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RINGS CHARACTERIZED BY DIRECT SUMS OF CS MODULES

Dinh Van Huynh,
Institute of Mathematics
P.O.Box 631, Boho
Hanoi, Vietnam

S. K. Jain and S. R. López-Permouth
Department of Mathematics
Ohio University
Athens, OH 45701, USA

ABSTRACT. It is shown that a ring for which every CS right module is Σ-CS is right artinian. As a consequence, it is also shown that over a ring R every direct sum of CS right R-modules is CS iff R is right artinian and the composition length of every uniform right R-module is at most 2.

We consider associative rings and all modules are unitary modules. A module M is called a CS module if every submodule of M is essential in a direct summand of M. CS modules are a generalization of injective modules. As is well-known, a ring R is right noetherian iff every direct sum of injective right R-modules is injective (Matlis-Papp Theorem, see [2]). In this note we investigate the structure of rings over which direct sums of CS right modules are CS. A module M is called a (countably) Σ-CS module if the direct sum of (countably many) arbitrary many copies of M is CS. For more details about CS modules we refer to [1].

Recently, J.L. Gómez Pardo and P.A. Guil Asensio [3] have obtained an interesting result which says that any Σ-CS module is a direct sum of uniform modules. As a consequence of this they have shown that a ring R is right noetherian if every CS right R-module is Σ-CS. We will improve the latter result by showing in Theorem 2 that such a ring is actually right artinian.

It was shown in [5] that if R is a right nonsingular ring, then R is right artinian and every uniform right R-module has composition length at most 2 if and only if every direct sum of CS right R-modules is CS. In this note we will show that the result holds even if one removes the condition of right nonsingularity (Corollary 3).

We start with the following lemma.

Lemma 1. *Let R be a semiprime right Goldie ring. If every uniform right ideal of R is countably Σ-CS, then R is semisimple artinian.*

Proof. Let $E = E(R_R)$ be the injective hull of R_R. Then $E = E_1 \oplus \cdots \oplus E_n$ where each E_i is uniform and injective. We claim that $E = R$, and therefore R is semisimple artinian. Assume, on the contrary, that $E \neq R$. Let S be the right socle of R. As R is semiprime and right Goldie, it is known that S is a ring direct summand of R (see for example Theorem 6.15 on page 106 of [4]). Write $R = S \oplus T$. Clearly S is a semisimple artinian ring and T has a zero right socle. Hence without loss of generality we may assume that $\mathrm{Soc}(R_R) = 0$. Now, there is at least one E_i, say E_1, which is not contained in R. Let $U = R \cap E_1$. Then U is uniform, $Soc(U) = 0$ and U_R is not injective . By hypothesis, U_R is countably Σ-CS, i.e. $U_R^{(\mathbb{N})}$ is CS.

As R is semiprime right Goldie, we can use the known fact that every uniform right ideal L of R with $\mathrm{Soc}(L_R) = 0$ is isomorphic to a proper submodule of L_R (see for example Corollary 3.3.3 on page 73 of [6]). Hence U is isomorphic to a proper submodule V of U. Let f_1 be an isomorphism $V \to U$. Then f_1 can be extended to a homomorphism g_1 of U to E_1. Set $U_1 = g_1(U)$. Since U is uniform, we must have $\mathrm{Ker}(g_1) = 0$, therefore $U_1 \cong U$. Moreover $U_1 = g_1(U) \supset g_1(V) = f_1(V) = U$, i.e. U_1 contains U properly. Next, because $V \cong U \cong U_1$, there is an isomorphism f_2 of V onto U_1. Let g_2 be an extension of f_2 to a monomorphism of U_1 into E_1. Then $U_2 = g_2(U_1) \cong U_1 \cong U$, and U_2 contains U_1 properly. Since U_R is not injective, $U_2 \neq E_1$. In this way, by induction, we produce an infinite ascending chain of submodules:

$$U_1 \subset U_2 \subset \cdots \subset U_i \subset \cdots, \tag{1}$$

where $U_i \cong U$ for all i. As $U^{(\mathbb{N})}$ is CS, it follows that $T = \oplus_{i=1}^{\infty} U_i$ is also CS. Note that T is a nonsingular right R-module. Now we use a standard argument to derive a contradiction.

Set $T' = \cup_{i=1}^{\infty} U_i$. Then T'_R is a nonsingular module. Let φ be the epimorphism from $T = \oplus_{i=1}^{\infty} U_i$ onto T'

via $\varphi(x_1, x_2, \cdots, x_m, 0, 0, ...) = x_1 + x_2 + \cdots + x_m \in U_m$ for all $m \in \mathbb{N}$. Then for each $n \in \mathbb{N}$,

$$\varphi(\oplus_{i=1}^n U_i) = \varphi(U_n) = U_n.$$

We have $T' \cong T/Ker(\varphi)$. As T' and T are both nonsingular right R-modules, Ker (φ) is closed in T and hence it is a summand of T. Set $T = Ker(\varphi) \oplus W$ where $W \cong T'$. Then W is also uniform and nonsingular. There exists a $l \in \mathbb{N}$ such that $W_1 = W \cap T(l) \neq 0$, and so W_1 is essential in W, where $T(l) = U_1 \oplus \cdots \oplus U_l$. On the other hand, since $T(l)$ is CS, W_1 is also essential in a direct summand W_1^* of $T(l)$. The nonsingularity of T then implies that $W = W_1^* \subseteq T(l)$. It follows that $T' = \varphi(T) = \varphi(W) \subseteq \varphi(T(l)) = U_l$, a contradiction to (1). Thus $E = R$. □

We remark that for the ring $\mathbb{Z}$, every ideal U of $\mathbb{Z}$ is *finitely* Σ-CS, that is, U^n is CS for any $n \in \mathbb{N}$. But $\mathbb{Z}$ is not artinian. Hence the condition in Lemma 1 can not be weakened.

Theorem 2. *If R is a ring such that every CS right R-module is Σ-CS, then R is right artinian.*

Proof. By [3], R is right noetherian. Let N be the prime radical of R. Then every uniform right ideal of R/N is Σ-CS as a right R-module. Consequently, it is also Σ-CS as a right R/N-module. By Lemma 1, R/N is semisimple artinian. Since N is nilpotent, we conclude that R is right artinian. □

Notice that the converse of Theorem 3 is not true in general. As an example, let $R = \begin{bmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$. Then R is right CS, right and left artinian. However $(R \oplus R)_R$ is not CS.

As an application we can prove the following

Corollary 3. *For a ring R the following conditions are equivalent:*
(a) R is right artinian and every uniform right R-module has composition length at most 2.
(b) Every direct sum of CS right R-modules is CS.

Proof. $(a) \Rightarrow (b)$. Since R is right artinian, every CS right R-module is a direct sum of uniform right R-modules (cf. [7]). Let A be a direct sum of CS right R-modules. It follows that $A = \oplus_{i \in I} A_i$, where each A_i is uniform. By (a), $l(A_i) \leq 2$ for every $i \in I$. If $l(A_j) = 2$ then $A_j = E(A_j)$ since the length of $E(E_j)$ must be also 2, where $E(A_j)$ is the injective hull of A_j. Therefore, A_j is injective whenever $l(A_j) = 2$. Hence A_R is CS by [1, Lemma 8.14], proving (b).

$(b) \Rightarrow (a)$. (b) implies that every CS right R-module is Σ-CS. Hence R is right artinian by Theorem 2. In addition, by [5, Lemma 12], every uniform right R-module has composition length at most 2. □

We conclude by mentioning that it was shown in [4] that over the ring $R = \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{bmatrix}$ every direct sum of CS right R-modules is CS, but R itself is not right CS.

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