Nonnegative Generalized Inverses of Powers of Nonnegative Matrices*

A. Berman
Technion — Israel Institute of Technology
Haifa 32000, Israel

and

S. K. Jain
Ohio Univerity
Athens, Ohio 45701

Submitted by Richard A. Brualdi

ABSTRACT

We study the nonnegativity of the Moore-Penrose inverse of the powers as well as the product of nonnegative matrices.

1. INTRODUCTION

Let A be an $m\times n$ real matrix. Consider the equations: (1) $AXA=A$, (2) $XAX=X$, (3) $(AX)^T=AX$, (4) $(XA)^T=XA$, and (5) $AX=XA$, where X is an $n\times m$ real matrix and T denotes the transpose. For any nonempty subset λ of $\{1,2,3,4,5\}$, X is called a λ-inverse of A if it satisfies equations (i) for all $i\in\lambda$. Let $A^{(\lambda)}$ denote a λ-inverse of A. A matrix is called λ-monotone if it has a nonnegative λ-inverse. If $\lambda=\{1,2,3,4\}$, then $A^{(\lambda)}$ is unique and is called the Moore-Penrose inverse of A, denoted by $A^{\dagger}$. A $\{1,2\}$-inverse of A

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which satisfies (5) is necessarily square and is called a group inverse of A. The group inverse of a matrix A, if it exists, is unique and is denoted by $A^{\#}$. Recall that $A^{\#}$ exists if and only if index $A=1$, that is, rank $A=$ rank A^2. More generally, let A be any square matrix of index k, that is, rank $A^{k-1}<$ rank $A^k=$ rank A^{k+1}. Then X is called the Drazin inverse of A if X satisfies (2) and (5) and also $A^{k+1}X=A^k$. The Drazin inverse of any square matrix always exists and is unique. It is denoted by $A^{(d)}$. Note that for index $A=1$, $A^{(d)}=A^{\#}$. A comprehensive discussion of the theory of generalized inverses {λ-inverses and Drazin inverses) is given, for example, in [1]. Throughout this paper $A\geqslant 0$ will mean that A is nonnegative, i.e., all the entries of A are nonnegative. $A\ngeqslant 0$ will denote that the real matrix A is not nonnegative. As usual, Z^+ will denote the set of positive integers.

In Section 2 of this paper the nonnegativity of the Moore-Penrose inverse of lower powers of a matrix is proved if a "high enough" power of the matrix possesses a nonnegative Moore-Penrose inverse. More precisely, we consider this "going down" property and prove that if A is a nonnegative matrix with index k and if $(A^m)^{\dagger}\geqslant 0$, $m\geqslant k$, then $(A^l)^{\dagger}\geqslant 0$, $k\leqslant l\leqslant m$ (Theorem 2.4). In Section 3 we consider the "going up" property and prove that if $A\geqslant 0$, index $A=k$, and $(A^k)^{\dagger}\geqslant 0$, then $(A^l)^{\dagger}\geqslant 0$ for every $l\geqslant k$ if and only if $A^{(d)}\geqslant 0$; moreover, if $(A^m)^{\dagger}\geqslant 0$ for some $m>k$, then $(A^l)^{\dagger}\geqslant 0$ for every $l\geqslant k$ (Theorem 3.7). Our main results depend upon a series of lemmas which are also of independent interest (see, e.g., Lemmas 3.3 and 3.4). The last section deals with the question of the nonnegativity of the Moore-Penrose inverse of the product of two nonnegative matrices.

2. GOING DOWN

We begin with the following well-known lemma.

LEMMA 2.1.

(i) *If A and B are matrices such that AB is defined and* $\operatorname{rank}(AB)=\operatorname{rank}A$, *then $B(AB)^{(1,3)}$ is a* {1,3}*-inverse of A.*

(ii) *If A and B are matrices such that BA is defined and* $\operatorname{rank}(BA)=\operatorname{rank}A$, *then $(BA)^{(1,4)}B$ is a* {1,4}*-inverse of A.*

Proof. (i): By the rank condition, $A=ABX$ for some X. Thus $(AB)(AB)^{(1,3)}AB=AB \Leftrightarrow AB(AB)^{(1,3)}A=A$, proving that $B(AB)^{(1,3)}$ is a nonnegative {1,3}-inverse of A. The proof of (ii) is similar. ∎

As an immediate consequence of the above lemma we have

LEMMA 2.2. *If A and B are matrices such that both AB and BA are defined and* $\operatorname{rank}(AB) = \operatorname{rank} A = \operatorname{rank}(BA)$, *then*

$$A^{\dagger} = (BA)^{(1,4)}BAB(AB)^{(1,3)}.$$

In particular,

$$A^{\dagger} = (BA)^{\dagger}BAB(AB)^{\dagger}.$$

Proof. The proof follows from the formula $A^{\dagger} = A^{(1,4)}AA^{(1,3)}$. ■

A special case of Lemma 2.2 is

LEMMA 2.3. *If A is a square matrix of index k, then*

$$(A^{l})^{\dagger} = (A^{m})^{\dagger}A^{2m-l}(A^{m})^{\dagger}, \qquad k \leqslant l \leqslant m.$$

Proof. Since $\operatorname{rank} A^{k} = \operatorname{rank} A^{l} = \operatorname{rank} A^{m} = \operatorname{rank}(A^{l}A^{m-l})$, we replace A by A^{l} and B by A^{m-l} in Lemma 2.2 and obtain the desired result. ■

We are now ready to prove

THEOREM 2.4. *Let A be a nonnegative square matrix such that* index $A = k$, *and let* $(A^{m})^{\dagger} \geqslant 0$ *for some* $m \in Z^{+}$, $m \geqslant k$. *Then* $(A^{l})^{\dagger} \geqslant 0$ *for every* $l \in Z^{+}$, $k \leqslant l \leqslant m$.

Proof. The proof follows from Lemma 2.3. ■

In the special case when index $A = 1$, we get

THEOREM 2.5. *Let A be a nonnegative square matrix such that* index $A = 1$ *and* $(A^{m})^{\dagger} \geqslant 0$ *for some positive integer m. Then* $(A^{l})^{\dagger} \geqslant 0$ *for every positive integer* $l \leqslant m$.

The following example shows that for every $k > 1$ there exists a matrix A of index k such that $(A^{k})^{\dagger} \geqslant 0$ but $(A^{l})^{\dagger}$ is not nonnegative for all l, $1 \leqslant l \leqslant k-1$. First recall that for a nonnegative matrix A, $A^{\dagger} \geqslant 0$ if and only if there exists a permutation matrix P such that

$$PA = \begin{pmatrix} B_1 \\ \vdots \\ B_r \\ 0 \end{pmatrix},$$

where each B_i has rank 1 and the rows of B_i are orthogonal to the rows of B_j, $i \neq j$ (the zero block may be absent) [5].

Example 2.6. For $k = 2$, we take

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the above characterization, $A^{\dagger}$ is not nonnegative. But A^2 is a nonnegative matrix of rank 1. Thus $(A^2)^{\dagger} \geqslant 0$.

For $k > 2$, take $A = (a_{ij})$ to be the nilpotent matrix with

$$a_{ij} = \begin{cases} 0 & \text{if} \quad i \geqslant j \\ 1 & \text{if} \quad i < j \end{cases}.$$

The converse of Theorem 2.5, i.e., that $A^{\dagger} \geqslant 0$ implies $(A^l)^{\dagger} \geqslant 0$, $l > 1$, is not necessarily true even when index $A = 1$, as is shown by the following example.

Example 2.7. Consider

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here index $A = 1$, $A^{\dagger} \geqslant 0$, but $(A^2)^{\dagger} \ngeqslant 0$. It can be verified that $A^{\#} \ngeqslant 0$. In the section which follows we shall show that $(A^2)^{\dagger} \geqslant 0$ if and only if $A^{\#} \geqslant 0$.

3. GOING UP

We first prove a series of lemmas. Lemmas 3.3 and 3.4 express respectively the Moore-Penrose inverse of some power of any matrix in terms of the group inverse and vice versa. Recall that if $A = FG$ is a full rank factorization of A then $A^{\dagger} = G^{\dagger}F^{\dagger}$, $F^{\dagger}F = I = GG^{\dagger}$, and furthermore, if A is a square matrix of index 1, then $(GF)^{-1}$ exists. Moreover, it is well known that a nonnegative matrix having a nonnegative $\{1\}$-inverse always possesses a nonnegative full rank factorization (see [3, p. 113]).

LEMMA 3.1. *Let A be a nonnegative matrix which possesses a nonnegative full rank factorization, $A = FG$. Then,*

$$A^{\dagger} \geqslant 0 \quad \textit{iff} \quad F^{\dagger} = (F^TF)^{-1}F^T \geqslant 0 \ \textit{and} \ G^{\dagger} = G^T(GG^T)^{-1} \geqslant 0.$$

Proof. By [1, Theorem 5, p. 23],

$$A^{\dagger} = G^T(F^TFGG^T)^{-1}F^T = G^T(GG^T)^{-1}(F^TF)^{-1}F^T.$$

Thus $A^{\dagger} \geqslant 0 \Rightarrow GA^{\dagger} \geqslant 0$ and so $(F^TF)^{-1}F^T \geqslant 0$, i.e., $F^{\dagger} \geqslant 0$. Similarly, $G^{\dagger} = G^T(GG^T)^{-1} \geqslant 0$. ■

The following sublemma is immediate.

SUBLEMMA 3.2. *Let $A = FG$ be a full rank factorization of a square matrix A, and suppose* index $A = 1$. *Then for any positive integer m*

$$A^m = F(GF)^{m-1}G \quad \textit{and} \quad (A^m)^{\dagger} = G^{\dagger}(GF)^{1-m}F^{\dagger}.$$

Proof. The proof follows from the fact that $F[(GF)^{m-1}G]$ is a full rank factorization of A^m because GF is invertible. ■

LEMMA 3.3. *Let A be a square matrix of index* 1. *Then*

$$(A^m)^{\dagger} = A^{\dagger}A(A^{\#})^{m-1}A^{\dagger}, \qquad m \in Z^+.$$

Proof. By [3, p. 163], $A^{\#} = F(GF)^{-2}G$ and so $(A^{\#})^p = F(GF)^{-(p+1)}G$. But then $A^{\dagger}A(A^{\#})^{m-1}A^{\dagger} = G^{\dagger}F^{\dagger}FGF(GF)^{-m}GG^{\dagger}F^{\dagger} = G^{\dagger}(GF)^{1-m}F^{\dagger}$. This proves, by Sublemma 3.2, that

$$(A^m)^\dagger = A^\dagger A (A^\#)^{m-1} A^\dagger. \qquad \blacksquare$$

LEMMA 3.4. *If A is a square matrix of index* 1, *then*

$$A^\# = \left[A^r (A^s)^\dagger\right]^m A^t$$

for all $r, s, t \in Z^+$ such that $m(s-r) = t+1$. More generally,

$$A^\# = A^{r_1}(A^{s_1})^\dagger A^{r_2}(A^{s_2})^\dagger \cdots A^{r_m}(A^{s_m})^\dagger A^t$$

for all $m, r_i, s_i, t \in Z^+$ such that $\sum_{i=1}^m (s_i - r_i) = t+1$. (Clearly, each $s, s_1, \dots, s_m$ must be greater than 1.)

Proof. Let $A = FG$ be a full rank factorization of A. Therefore, by Sublemma 3.2

$$A^n = F(GF)^{n-1}G$$

and

$$(A^n)^\dagger = G^\dagger (GF)^{1-n} F^+, \qquad n \in Z^+.$$

Thus

$$A^r (A^s)^\dagger = F(GF)^{r-1} G G^\dagger (GF)^{1-s} F^\dagger = F(GF)^{r-s} F^\dagger.$$

This gives

$$\left(A^r (A^s)^\dagger\right)^m = F(GF)^{m(r-s)} F^\dagger,$$

and so

$$\begin{aligned}\left(A^r (A^s)^\dagger\right)^m A^t &= F(GF)^{m(r-s)} F^\dagger F (GF)^{t-1} G \\ &= F(GF)^{m(r-s)+t-1} G = F(GF)^{-2} G,\end{aligned}$$

since $m(s-r)=t+1$. Because $A^{\#}=F(GF)^{-2}G$, this proves

$$A^{\#}=\left(A^{r}(A^{s})^{\dagger}\right)^{m}A^{t} \qquad \text{whenever} \quad m(s-r)=t+1.$$

The proof of the latter statement is exactly similar. ∎

THEOREM 3.5. *Let A be a nonnegative square matrix with* index $A=1$, *and $A^{\dagger}\geqslant 0$. Then the following are equivalent*:

(i) $(A^2)^{\dagger}\geqslant 0$,
(ii) $A^{\#}\geqslant 0$,
(iii) $(A^l)^{\dagger}\geqslant 0$ *for all* $l\in Z^+$.

Proof. (i) ⇒ (ii): Put $s=2$ and $r=1$ in Lemma 3.4 and obtain the desired implication.
(ii) ⇒ (iii): This follows from Lemma 3.3.
(iii) ⇒ (i): This is obvious. ∎

Since $(A^2)^{\dagger}\geqslant 0$ implies $A^{\dagger}\geqslant 0$ when index $A=1$ (Theorem 2.5), Theorem 3.5 can also be restated as

THEOREM 3.5′. *Let A be a nonnegative square matrix with index $A=1$. Then the following statements are equivalent*:

(i) $(A^2)^{\dagger}\geqslant 0$,
(ii) $A^{\#}\geqslant 0$ *and* $A^{\dagger}\geqslant 0$,
(iii) $(A^l)^{\dagger}\geqslant 0$ *for all* $l\in Z^+$.

Combining Theorems 2.5 and 3.5, we get

COROLLARY 3.6. *Let $A\geqslant 0$ and let $A^{\#}\geqslant 0$. If for some fixed $k\in Z^+$ one has $(A^k)^{\dagger}\geqslant 0$, then $(A^l)^{\dagger}\geqslant 0$ for all $l\in Z^+$.*

Examples 2.6 and 2.7 show that the conclusion of Corollary 3.6 need not be true if $A^{\#}$ is not nonnegative.

For nonnegative square matrices A with index $A>1$, Theorem 3.5 has the following extension.

THEOREM 3.7. *Let A be a nonnegative square matrix with* index $A=k$ *and $(A^m)^{\dagger}\geqslant 0$ for some $m\in Z^+$, $m\geqslant k$. Then the following are equivalent*:

(i) $(A^{m+1})^{\dagger}\geqslant 0$,
(ii) $A^{(d)}\geqslant 0$,
(iii) $(A^l)^{\dagger}\geqslant 0$ *for all* $l\in Z^+$, $l\geqslant k$.

Proof. First note that $(A^{(d)})^k = (A^k)^{(d)} = (A^k)^{\#}$ [1, p. 123].

(i) $\Leftrightarrow$ (iii): Let $A^m = FG$ be a nonnegative full rank factorization of A^m. Since index $A = k$ and $m \geqslant k$, $A^{m+1} = (AF)G = F(GA)$ are nonnegative full rank factorizations of A^{m+1}. So by Lemma 3.1, $(AF)^{\dagger} \geqslant 0$ and $(GA)^{\dagger} \geqslant 0$. But then the rank factorization $A^{m+2} = (AF)(GA)$ of A^{m+2} yields $(A^{m+2})^{\dagger} \geqslant 0$. Continuing likewise or by induction on l, it follows that $(A^l)^{\dagger} \geqslant 0$ for all $l \in Z^+$, $l \geqslant m$. Theorem 2.4 then shows that $(A^l)^{\dagger} \geqslant 0$ for all $l \in Z^+$, $l \geqslant k$.

(ii) $\Rightarrow$ (iii): $A^{(d)} \geqslant 0 \Rightarrow (A^m)^{\#} \geqslant 0$, and since $(A^m)^{\dagger} \geqslant 0$, it follows by Theorem 3.5 applied to the matrix A^m (of index 1) that $(A^{2m})^{\dagger} \geqslant 0$. Theorem 2.4 then yields $(A^{m+1})^{\dagger} \geqslant 0$, and so $(A^l)^{\dagger} \geqslant 0$ for all $l \geqslant k$.

(iii) $\Rightarrow$ (ii): By (iii), $(A^k)^{\dagger} \geqslant 0$ and $(A^{2k})^{\dagger} \geqslant 0$, and so by Theorem 3.5, $(A^k)^{\#} = (A^{(d)})^k \geqslant 0$, proving $A^{(d)} \geqslant 0$. ∎

Analogous to Theorem 3.5′, Theorem 3.7 has the following equivalent statement.

THEOREM 3.7′. *Let A be a nonegative square matrix with* index $A = k$. *Then the following are equivalent*:

(i) $(A^m)^{\dagger} \geqslant 0$ *for some fixed* $m \in Z^+$, $m > k$,
(ii) $A^{(d)} \geqslant 0$ *and* $(A^k)^{\dagger} \geqslant 0$,
(iii) $(A^l)^{\dagger} \geqslant 0$ *for all* $l \in Z^+$, $l \geqslant k$.

We remark that if $A \geqslant 0$ and index $A = k$ and if $(A^{2k+1})^{\dagger} \geqslant 0$, then it follows from the well-known formula $A^{(d)} = A^k(A^{2k+1})^{\dagger}A^k$ [1, p. 174] that $A^{(d)} \geqslant 0$. But Theorem 3.7 shows $A^{(d)} \geqslant 0$ under a weaker assumption, namely, $(A^{k+1})^{\dagger} \geqslant 0$.

4. MONOTONICITY OF *AB*

In this section we consider the nonnegativity of the Moore-Penrose inverse of the product of nonnegative matrices, and the nonnegativity of the Moore-Penrose inverse of each of the matrices. Clearly, for nonnegative matrices A, B, $(AB)^{\dagger} \geqslant 0$ if and only if $A^{\dagger} \geqslant 0$ and $B^{\dagger} \geqslant 0$ when the matrices are nonsingular, but this is far from being true in general. However, we have the following

PROPOSITION 4.1. *If A, B are nonnegative matrices such that* rank $A =$ *rank* $AB =$ rank BA, $(AB)^{\dagger} \geqslant 0$, *and* $(BA)^{\dagger} \geqslant 0$, *then* $A^{\dagger} \geqslant 0$.

Proof. The proof follows by Lemma 2.2. ∎

Observe that Theorem 2.4 is a special case of Proposition 4.1. Another special case of Proposition 4.1 is

COROLLARY 4.2. *If A and B are commuting nonnegative matrices such that $(AB)^{\dagger} \geqslant 0$ and* rank AB = rank A, *then $A^{\dagger} \geqslant 0$.*

The following example shows that the hypothesis in Proposition 4.1 cannot be weakened.

EXAMPLE 4.3. *Let*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here $\operatorname{rank}A = \operatorname{rank}AB = \operatorname{rank}BA$ $(= \operatorname{rank}B)$. *Also* $(AB)^{\dagger} \geqslant 0$ *but* $A^{\dagger} \ngeqslant 0$. *Note also that* $(BA)^{\dagger} \ngeqslant 0$.

The dual of Proposition 4.1 is

PROPOSITION 4.4. *If A, B are nonnegative matrices*, rank B = rank AB = rank BA, $(AB)^{\dagger} \geqslant 0$, *and* $(BA)^{\dagger} \geqslant 0$, *then* $B^{\dagger} \geqslant 0$.

We now consider the converse of each of the Propositions 4.1 and 4.4. In other words, if A, B, $A^{\dagger}$, $B^{\dagger}$ are nonnegative matrices satisfying rank conditions, is it true that $(AB)^{\dagger} \geqslant 0$ and $(BA)^{\dagger} \geqslant 0$? It is shown later (Example 4.6) that this does not hold in general. The theorem which follows gives a necessary and sufficient condition for the nonnegativity of both $(AB)^{\dagger}$ and $(BA)^{\dagger}$.

LEMMA 4.5. *Let $A = FG$ and $B = HK$ be full rank factorizations of A and B respectively. Suppose* rank A = rank B = rank AB = rank BA. *Then*

$$(AB)^{\dagger} = K^{T}(KK^{T})^{-1}(GH)^{-1}(F^{T}F)^{-1}F^{T}.$$

In particular, if $A^{\dagger}$, $B^{\dagger}$, and $(AB)^{\dagger} \geqslant 0$ and if $A = FG$ and $B = HK$ are chosen to be nonnegative full rank factorizations, then $(AB)^{\dagger} \geqslant 0$ if and only if $(GH)^{-1} \geqslant 0$.

Proof. Clearly $AB = F(GHK)$ is a full rank factorization of AB. Thus

$$(AB)^{\dagger} = (GHK)^{\dagger}F^{\dagger} = (GHK)^{T}\left[(GHK)(GHK)^{T}\right]^{-1}(F^{T}F)^{-1}F^{T}$$

$$= K^{T}(KK^{T})^{-1}(GH)^{-1}(F^{T}F)^{-1}F^{T} = K^{\dagger}(GH)^{-1}F^{\dagger}.$$

To prove the latter statement, it is clear that if $(AB)^{\dagger} \geqslant 0$ then $(GH)^{-1} \geqslant 0$, because $F \geqslant 0$ and $K \geqslant 0$. Conversely, if $(GH)^{-1} \geqslant 0$ then $(AB)^{\dagger} \geqslant 0$, because $A^{\dagger} \geqslant 0$ and $B^{\dagger} \geqslant 0$ imply $(F^{T}F)^{-1}F^{T} \geqslant 0$ and $K^{T}(KK^{T})^{-1} \geqslant 0$. ∎

THEOREM 4.6. *Let $A, B, A^{\dagger}, B^{\dagger}$ be nonnegative matrices such that* rank A = rank B = rank AB = rank BA. *Then the following are equivalent*:

(i) $(AB)^{\dagger} \geqslant 0$ *and* $(BA)^{\dagger} \geqslant 0$,
(ii) $(AB)^{\#} \geqslant 0$,
(iii) $(BA)^{\#} \geqslant 0$.

Proof. Let $A = FG$ and $B = HK$ be nonnegative full rank factorizations of A and B respectively. Then $AB = F(GHK)$ is a full nonnegative rank factorization of AB. By Lemma 4.5, $(AB)^{\dagger} \geqslant 0$ if and only if $(GH)^{-1} \geqslant 0$. Similarly, $(BA)^{\dagger} \geqslant 0$ if and only if $(KF)^{-1} \geqslant 0$. Furthermore, since GH and KF are nonsingular, it follows that $GHKF$ is nonsingular and so $(AB)^{\#}$ exists. By [3], $(AB)^{\#} \geqslant 0$ if and only if $(GHKF)^{-1} \geqslant 0$, which is equivalent to $(GH)^{-1} \geqslant 0$ and $(KF)^{-1} \geqslant 0$, since $GH \geqslant 0$ and $KF \geqslant 0$. This proves $(AB)^{\dagger} \geqslant 0$ and $(BA)^{\dagger} \geqslant 0$ if and only if $(AB)^{\#} \geqslant 0$. This proves (i) ⇔ (ii). The equivalence of (i) and (iii) follows by interchanging the roles of A and B. ∎

We conclude with the promised example

EXAMPLE 4.6. *Let*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad BA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

So $\operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} AB = \operatorname{rank} BA$, $A^{\dagger} \geqslant 0$, $B^{\dagger} \geqslant 0$, *but* $(AB)^{\dagger} \ngeqslant 0$.

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