Advances in Algebra
World Scientific Publishing Co
401-408 (2003)

2000-089

SURVEY OF SOME RECENT RESULTS ON CS-GROUP ALGEBRAS AND OPEN QUESTIONS

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1. Introduction

Let K be a ring and G be a group. Let $R = KG$ be the group ring of the group G. We shall call R to be a group algebra if K is a field. It is well known that R is right selfinjective if and only if K is a right selfinjective ring and G is a finite group. For any ring R, we know that if M is an injective R-module then each essentially closed submodule is a direct summand of M. This property is known as property (C_1) in the literature and the modules satisfying this property are known as CS-modules or extending modules.

The subject of CS-modules has been of considerable interest to many authors and a number of papers have been written. (See for example [5], [6], [7], [8], [10], and so on.) The question of CS-group algebras was initiated in 2000 by Jain et. al. in [8] where the group algebra of infinite dihedral group was considered. There is very little known about CS-group algebras, in general and several interesting questions are open. We will give a survey of recent results in the area and state some of the open problems.

2. Preliminaries

Throughout, unless otherwise stated, K will denote a field and $R = KG$, the group algebra of the group G over K. The augmentation ideal $\omega(R)$ of R is defined to be the ideal generated by $\{1 - g \mid g \in G\}$. It is known that $\omega(R)$ is a maximal ideal of R. R is a prime ring if and only if G has no finite normal subgroups. R is a semiprime ring if and only if either characteristic of K is 0 or if characteristic of K is p then G has no finite normal subgroup whose order is divisible by p. If R is local then characteristic of K is p and G is a p-group ([11], Lemma

[1]Plenary talk at ICM Satellite Conference in Algebra, August 2002.

2000 AMS Subject Classification. *Primary:*16S34, 16R20; *Secondary:*16L30, 16E60.

Key words and phrases. CS group rings, finitely $\sum$-CS group rings, prime rings, semiprime rings, polycyclic-by-finite groups, locally finite groups.

1.13, p415). R is artinian (even perfect) if and only if G is finite. The characterization of semiperfect or semilocal group algebras is not easy and is known only in some special cases.

A group G is called polycyclic if G has a finite subnormal series

$$(e) = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that each quotient G_{i+1}/G_i is cyclic. If G_{i+1}/G_i is either cyclic or finite then G is called polycyclic-by-finite. For example, the infinite dihedral group

$$D_\infty = \{\langle a, b\rangle \mid o(a) \text{ is infinite, } o(b) = 2,\ ab = ba^{-1}\}$$

is a polycyclic-by-finite group as

$$(e) \lhd \langle a\rangle \lhd D_\infty.$$

G is called locally finite if every finitely generated subgroup of G is finite. G is called an FC-group if every element of G has finitely many conjugates.

For a group G, $G^{'}$ will denote its commutator subgroup, $Z(G)$, its center and G^+, the set of its torsion elements. We will denote by $\Delta(G)$, the subgroup of all those elements of G that have finitely many conjugates, that is,

$$\Delta(G) = \{x \in G \mid [G : C_G(x)] < \infty\}.$$

Observe that G is an FC-group if and only if $G = \Delta(G)$. $\Delta^+(G)$ will denote those elements of $\Delta(G)$ that are of finite order, that is,

$$\Delta^+(G) = \{x \in G \mid [G : C_G(x)] < \infty \text{ and } o(x) < \infty\}.$$

For any two R-modules M and N, M is said to be N-injective if for any R-homomorphism $\phi : N \to E(M)$, where $E(M)$ is the injective hull of M, $\phi(N) \subset M$. M is said be injective if M is N-injective for all right R-modules N. A submodule K of a right R-module M is said to be essential in M, denoted by $K \subset_e M$, if for any nonzero submodule L of M, $K \cap L \neq 0$. M is called a CS (or extending) module if every submodule of M is essential in a direct summand of M, equivalently, if every essentially closed submodule of M is a direct summand of M. M is called finitely $\sum$-CS if direct sum of finite number of copies of M is CS. M is called CS with respect to uniform submodules if every uniform submodule of M is essential in a direct summand of M, equivalently, if every uniform closed submodule of M is a direct summand of M. M is said to satisfy condition (C_3) if for any two summands M_1 and M_2 of M with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is also a summand of M. A CS module is called quasi-continuous if it satisfies (C_3). It is known that if $M \times N$ is quasi-continuous then M and N are injective relative to each other.

A ring R is said to be right CS (or CS with respect to uniform right ideals) if the right R-module R_R is CS (resp. CS with respect to uniform R-submodules). R is called right selfinjective if R_R is injective. R is called right PP if every principal right ideal of R is projective. It is well known that all right PP rings are right nonsingular and that every right nonsingular right CS ring is right PP. For a ring R, $J(R)$ will denote the Jacobson radical of R and $Z_r(R)$, the right singular ideal $\{r \in R \mid rI = 0$ for some essential right ideal I of $R\}$ of R. For an element a of a ring R, $r.ann_R(a)$ will denote the right annihilator of a in R.

3. Prime and Semiprime Group Algebras

We begin this section with a general result on nonsingular CS-group rings.

Proposition 3.1. *Let R be a ring with no nontrivial idempotents and G any group such that RG is nonsingular and CS. Then for every finite subgroup H of G, $o(H)$ is invertible in R. In particular, order of every torsion element in G is invertible in R.*

Proof. It is sufficient to prove the result for finite cyclic subgroups of G. Let $H = \langle h \rangle$ be a finite cyclic subgroup of G and let $o(H) = n$. Then $\omega(RH) = (H-1)RH = \sum_{i=1}^{n-1}(h^i - 1)RH$. Thus $r.ann_{RH}(\omega(RH)) = r.ann_{RH}((h-1)RH) = r.ann_{RH}(h-1) = (\sum_{i=0}^{n-1} h^i)RH = (\sum_{i=0}^{n-1} h^i)R$. Since RG is nonsingular and CS, RG is PP. Consequently RH is PP. Thus $r.ann_{RH}(h-1)$ is generated by an idempotent. Hence there exist an idempotent e in RH such that $(\sum_{i=0}^{n-1} h^i)R = eRH$. Let $e = (\sum_{i=0}^{n-1} h^i)r$. Then $e^2 = (\sum_{i=0}^{n-1} h^i)^2 r^2 = n(\sum_{i=0}^{n-1} h^i)r^2$. Since e is an idempotent, $n(\sum_{i=0}^{n-1} h^i)r^2 = (\sum_{i=0}^{n-1} h^i)r$. It follows that $nr^2 = r$. If $nr = 0$ then $r = nr^2 = 0$. Thus $e = 0$, a contradiction as $r.ann_{RH}(\omega(RH)) \neq 0$. Thus $nr \neq 0$. Also $n^2r^2 = nr$, that is, nr is an idempotent. Since R has no nontrivial idempotents, $nr = 1$, as desired. ∎

We now consider prime and semiprime noetherian group algebras which are CS. We state the following result.

Theorem 3.2. *([8], Theorem 3.6, Lemma 3.8, Theorem 3.9) Let $R = KD_\infty$, the group algebra of the infinite dihedral group. Then*

(i) R is a right CS-ring if and only if char$(K) \neq 2$.

(ii) The center $C = Z(R)$ of R is a Dedekind domain and R_C is also CS.

Recall that the infinite dihedral group D_∞ in Theorem 3.2 is a polycyclic-by-finite group. Also KD_∞ is noetherian. Using theory of PI group algebras, it can be seen that KD_∞ has PI ([11], Corollary 3.8, p196 and Corollary 3.10, p197). Since char$(K) \neq 2$, $gl.\dim(KD_\infty) < \infty$ ([11], Theorem 3.13, p450). Thus $gl.\dim(KD_\infty)$ is equal to the Hirsch number $h(D_\infty)$ of D_∞ ([11], p450). Since $h(D_\infty)$ is the number of infinite cyclic quotients in any subnormal series of D_∞, $h(D_\infty) = 1$. It follows, then, that the ring R in Theorem 3.2 is also hereditary. Thus KD_∞ is a prime right-left CS, noetherian, hereditary ring with PI.

Behn, in his Ph.D. dissertation [1], considered prime CS group algebras R of polycyclic-by-finite groups. It can be shown that the prime group algebra KG of a polycyclic-by-finite group is noetherian and has PI.

Theorem 3.3. *([2], Theorem 3.3.10) Let G be a polycyclic-by-finite group and suppose $R = KG$ is prime. Then R is CS if and only if G is either torsion free or $G \simeq D_\infty$ and char(K) $\neq 2$.*

Question 1. Characterize group G if $R = KG$ is a semiprime PI noetherian CS-group algebra.

(The hypothesis in Question 1 implies G is polycyclic-by-finite. (See Lemma 3.4))

As observed above the group algebra KD_∞ is indeed hereditary. It, therefore, follows that if $R = KG$ is a prime CS group algebra of a polycyclic-by-finite group which is not a domain then it is hereditary. By ([5], Corollary 12.18), this is equivalent to R is finitely $\sum$-CS, that is, R^n is CS as R-module for all $n > 0$. This leads us to raise the following question.

Question 2. Is semiprime PI noetherian CS-group algebra which is not a direct sum of domains also finitely $\sum$-CS, that is, R^n is CS as R-module for all $n > 0$?

We know that semiprime noetherian ring R is finitely $\sum$-CS if and only if it is right hereditary ([5], Corollary 12.18). With respect to Question 2, one may ask to characterize groups G such that $R = KG$ is a semiprime noetherian PI hereditary ring. Hereditary group algebras have been completely characterized by Dicks ([4]). The structure of group G, in this case, is described in terms of fundamental group of connected graphs of finite groups.

Since the description of fundamental groups is rather too abstract, we give below a characterization of semiprime hereditary group algebras. In this special case the group G is quite easy to describe. We first give a Lemma that is of independent interest.

Lemma 3.4. *If KG is right (or left) noetherian and has PI then G is polycyclic-by-finite.*

Proof. First we note that G is noetherian because KG is noetherian.

Case I: $\mathrm{char}(K) = 0$.

Since KG has PI, G has a normal abelian subgroup A of finite index which is finitely generated and so it is polycyclic ([11], Corollary 3.8, p196). Since $[G : A] < \infty$, G is polycyclic-by-finite.

Case II: $\mathrm{char}(K) = p$.

Since KG has PI, G has a normal p-abelian subgroup A of finite index which is finitely generated ([11], Corollary 3.10, p197). Consequently $\frac{A}{A'}$ is finitely generated, abelian and hence polycyclic. Also as A is p-abelian, A' is finite. Since $[G : A] < \infty$, $(1) \trianglelefteq A' \trianglelefteq A \trianglelefteq G$ is a finite subnormal series in which each factor is either finite or polycyclic. Thus G is polycyclic-by-finite. ∎

The theorem that follows gives a characterization of a semiprime noetherian hereditary group algebras with PI.

Theorem 3.5. *Let K be a field with $\mathrm{char}(K) \neq 2$ and let KG be a semiprime, noetherian, PI group algebra. Then the following are equivalent.*

(i) KG is hereditary.

(ii) Either G is finite or G has an infinite cyclic subgroup A of finite index, say, n such that in the case $\mathrm{char}(K) = p$, p does not divide n and G has no element of order p.

Proof. (i) $\Rightarrow$ (ii).

Since KG is hereditary, $gl.\dim(KG)$ is 0 or 1. If $gl.\dim(KG) = 0$ then G is a finite group with no elements of order p. So let $gl.\dim(KG) = 1$. KG, being hereditary, is nonsingular. Also KG is finitely $\sum$-CS and hence CS. We show that G has no element of order p if $\mathrm{char}(K) = p$. Let, if possible, G has an element, say x, of order p. Let $H = \langle x \rangle$, the cyclic subgroup generated by x. Since KG is nonsingular and CS, it must be PP. Thus, by ([2], Lemma 3.3.1), KH is PP. Hence KH is nonsingular. However, the right nonsingular ideal $Z_r(KH)$ of KH is $\omega(KH) \neq 0$ ([11], Exercise 30, p467), a contradiction. Thus G has no element of order p, if $\mathrm{char}(K) = p$. By ([11], p450), Hirsch length $h(G)$ of G is equal to 1. Thus by ([11], Lemma 2.5, p422) G has a characteristic infinite cyclic subgroup A of finite index. If $\mathrm{char}(K) = 0$ then we have nothing else to prove. So let us assume that $\text{char}(K) = p$. Since A is infinite cyclic, $Aut(A) \cong C_2$, the cyclic group of order 2. Define $\sigma : G \longrightarrow Aut(A)$ given by $\sigma(x) = \sigma_x$ where $\sigma_x : A \longrightarrow A$, $\sigma_x(a) = xax^{-1}$. It can be checked that σ is a group homomorphism and $\ker(\sigma) = C_G(A)$. Thus $[G : C_G(A)] \leq 2$.

Case I : $[G : C_G(A)] = 1$.

In this case $G = C_G(A)$. Thus $A \subset Z(G)$. Since $[G : A] < \infty$, $[G : Z(G)] < \infty$. Thus G' is finite and $p \nmid o(G')$. Since for any $x, g \in G$, $g^{-1}xg = xx^{-1}g^{-1}xg \in xG'$ and G' is finite, it follows that G is an FC-group, that is, $G = \Delta(G)$. Hence G^+, the group of all torsion elements of G, is a finite normal subgroup of G and $\frac{G}{G^+}$ is finitely generated torsion free abelian ([11], Lemma 1.5, p116). Thus $G' \subset G^+$.

Consider the subnormal series

$$(1) \trianglelefteq G' \trianglelefteq G^+ \trianglelefteq G.$$

Since $\frac{G}{G'}$ is finitely generated torsion free abelian, $\frac{G}{G^+}$ is free abelian. But G^+ is finite. Therefore $h(\frac{G}{G^+}) = h(G) = 1$. Therefore $\frac{G}{G^+}$ is an infinite cyclic group. Let $\frac{G}{G^+} = \langle xG^+ \rangle$. Then $G = G^+ \langle x \rangle$ and $p \nmid o(G^+) = [G : \langle x \rangle]$. This completes the proof in this case.

Case II : $[G : C_G(A)] = 2$.

Let $G_0 = C_G(A)$. Then $C_{G_0}(A) = G_0$. Note that KG_0 is semiprime, noetherian, PI, and hereditary because $gl.\dim(KG_0) = h(G_0) = 1$ ([11], Lemma 2.10, p426). By Case I, $G_0 = G_0^+ \langle x \rangle$ and $p \nmid [G_0 : \langle x \rangle]$. Since $[G : \langle x \rangle] = 2[G_0 : \langle x \rangle]$ and $\text{char}(K) \neq 2$, $p \nmid [G : \langle x \rangle]$ and we are done in this case as well.

(ii) $\Rightarrow$ (i)

We only need to consider the case when G is infinite. Let A be an infinite cyclic subgroup of G with $[G : A] < \infty$. Then KA is a PID. Thus $gl.\dim(KA) = 1$. Since $[G : A] < \infty$, and G has no element of order p if $\text{char}(K) = p$, $gl.\dim(KG) < \infty$ ([11], Theorem 3.12, p442). Indeed $gl.\dim(KG) = gl.\dim(KA) = 1$ ([11], p450). Thus KG is hereditary. ∎

Theorem 3.6. *Suppose $R = KG$ is a semiprime noetherian hereditary group algebra with PI. Then*

(i) the center $Z(R)$ of R is a direct sum of Dedekind domains.

(ii) R is CS as a module over its center $Z(R)$.

Proof. (i) Since $R = KG$ is semiprime, the classical quotient ring $Q_{cl}(R)$ of R is semisimple artinian. Let $Q_{cl}(R) = e_1Q_{cl}(R) \oplus e_2Q_{cl}(R) \oplus \cdots \oplus e_nQ_{cl}(R)$ where for each i, e_i is a central idempotent and $e_iQ_{cl}(R)$ is simple artinian. By ([11], Exercise 32, p167), $e_i \in R$. Hence

$R = e_1R \oplus e_2R \oplus \ldots \oplus e_nR$. Since R is hereditary, e_iR is hereditary. Also e_iR is a prime ring satisfying a polynomial identity ([11] Corollaries 3.8 and 3.10, p196-197). Thus e_iR is a prime PI noetherian hereditary ring . So by ([9], Theorem 13.9.16, p483), the center $Z(e_iR)$ is a Dedekind domain. Since $Z(R) = Z(e_1R) \oplus Z(e_2R) \oplus \ldots \oplus Z(e_nR)$, the result follows.

(ii) The proof follows from the fact that $Z(e_iR)$ is a Dedekind domain and e_iR is CS over $Z(e_iR)$. ∎

4. Local CS-Group Algebras

In this section we state recent results on local group algebras which are either CS or finitely $\sum$-CS. We begin with the following result.

Theorem 4.1. *([3], Lemma 4.1) Let K be a field and G be any group. The group algebra KG of the group G is local right CS if and only if $char(K) = p$ and G is a locally finite p-group.*

Lemma 4.2. *([3], Lemma 3.3) Suppose R is a local right uniform ring with nil radical and S is a uniform direct summand of $(R \times R)_R$. Then $S = (1, b)R$ for some $b \in R$ or $(a, 1)R$ for some $a \in R$.*

Lemma 4.3. *([3], Theorem 3.10) Under the conditions of Lemma 4.2, $R \times R$ has (C_3), that is, if S_1 and S_2 are direct summands of $R \times R$ with $S_1 \cap S_2 = 0$ then $S_1 \oplus S_2$ is also a summand of $R \times R$.*

Theorem 4.4. *Suppose $R = KG$ is a local group algebra. Then $M_2(R)$ is CS if and only if R is selfinjective.*

Proof. By ([5], Lemma 12.8), $(R \times R)_R$ is CS. Since radical of R is nil, $(R \times R)_R$ has (C_3) (by Lemma 4.3). Thus $(R \times R)_R$ is quasi-continuous. Consequently R is selfinjective. (Note that in this case R is selfinjective is equivalent to saying that G is a finite p-group.) ∎

Theorem 4.5. *Suppose $R = KG$ is a local group algebra. Then $M_n(R)$ $(n > 1)$ is CS if and only if $M_2(R)$ is CS.*

A ring R is called semilocal if $\frac{R}{J(R)}$ is semisimple artinian. For a semilocal group algebra $R = KG$, one can ask the following questions.

Question 3. Characterize groups G such that semilocal group algebra KG is finitely $\sum$-CS.

Question 4. Characterize groups G such that semilocal group algebra KG is CS.

In case G is a nilpotent group, then the following result is known.

Theorem 4.6. *([3], Theorem 4.3) Let K be a field and G be a nilpotent group such that the group algebra $R = KG$ is semiperfect. Then $M_n(R)$ $(n > 1)$ is CS if and only if R is selfinjective.*

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