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On Weak Monotonicity of the Powers of Nonnegative Matrices

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The purpose of this paper is to study conditions under which the powers of a square matrix A are weak monotone. Necessary and sufficient conditions under which the product of any two weak monotone matrices is weak monotone are also obtained. Several examples are given to illustrate the necessity of the conditions stated.

1. INTRODUCTION

An $m \times n$ *matrix* A is called weak monotone (w.m.) if $Ax \geqslant 0$ implies $x \in R^n_+ + N(A)$. In other words A is weak monotone if and only if the linear system $Ax = b$, $b \geqslant 0$, has a nonnegative solution. The importance of the concept of weak monotonicity can hardly be overemphasized because of its various applications to the problems requiring nonnegative solutions of linear systems. In general, there are no convenient methods to test for weak monotonicity of any matrix. Our results infer the weak monotonicity of a matrix A if some power of A is weak monotone and vice versa. In particular, if some power of a nonnegative matrix A has rank 1, then A must be weak monotone, and for a matrix A with index 1, A^n is weak monotone whenever A is weak monotone provided $AA^{\#} \geqslant 0$.

2. NOTATION AND DEFINITIONS

$R^n_+ \;= \{x \in R^n \mid x \geqslant 0\}$
$R(A) = \{Ax \mid x \in R^n\}$
$N(A) = \{x \in R^n \mid Ax = 0\}$
$A\{1\} = \{X \in R^{n \times m} \mid AXA = A\}$
$A^{(1)}$ = a member of $A\{1\}$
$A^{\#}$ = group inverse of A
$A^{(d)}$ = Drazin inverse of A
A^T = transpose of A
$A \geqslant 0$ means each entry of A is nonnegative.

A singular square matrix A is of index k if k is the smallest positive integer such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$. A nonsingular matrix is said to have index 0. If A is an $m \times n$ matrix of rank r, then $A = FG$, where F and G are respectively $m \times r$ and $r \times n$ matrices of rank r, is called a full rank factorization of A. Such a factorization is called nonnegative if both F and G are nonnegative.

If A, X are $n \times n$ matrices such that (1^k) $A^{k+1}X = A^k$, (2) $XAX = X$, (3) $AX = XA$, where $k = \text{index } A$, then X is called the Drazin inverse of A. The Drazin inverse of a matrix always exists and is unique. It is denoted by $A^{(d)}$. If $k = 1$, then the Drazin inverse $A^{(d)}$ is called the group inverse, denoted by $A^{\#}$. If $X \in A\{1\}$, i.e., X satisfies $AXA = A$, then X is called a $\{1\}$-inverse of A. Furthermore, if $X \geqslant 0$ then A is called $\{1\}$-monotone.

An $n \times n$ matrix A is called monotone if $Ax \geqslant 0$ implies $x \geqslant 0$. It may be remarked that if a nonsingular matrix A is weak monotone, then A is monotone. Equivalently, A is monotone if and only if $A^{-1} \geqslant 0$.

3. PRELIMINARY RESULTS

We begin with a list of some of the simple results referenced throughout the paper.

3.1. PROPOSITION *Let A be any matrix. Under any one of the following conditions, A is weak monotone.*

(i) $A = A^2$
(ii) $A \geqslant 0$ *and* rank $A = 1$
(iii) *A is $\{1\}$-monotone*
(iv) *A has a full rank factorization $A = FG$ where both F and G are weak monotone*
(v) *A^T is weak monotone.*

Proof The proofs of (i)–(iv) follow immediately from the definition of weak monotonicity. The proof of (v) depends upon Farkas' Theorem [3, Theorem 1′] which states that for any $\alpha \in R(A)$, there exists $c \geqslant 0$ such that $Ac = \alpha$ if and only if $\beta\alpha \geqslant 0$ whenever $\beta A \geqslant 0$ for any row vector β. Now, let $\alpha = A^T x \geqslant 0$. If A^T is not weak monotone, then there does not exist any nonnegative γ such that $A^T\gamma = \alpha$. So by Farkas' Theorem, there exists β such that $\beta A^T \geqslant 0$, but $\beta\alpha = \beta A^T x \leqslant 0$. Furthermore, since A is weak monotone and $A\beta^T > 0$, there exists $z > 0$ such that $A\beta^T = Az$. Thus $x^T A\beta^T = x^T Az$, a contradiction since $x^T A \beta^T < 0$ and $x^T Az \geqslant 0$.

The next proposition is a converse of Proposition 3.1(iv) for nonnegative matrices.

3.2. PROPOSITION *Let A be a nonnegative weak monotone matrix. Then there exists a full rank factorization $A = FG$ such that at least one of the factors F or G is weak monotone. In case A has a nonnegative full rank factorization, then both F and G are weak monotone.*

The proof is straightforward.

A consequence of Propositions 3.1 and 3.2 is

3.3. PROPOSITION *Let $A = FG$ be a nonnegative full rank factorization of A. Then A is weak monotone if and only if A is $\{1\}$-monotone.*

Proof By Proposition 3.2, both F and G are weak monotone matrices. We now show that there exists a nonnegative left inverse of F. Let F_L be any left inverse of

F. Write $F_L^T = [x_1 x_2 \cdots x_r]$, where x_i denotes the ith column. Since F^T is w.m. and $F^T x_i \geqslant 0$, there exists $w_i \geqslant 0$ such that $F^T x_i = F^T w_1$. Define $F_L' = [w_1 w_2 \cdots w_r]^T$. Clearly F_L' is a nonnegative left inverse of F. Similarly, one can show that G has a nonnegative right inverse G_R'. Then $G_R' F_L'$ is a nonnegative $\{1\}$-inverse of A. Sufficiency follows from Proposition 3.1.

4. MAIN RESULTS

In this section we study the weak monotonicity of the powers of A.

4.1. LEMMA *Let A be a matrix with index* 1.

(a) *If $A \geqslant 0$ and A^n is w.m. for some $n > 0$, then A is w.m.*
(b) *$AA^\# \geqslant 0$ and A is w.m. then A^n is w.m. for all $n > 0$.*

Proof (a) We may assume $n \geqslant 2$. First we show that if A^n is w.m., then A^{n-1} is also w.m. Now $A^{n-1}x \geqslant 0$ implies $A^n x \geqslant 0$. But then $A^n x = A^n t$, $t \geqslant 0$, since A^n is w.m. This implies $A^\# A^n x = A^\# A^n t$ and so $A^{n-1}x = A^{n-1}t$. Hence if A^n is w.m., then A is w.m.

(b) We first show that for any $y \geqslant 0$, $A^\# y \geqslant 0$ whenever $AA^\# \geqslant 0$ and A is w.m. Assume that $AA^\# \geqslant 0$, $y \geqslant 0$, and A is w.m. Then $AA^\# y \geqslant 0$ so there is some $z \geqslant 0$ such that $AA^\# y = Az$ since A is w.m. Then $A^\# AA^\# y = A^\# Az$ and so $A^\# y = A^\# Az \geqslant 0$. Now we will show that if A^{n-1} is w.m., then A^n is w.m. For this, suppose $A^n x \geqslant 0$. Then $A^n x = A^{n-1}(Ax) = A^{n-1}y$ for some $y \geqslant 0$ since A^{n-1} is w.m. Multiplying by $AA^\#$ yields

$$A^n x = A^n (A^\#) y. \tag{1}$$

This shows that A^n is w.m. since $A^\# y \geqslant 0$.

4.2. *Remark* The following example shows that if the hypothesis "$AA^\# \geqslant 0$" is removed, then A^n need not be w.m. whenever A is w.m.

4.3. *Example* Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

be a nonnegative full rank factorization. Then

$$A^\# = F(GF)^{-2} \cdot G = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad AA^\# = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \ngeqslant 0.$$

It is easy to show that A is w.m. However, A^2 is not w.m. For, if $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, then $A^2x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, but $A^2x = A^2t = \begin{bmatrix} t_1 + t_2 + t_3 \\ t_1 + t_2 + t_3 \\ t_1 + t_3 \end{bmatrix}$ and $t \geqslant 0$ are not consistent.

4.4. *Remark* Lemma 4.1(a) is not necessarily true if index $A > 1$ as shown by the following example.

4.5. *Example* Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here rank $A = 2$, rank $A^2 = \text{rank } A^3 = 1$ and so index $A = 2$. Since A^2 is of rank 1, A^2 is w.m. However, A is not w.m., since for $x = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $Ax \geqslant 0$, but there does not exist any $y \geqslant 0$ such that $Ax = Ay$.

4.6. LEMMA *Let A be a nonnegative weak monotone matrix of index k. If A^{l+1} is weak monotone, then A^l is also monotone for $l \geqslant k$.*

Proof Let $A^l x \geqslant 0$. Then $A^{l+1}x \geqslant 0$. Since A^{l+1} is w.m., there exists $y \geqslant 0$ such that $A^{l+1}x = A^{l+1}y$. Therefore $A^{(d)}A^{l+1}x = A^{(d)}A^{l+1}y$, and so $A^l x = A^l y$.

4.7. LEMMA *Let A be any nonnegative square matrix of index k with $k > 1$ and $AA^{(d)} \geqslant 0$. Then the weak monotonicity of A^l implies that A^{l+1} is weak monotone, for $l \geqslant k$.*

Proof Suppose $A^{l+1}x \geqslant 0$. Since A^l is w.m., there exists $y \geqslant 0$ such that $A^{l+1}x = A^l y$. This implies $A^{l+1}x = A^l A^l (A^l)^{(d)} y$ and so,

$$A^{l+1}x = A^{l+1}((A^l)^{\#} A^{l-1} y). \tag{2}$$

Let $t = (A^l)^{\#} A^{l-1} y$. We will show that $t \geqslant 0$. Clearly $(A^l)^{\#} A^l A^{l-1} y \geqslant o$. Because A^l is w.m., there exists $z \geqslant 0$ such that

$$(A^l)^{\#} A^l A^{l-1} y = A^l z.$$

From the above relation we get

$$\begin{aligned} t = (A^l)^{\#} A^{l-1} y &= (A^l)^{\#} A^l z \\ &= (A^{(d)})^l A^l z = A^{(d)} A z \geqslant 0, \end{aligned}$$

proving $t \geqslant 0$ and $A^{l+1}x = A^{l+1}t$ by (2).

The following lemma is an improvement of Lemma 4.6.

4.8. LEMMA *Let A be a nonnegative matrix of index k ($k > 1$) such that $AA^{(d)} \geqslant 0$. Then whenever A^{k-1} is weak monotone, A^k is also weak monotone.*

Proof Suppose $A^k x \geqslant 0$. This implies $A^k x = A^{k-1} y$, $y \geqslant 0$ and so $A^k x = A^k A^{(d)} y$. Now since $A^{k-1} A^{(d)} y \geqslant 0$ and A^{k-1} is w.m., there exists $z \geqslant 0$ such that $A^{k-1} A^{(d)} y = A^{k-1} z$ and so $A^k A^{(d)} y = A^k z$. Hence $A^k x = A^k z$.

4.9. COROLLARY *Let A be a nonnegative matrix of index k ($k > 1$) such that $AA^{(d)} \geqslant 0$. Then A^l is weak monotone if and only if A^{l+1} is weak monotone for $l \geqslant k$.*

4.10. THEOREM *Let A be a weak monotone matrix such that $AA^{(d)} \geqslant 0$ with index $A = k \geqslant 1$. Then A^k is weak monotone (and hence A^l is weak monotone for all $l \geqslant k$).*

Proof For $k = 1$ the result follows from Lemma 4.1(b), so we assume $k > 1$. Suppose $A^k x \geqslant 0$. Then $A^k x = A y_0$, for some $y_0 \geqslant 0$. This implies $A^k x = AA^{(d)} A y_0 = A^k((A^{(d)})^k A y_0)$. Now, since $AA^{(d)} y_0 \geqslant 0$ and A is w.m., we have $AA^{(d)} y_0 = A y_1$, for some $y_1 \geqslant 0$. This gives $A(A^{(d)})^k y_0 = (A^{(d)})^{k-1} A y_1$. Because $AA^{(d)} y_1 \geqslant 0$ and A is w.m., we obtain $(A^{(d)})^{k-1} A y_1 = (A^{(d)})^{k-2} A y_2$. Continuing this process we get a sequence of nonnegative vectors $y_0, y_1, \ldots, y_{k-1}$ such that $(A^{(d)})^k A y_0 = (A^{(d)})^{k-1} A y_1 = (A^{(d)})^{k-2} A y_2 = \cdots = A^{(d)} A y_{k-1} \geqslant 0$. Hence $A^k x = A^k t$ where $t = (A^{(d)})^k A y_0 \geqslant 0$.

The following example shows that under the hypothesis of Theorem 4.10, A^l need not be weak monotone for $l < k$.

4.11. *Example* Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Then rank $A = 3$, rank $A^2 = 2$, rank $A^3 = 1$, and $A^{(d)} = A^3/16 \geqslant 0$ and so $AA^{(d)} \geqslant 0$. By choosing

$$x = \begin{bmatrix} 0 \\ -1 \\ 1 \\ * \\ * \\ * \end{bmatrix}, \qquad A^2 x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \geqslant 0.$$

However, there does not exist a $t \geqslant 0$ such that $A^2 x = A^2 t$. Hence A^2 is not w.m. Though it is somewhat messy, one can show that A is a weak monotone matrix. Note that A^3 is weak monotone since A^3 is a nonnegative rank 1 matrix.

5. MONOTONICITY OF THE PRODUCT

In this section we study the weak monotonicity of the product AB when A and B are given w.m. matrices. As corollaries to the results in this section we may obtain some of the lemmas discussed in Section 4.

5.1. PROPOSITION *Let A and B be weak monotone matrices such that* $\text{index}(AB) = \text{index}(BA) = 1$, $(AB)(AB)^{\#} \geqslant 0$ *and* $(BA)(BA)^{\#} \geqslant 0$. *Then AB and BA are weak monotone matrices.*

Proof Suppose $ABx \geqslant 0$. Then $ABx = Ay$, for some $y \geqslant 0$. Now since $(BA)(BA)^{\#} y \geqslant 0$ and B is w.m., there exists $z \geqslant 0$ such that $BA(BA)^{\#} y = Bz$ and so $BAy = BABz$. Since $ABx = Ay$, it follows that $BABx = BABz$ and therefore $ABx = ABz$. Thus AB is weak monotone. By switching the roles of A and B we also obtain BA is weak monotone.

5.2. *Remark* One may ask here a natural question: Is the converse of Proposition 5.1 true? In other words, if AB and BA are w.m. matrices and $(AB)(AB)^{\#} \geqslant 0$ and $(BA)(BA)^{\#} \geqslant 0$, is it true that A and B are weak monotone? We answer this question in the negative with the following example.

5.3. *Example* Let A be the matrix as in Example 4.11. Take $U = A^2$ and $V = A^2$. Then $(UV)(UV)^{\#} = (A^4)(A^4)^{\#} = AA^{(d)} \geqslant 0$ and $(VU)(VU)^{\#} \geqslant 0$. Also $UV = VU = A^4$ is w.m.. However, U is not w.m.

The following is a simple fact.

5.4. PROPOSITION *Suppose A and B are weak monotone matrices and assume A has a left inverse (or B has a right inverse). Then AB is weak monotone.*

The following is a generalization of Proposition 5.1.

5.5. THEOREM *Let A and B be weak monotone matrices such that* $(AB)(AB)^{(d)} \geqslant 0$ *and* $(BA)(BA)^{(d)} \geqslant 0$. *Then* $(AB)^k$ *and* $(BA)^l$ *are weak monotone matrices, where* $k = \text{index}(AB)$ *and* $l = \text{index}(BA)$.

Proof Without loss of generality we may assume $k \geqslant l$. Suppose $(AB)^k x \geqslant 0$. Since A is w.m., there exists $y \geqslant 0$ such that

(i) $(AB)^k x = Ay$.

Since $(BA)(BA)^{(d)} y \geqslant 0$ and B is w.m., there exists $z \geqslant 0$ such that

(ii) $(BA)(BA)^{(d)} y = Bz$.

By (i), $(BA)^{(d)} B(AB)^k x = (BA)^{(d)}(BA)y$, and so $(BA)^{(d)}(BA)^k Bx = Bz$. Multiplying both sides by ABA yields $(AB)^{k+1} x = (AB)^2 z$, and so

$$(AB)^k x = (AB)^{(d)}(AB)^2 z = (AB)^k(((AB)^{(d)})^k (AB)z).$$

Let $t = ((AB)^{(d)})^k (AB)z$. We will show $t \geqslant 0$. Let $z_0 = z$. Since $(AB)(AB)^{(d)} z_0 \geqslant 0$ and A is w.m., there exists $z_1 \geqslant 0$ such that $(AB)(AB)^{(d)} z_0 = Az_1$.

Thus we get $(AB)^{(d)} z_0 = (AB)^{(d)} Az_1$. Now since $(BA)(BA)^{(d)} z_1 \geqslant 0$ and B is w.m., there exists $z_2 \geq 0$ such that $(BA)(BA)^{(d)}z_1 = Bz_2$. This implies $A(BA)^k z_1 = A(BA)^k Bz_2 = (AB)^{k+1}z_2$, that is, $(AB)^k Az_1 = (AB)^{k+1}z_2$. Multiplying by $((AB)^{(d)})^{k+1}$ we obtain $(AB)^{(d)}Az_1 = (AB)^{(d)}(AB)z_2$.

Next by starting with z_2 and using the same procedure, we obtain nonnegative vectors z_3 and z_4 such that $(AB)^{(d)}z_2 = (AB)^{(d)}Az_3$ and $(AB)^{(d)}Az_3 = (AB)^{(d)}(AB)z_4$. Thus we get a sequence of nonnegative vectors $z_1, z_2, z_3, \ldots$ such that

$$\begin{array}{ll}
(AB)^{(d)}z_0 = (AB)^{(d)}Az_1, & (AB)^{(d)}Az_1 = (AB)^{(d)}(AB)z_2; \\
(AB)^{(d)}z_2 = (AB)^{(d)}Az_3, & (AB)^{(d)}Az_3 = (AB)^{(d)}(AB)z_4; \\
(AB)^{(d)}z_4 = (AB)^{(d)}Az_5, & (AB)^{(d)}Az_5 = (AB)^{(d)}(AB)z_6; \\
\vdots & \vdots \\
(AB)^{(d)}z_{i-1} = (AB)^{(d)}Az_i, & (AB)^{(d)}Az_i = (AB)^{(d)}(AB)z_{i+1}; \\
\cdots\cdots\cdots & \cdots\cdots\cdots
\end{array}$$

Therefore,

$$(AB)^{(d)}z_0 = (AB)^{(d)}(AB)z_2 = (AB)^{(d)}(AB)^2 z_4 = \cdots = (AB)^{(d)}(AB)^s z_{2s} \qquad (s \geq 1).$$

This implies $((AB)^{(d)})^k(AB)z_0 = ((AB)^d)^k(AB)^{s+1}z_{2s}$ $(s \geq 1)$. In particular when $s = k-1$,

$$t = ((AB)^{(d)})^k(AB)z_0 = ((AB)^{(d)})^k(AB)^k z_{2(k-1)}$$
$$= (AB)^{(d)}(AB)z_{2(k-1)} \geq 0.$$

Hence $(AB)^k x = (AB)^k t$ where $t \geq 0$, and so $(AB)^k$ is w.m. Similarly, $(BA)^l$ is w.m.

5.6. *Remark* Even though Theorem 4.10 is a corollary of Theorem 5.5, we chose to include the proof of Theorem 4.10 because it is somewhat less involved. The following example shows that with the hypothesis in Theorem 5.6, $(AB)^l$ may not be w.m. if $l < k$.

5.7. *Example* Consider the matrix A in Example 4.11. Let $B = A$ in Theorem 5.6. Then A and B are w.m., $k = \text{index}(AB) = \text{index}(BA) = \text{index}(A^2) = 2$, and $(AB)(AB)^{(d)} = (A^2)(A^2)^{(d)} = AA^{(d)} \geq 0$. Similarly, $(BA)(BA)^{(d)} \geq 0$. However, $(AB)^{k-1} = (BA)^{k-1} = A^2$ is not w.m.

5.8. *Remark* Let b be a nonnegative vector. Call a matrix A to be b-weak monotone if $Ax \geq b$ implies $Ax = Ay$ for some $y \geq 0$. Clearly, weak monotonicity is the same as 0-weak monotonicity and if A is weak monotone, then A is b-weak monotone for all $b \geq 0$. It is also interesting to note that if A is b-weak monotone, then for any $\varepsilon > 0$, however small, A is also εb-weak monotone. However, for $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, it is trivial that A is (vacuously) b-weak monotone, but A is not weak monotone. Thus one may ask a natural question: Under what conditions on A or

b, b-weak monotonicity of A implies the weak monotonicity of A? In this connection it can be shown that if $b \in R(A)$, $b \geqslant 0$, then A is weak monotone whenever A is b-weak monotone. It may be of interest to investigate other conditions or special types of matrices for which b-weak monotonicity implies weak monotonicity.

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