s_2)$. Also,

$$\begin{aligned}
\gamma(sa) &= \frac{1}{2}[\pi_2(sa) + \pi_2(sab)b] \\
&= \frac{1}{2}[\pi_2(s)a + \pi_2(sba^{-1})b] \\
&= \frac{1}{2}[\pi_2(s)a + \pi_2(sb)a^{-1}b] \\
&= \frac{1}{2}[\pi_2(s)a + \pi_2(sb)ba] \\
&= \gamma(s)a.
\end{aligned}$$

Similarly, $\gamma(sb) = \gamma(s)b$. Thus $\gamma \in Hom_S(S, S)$. Let $V = \gamma(S)$. Then V is a right ideal of S. We will prove that $S = U \oplus V$. So, let $s \in S$. Write $s = (s - \gamma(s)) + \gamma(s)$. Since

$$\begin{aligned}
s - \gamma(s) &= s - \frac{1}{2}[\pi_2(s) + \pi_2(sb)b] \\
&= \frac{1}{2}[(s - \pi_2(s)) + (s - \pi_2(sb)b)] \\
&= \frac{1}{2}[(s - \pi_2(s)) + (sb - \pi_2(sb))b] \\
&= \frac{1}{2}[\pi_1(s) + \pi_1(sb)b],
\end{aligned}$$

and U is a right ideal of S, we have $s - \gamma(s) \in U$. Thus $S = U + V$. Also since for every $s \in S$, $s - \gamma(s) \in U$, we have $\gamma(s - \gamma(s)) = 0$, that is, $\gamma(s) = \gamma^2(s)$ for every $s \in S$.

To prove $U \cap V = (0)$, let $x \in U \cap V$. Then $x = \gamma(s)$ for some $s \in S$ and $\gamma(x) = 0$. Thus $\gamma^2(s) = 0$ and consequently, $x = \gamma(s) = \gamma^2(s) = 0$, as desired. □

Lemma 3.4. *If U is a closed right ideal of S, then U is a closed submodule of the right R-module S_R.*

Proof. Suppose $x \in cl(U_R)$, the closure of U_R in S_R. Then $x \in S$ and $xE \subset U$ for some essential right ideal E of R. Consequently, $x(ES) \subset US \subset U$. Also ES is an essential right ideal of S ([5], Exercise 27, p. 467). Thus $x \in cl(U_S) = U$, because U is a closed right ideal of S. This completes the proof. □

Lemma 3.5. *If $char(K) = 2$, then S has no nontrivial idempotents.*

Proof. For $\alpha = \sum k_i a^i \in R$, let $\alpha^* = \sum k_i a^{-i}$. Since $ab = ba^{-1}$, it follows that $\alpha b = b\alpha^*$ for every $\alpha \in R$. Now if $\alpha + b\beta \in S$ is a nontrivial idempotent in S, then using $\alpha b = b\alpha^*$ and $(\alpha + b\beta)^2 = \alpha + b\beta$ we get $\alpha^2 + \beta^*\beta + b(\alpha^*\beta + \beta\alpha) = \alpha + b\beta$. Thus $\alpha^2 + \beta^*\beta = \alpha$ and $\alpha^*\beta + \beta\alpha = \beta$. Since R is a PID and $\alpha + b\beta$ is an idempotent in S, $\beta \neq 0$. Consequently, using R is a domain, the relation $\alpha^*\beta + \beta\alpha = \beta$ yields $\alpha^* + \alpha = 1$. Since $\alpha \in R$, $\alpha = \sum k_i a^i$ where $k_i \in K$. Since $\alpha^* + \alpha = 1$ and $char(K) = 2$, we have $0 = 1$, a contradiction.Thus S has no nontrivial idempotents. □

Theorem 3.6. *S is a right CS-ring if and only if $char(K) \neq 2$.*

Proof. First assume that $char(K) \neq 2$. Let U be a closed right ideal of S. By Lemma 3.4, U_R is a closed submodule of the right R-module S_R. Since S_R is CS (Lemma 3.2), U_R is a direct summand of S_R. But then by Lemma 3.3, U is a summand of S. Hence S is a right CS-ring. Conversely, let S be right CS. If possible, let $char(K) = 2$. By Lemma 3.5, S has no nontrivial idempotents. Since S is right CS, every nonzero right ideal of S is essential in S. Thus S and hence the right maximal quotient ring $Q^r_{max}(S)$ of S is uniform. Since S is right nonsingular, it follows that $Q^r_{max}(S)$ is a division ring. Hence S is a domain, a contradiction because $(1 + b)^2 = 0$ and $1 + b \neq 0$. Thus $char(K) \neq 2$. □

In what follows $Z(S)$ will denote the center of the ring S. Unless otherwise stated $char(K) \neq 2$ and $e = \frac{1}{2} + \frac{1}{2}b$. Notice that e is an idempotent in S. For $\alpha = \sum k_i a^i$, we will write $\alpha^* = \sum k_i a^{-i}$. In the following lemma we determine $Z(S)$.

Lemma 3.7. *For any field K, $Z(S) = \{\alpha \in R \mid \alpha = \alpha^*\}$.*

Proof. Clearly, $\{\alpha \in R \mid \alpha = \alpha^*\} \subset Z(S)$. To prove the reverse inclusion, let $s = \alpha + \beta b \in Z(S)$. Then $sx = xs$ for every $x \in S$. In particular, $sa = as$ and $sb = bs$. Now $sa = as$ gives $\alpha a + \beta ba = a\alpha + a\beta b$. Since $\alpha a = a\alpha$, we get $\beta a^{-1} b = a\beta b$, that is, $(a^2 - 1)\beta = 0$. Thus, $\beta = 0$. Consequently, $s = \alpha \in R$. Again as $sb = bs$, we have $\alpha b = b\alpha = \alpha^* b$. Thus $\alpha = \alpha^*$ and the proof is complete. □

Lemma 3.8. *$Z(S)$ is a Dedekind domain.*

Proof. Clearly, $Z(S) \simeq eZ(S) = eSe$. Since S is right noetherian, eSe is right noetherian ([7], Lemma 2.7.12). Thus $Z(S)$ is right noetherian. Since S is a prime

$\alpha, \beta \in Z(S)$, $\alpha = \alpha^*$, $\beta = \beta^*$. Hence $\alpha = \alpha^* = (\gamma\beta)^* = \beta^*\gamma^* = \beta\gamma^*$ so that $\gamma^* = \alpha\beta^{-1} = \gamma$. Since $\gamma \in R$, we have $\gamma \in Z(S)$. Hence $R \cap L = Z(S)$. Since R is a Dedekind domain, it follows, by ([8], Theorem 20, p.283), that $Z(S)$ is a Dedekind domain □

Theorem 3.9. *R and S are CS as right $Z(S)$-modules.*

Proof. Clearly, $Z(S) \cap aZ(S) = (0)$. Further, since $a^{-n} = (a^n + a^{-n}) - a^n$ and $a^n = a^{n-1}(a + a^{-1}) - a^{n-2}$ we have $R = Z(S) \oplus aZ(S) \simeq Z(S) \times Z(S)$. Also as $S = R \oplus bR$, we have $S \simeq Z(S) \times Z(S) \times Z(S) \times Z(S)$. By Theorem 3.8, $Z(S)$ is a Dedekind domain. The result now follows by Lemma 3.1. □

Remark 1. The uniform dimension of R as a $Z(S)$-module is 2 and that of S as a $Z(S)$-module is 4.

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