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The Structure of Right Continuous Right π-rings.

K.I. Beidar
Department of Mathematics,
National Cheng-Kung University, Tainan, Taiwan
e-mail: beidar@mail.ncku.edu.tw
and
S.K. Jain
Department of Mathematics, Ohio University,
Athens, Ohio, USA
e-mail: jain@math.ohiou.edu

Abstract

Recently the Jain, López-Permouth and Syed classified indecomposable non-local right continuous right π-rings. In the present article we describe the structure of right continuous right π-rings.

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1 Introduction

Throughout all rings have unity ($\neq 0$) and all modules are unital. Given a ring R and a right R-module M, we denote by $\mathrm{Soc}(M)$ and $J(R)$ the socle of M and the Jacobson radical of R respectively. Given a nonempty subset S of R, we set

$$r(R;S) = \{r \in R \mid Sr = 0\} \quad \text{and} \quad \ell(R;S) = \{r \in R \mid rS = 0\}.$$

We denote by $E(M)$ the injective hull of M. For a submodule N of a module M, the notations $N \subseteq_e M$ and $N \subseteq^{\oplus} M$ respectively denote that N is an essential submodule of M and N is a direct summand of M. Recall that a *closure* of N in M is a maximal essential extension of N in M. The submodule N is said to be *closed* in M if it has no proper essential extensions in M. Given a module L, M is said to be L-injective if for any submodule $K \subseteq L$, every $\alpha \in \mathrm{Hom}(K_R, M_R)$ is the restriction of some $\beta \in \mathrm{Hom}(L_R, M_R)$.

Consider now the following conditions for a module M.

(C1) For every submodule N of M there exists a direct summand L of M with $N \subseteq_e L$.

(C2) If a submodule N of M is isomorphic to a direct summand of M then N itself is a direct summand of M.

(C3) If A and B are direct summands of M with $A \cap B = 0$ then $A \oplus B$ is a direct summand of M.

A module M is called *continuous* (respectively, *quasi-continuous*) if it satisfies (C1) and (C2) (respectively, (C1) and (C3)).

It is well-known that a module M is quasi-continues if and only if

$$eM \subseteq M \quad \text{for every idempotent } e \in \mathrm{End}(E(M)_R). \tag{1}$$

A ring R is said to be a *right π-ring* if every right ideal in R is a quasi-continuous module.

Let n be an integer with $n \geq 1$, let $D_1, D_2, \dots, D_n$ be skew fields and let Δ be a right continuous right π-ring, all of whose idempotents are central, with essential ideal P such that Δ/P is a skew field and the right Δ-module Δ/P is not embedable into Δ_Δ. Next, let V_i be D_i-D_{i+1}-bimodule such that

$$\dim(\{V_i\}_{D_{i+1}}) = 1 \quad \text{for all } i = 1, 2, \dots, n-1,$$

and let V_n be a D_n-Δ-bimodule such that $V_n P = 0$ and $\dim(\{V_n\}_{\Delta/P}) = 1$. We denote by $G_n(D_1, \dots, D_n, \Delta, V_1, \dots, V_n)$ the ring of $(n+1) \times (n+1)$ matrices of the form

$$\begin{pmatrix} D_1 & V_1 & & & & \\ & D_2 & V_2 & & & \\ & & D_3 & V_3 & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & \cdot & D_n & V_n \\ & & & & & \Delta \end{pmatrix}$$

where it is understood that $V_i V_j = 0$ for all i, j.

Recall that a nonzero right R-module M is called *square free* if it does not contain the direct sum of two isomorphic nonzero submodules. A ring R is said to be square free if the module R_R is square free.

The study of rings, for which every right ideal is quasi-injective (known as right q-rings), was initiated by Jain, Mohamed and Singh [10] in 1969. Since then such rings were studied by a number of authors [1, 2, 5, 6, 7, 8, 11, 12, 13, 14]. Recently Jain, López-Permouth and Syed [9] extended this line of research by studying right π-rings. They described the structure of indecomposable non-local right continuous right π-rings and obtained the following useful general result.

Theorem 1.1 (**[9, Theorem 2.6]**) *A right π-ring R is the direct sum of a semisimple Artinian ring and a right square free right π-ring.*

The goal of the present paper is to describe the structure of right continuous right π-rings.

We show that every ring of the form $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$ is a right continuous right π-ring (Proposition 2.16). Our main result, describing the structure of right continuous right π-rings, is the following theorem.

Theorem 1.2 *A ring R is right continuous right π-ring if and only if R is the direct sum of finitely many rings of the form $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$, finitely many indecomposable non-local right continuous right π-rings, and a right continuous right π-ring all of whose idempotents are central.*

2 Proof of the Main Results

First we show that a right continuous right π-ring R has the form described in Theorem 1.2. We conclude from Theorem 1.1 that, proving Theorem 1.2, we may assume that the ring in question is a square free right continuous right π-ring.

Throughout this paper R is a square free right continuous right π-ring.

Because R is right continuous and square free, it follows at once from [15, Lemma 3.4, Proposition 3.5 and Theorem 3.11] that

$$R/J(R) \text{ is a strongly regular right continuous ring.} \tag{2}$$

Furthermore, note that

$$\text{if } P \text{ is a right primitive ideal of } R\text{, then } R/P \text{ is a skew field,} \tag{3}$$

because $R/J(R)$ is strongly regular, and so $R/P \cong (R/J(R))/(P/J(R))$ is a primitive strongly regular ring; thus R/P is a skew field.

The following result follows at once from [9, Lemma 3.2].

Lemma 2.1 *Let S be a minimal right ideal of R and let e be an idempotent of R such that $S \subseteq_e eR$. Suppose that $eR(1-e) \neq 0$. Then eRe is a skew field and $eR(1-e)$ is the only proper submodule of eR. In particular, $eR(1-e) = S$.*

Given two right R-modules M and N, we set

$$\mathrm{Tr}_N(M) = \sum_{f \in \mathrm{Hom}(M_R, N_R)} f(M).$$

Combining [9, Lemma 2.3(a)] with [9, Theorem 3.5] we get at once

Lemma 2.2 *Let A and B be two right ideals of the ring R such that $A \cap B = 0$. Then $\mathrm{Tr}_B(A)$ is a direct sum of finitely many of minimal right ideals of R.*

Our plan of proving Theorem 1.2 consists of several steps. First, we show that the ring R contains a finite set of nilpotent minimal right ideals that determines a direct summand R_1 of R where R_1 is the direct sum of indecomposable right continuous right π-rings. Furthermore, by setting $R = R_1 \oplus R'$, we shall show that the ring R' contains finitely many nilpotent minimal right ideals of R', determining a direct summand R_2 of R', where R_2 is the direct sum of rings of the form $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$. Finally, putting $R = R_1 \oplus R_2 \oplus R_3$, we complete the proof by showing that R_3 is a right continuous right π-ring all of whose idempotents are central.

We continue with the following two general statements.

Lemma 2.3 *Let A be a ring, let M and N be two right A-modules and let u be a nonzero idempotent of A. Suppose that $uA(1-u) = 0$, $Mu = M$ and $Nu = N$. Set $B = uA$. Then:*

(1) B is a ring with identity u and $B = uA = uAu$.

(2) Both M and N are right B-modules canonically.

(3) Every submodule of M_B is a submodule of M_A.

(4) Every right ideal of the ring B is a right ideal of the ring A.

(5) M_A is simple if and only M_B is so.

(6) $\mathrm{Hom}(M_B, N_B) = \mathrm{Hom}(M_A, N_A)$.

(7) $M_B \cong N_B$ if and only if $M_A \cong N_A$.

(8) If A is a right continuous right π-ring, then so is B.

(9) If $B \oplus M$ is a quasi-continuous right A-module, then M_B is injective.

Proof. Clearly $B = uA = uAu + uA(1-u) = uAu$ and so u is the identity of the ring B. Since $Mu = M$, $mu = m$ for all $m \in M$ and whence M is a right B-module. Analogously N is a right B-module. Given a submodule L of M_B, we have that $L = LB = LuA$ and so L is a submodule of M_A. Both (4) and (5) follow from (3) at once. Next, the inclusion $\mathrm{Hom}(M_B, N_B) \supseteq \mathrm{Hom}(M_A, N_A)$ is obvious. Let $f : M_B \to N_B$. Given $m \in M$ and $a \in A$, we have that $ua \in B$ and $m(1-u) = 0$. Therefore

$$\begin{aligned} f(ma) &= f(m\{ua + (1-u)a\}) = f(m\{ua\}) = f(m)\{ua\} \\ &= f(m)\{ua + (1-u)a\} = f(m)a \end{aligned}$$

forcing $f \in \mathrm{Hom}(M_A, N_A)$ and so $\mathrm{Hom}(M_B, N_B) = \mathrm{Hom}(M_A, N_A)$. Clearly (7) follows from (6).

Suppose that A is a right continuous right π-ring. Then (4) implies that B is also right continuous ring right π-ring.

Finally, assume that $B \oplus M$ is a quasi-continuous A-module. Then M_A is B_A-injective by [15, Proposition 2.10]. It now follows at once from (6) that M_B is an injective module. $\square$

Lemma 2.4 *Let $\{L_\gamma \mid \gamma \in \Gamma\}$ be a family of simple right A-modules, let $M = \oplus_{\gamma\in\Gamma} L_\gamma$ and let K be a submodule of M. Suppose that M is square free. Then there exists a subset $\Omega \subseteq \Gamma$ such that $K = \oplus_{\omega\in\Omega} L_\omega$.*

Proof. It is well-known that there exists a subset $\Lambda \subseteq \Gamma$ such that $M = K \oplus L$ where $L = \oplus_{\lambda\in\Lambda} L_\lambda$. Let $\pi : M \to L$ be the canonical projection of modules and set $\Omega = \Gamma \setminus \Lambda$. Given $\omega \in \Omega$, we have that $L_\omega \cap L = 0$. As L_ω is simple, either $L_\omega \cong \pi(L_\omega)$ or $\pi(L_\omega) = 0$. Since M is square free, we conclude that $\pi(L_\omega) = 0$ and so $L_\omega \subseteq \ker(\pi) = K$. Therefore $\oplus_{\omega\in\Omega} L_\omega \subseteq K$. Clearly $(\oplus_{\omega\in\Omega} L_\omega) \oplus L = M$ and so the equality $K \oplus L = M$ forces $K = \oplus_{\omega\in\Omega} L_\omega$. $\square$

In what follows S_γ, $\gamma \in \Gamma$, are all homogeneous components of $\mathrm{Soc}(R)$ and $e_\gamma \in R$, $\gamma \in \Gamma$, are idempotents such that $S_\gamma \subseteq_e e_\gamma R$ for all $\gamma \in \Gamma$. Since $\sum_{\gamma\in\Gamma} S_\gamma = \oplus_{\gamma\in\Gamma} S_\gamma$ and each S_γ is an essential submodule of $e_\gamma R_R$, it is easy to see that $\sum_{\gamma\in\Gamma} e_\gamma R = \oplus_{\gamma\in\Gamma} e_\gamma R$. It now follows at once from [15, Lemma 3.8] that we may assume without loss of generality that

$$e_\alpha e_\beta = 0 \text{ for all } \alpha, \beta \in \Gamma \text{ with } \alpha \neq \beta. \tag{4}$$

Given $\gamma \in \Gamma$, we note that

$$S_\gamma \text{ is both an ideal and a minimal right ideal of } R. \tag{5}$$

Indeed, S_γ is a sum of isomorphic minimal right ideals and because R is a square free, we conclude that S_γ is a minimal right ideal of R. As a homogeneous component of the socle of a ring, S_γ is an ideal of R.

Since $S_\alpha \subseteq_e e_\alpha R$, we conclude that

$$e_\alpha R \text{ is a uniform module for all } \alpha \in \Gamma. \tag{6}$$

Therefore

$$\text{given a central idempotent } e \text{ of } R, \text{ either } ee_\alpha = 0 \text{ or } ee_\alpha = e_\alpha, \tag{7}$$

because $e_\alpha R = (e_\alpha e)R \oplus [e_\alpha(1-e)R]$ and so either $e_\alpha e = 0$ or $e_\alpha(1-e) = 0$.

We set

$$\Omega = \{\gamma \in \Gamma \mid e_\gamma \text{ is not central}\}.$$

Lemma 2.5 *Let u, v be orthogonal idempotents of R such that $uRv \neq 0$. Then there exists a finite subset $\Omega_{u,v} \subseteq \Omega$ such that $uRv = \oplus_{\omega \in \Omega_{u,v}} S_\omega \subseteq \mathrm{Soc}(R)$. In particular, uRv is an ideal of R. Moreover $uRvRu = 0 = vRuRv$.*

Proof. Set $A = vR$ and $B = uR$. It is well-known that every homomorphism $A_R \to B_R$ is given by the left multiplication by an element of uRv. Therefore $\mathrm{Tr}_B(A) = uRvR$ and so Lemma 2.2 yields that $uRvR$ is direct sum of finitely many minimal right ideals of R. Since R_R is square free, it now follows from Lemma 2.4 that there exists a finite subset $\Omega_{u,v} \subseteq \Gamma$ such that $uRvR = \oplus_{\omega \in \Omega_{u,v}} S_\omega \subseteq \mathrm{Soc}(R)$.

Now consider the right ideal $uRvRuR$. We already know that $vRuR = \sum_{\omega \in \Omega_{v,u}} S_\omega \subseteq vR$. Given $\omega \in \Omega_{v,u}$ and $r \in uRv$, either $rS_\omega = 0$, or $rS_\omega \cong S_\omega$ because S_ω is a minimal right ideal. The latter case is ruled out by the facts that $rS_\omega \cap S_\omega \subseteq uR \cap vR = 0$ and that R is square free. Therefore $rS_\omega = 0$ for all $\omega \in \Omega_{v,u}$ and $r \in uRv$ forcing $uRvRuR = 0$. In particular, $uRvRu = 0$. Analogously, $vRuRv = 0$. Note that v and $1 - v$ are orthogonal idempotents and so $(1-v)RvR(1-v) = 0$ by the above result. Since $u(1-v) = u$, we see that $0 = u(1-v)RvR(1-v) = uRvR(1-v)$. Therefore

$$\sum_{\omega \in \Omega_{u,v}} S_\omega = uRvR = uRvRv + uRvR(1-v) = uRvRv = uRv.$$

It now follows from (5) that uRv is an ideal of R.

Let $\omega \in \Omega_{u,v}$ and assume that e_ω is central. By (6), $e_\omega R$ is a uniform module. As e_ω is central,

$$(e_\omega uR) \cap (e_\omega vR) \subseteq (uR) \cap (vR) = 0.$$

Therefore either $e_\omega u = 0$ or $e_\omega v = 0$. Say, $e_\omega v = 0$. We have $S_\omega \subseteq uRv$ and

$$S_\omega = e_\omega S_\omega \subseteq e_\omega uRv = uR(e_\omega v) = 0,$$

a contradiction. Therefore e_ω is not central and so $\omega \in \Omega$ forcing $\Omega_{u,v} \subseteq \Omega$. The proof is thereby complete. □

If $\Omega = \emptyset$, the Lemma 2.5 yields that $eR(1-e) = 0 = (1-e)Re$ for any $e = e^2 \in R$ (because $\Omega_{e,1-e} = \emptyset = \Omega_{1-e,e}$) and so all idempotents are central. In this case there is nothing to prove. Therefore we may assume without loss of generality that $\Omega \neq \emptyset$.

Lemma 2.6 *Let n be a positive integer, let $\omega_1, \omega_2, \ldots, \omega_n \in \Omega$ and let $u = e_{\omega_1} + e_{\omega_2} + \ldots + e_{\omega_n}$. Suppose that $uR(1-u) = 0$. Then u is a central idempotent.*

Proof. Set $u_i = e_{\omega_i}$ and $T_i = S_{\omega_i}$, $i = 1, 2, \ldots, n$. By Lemma 2.3(1),

$$uRu = uR. \tag{8}$$

Assume that $(1-u)Ru = 0$. Then the equality $uR(1-u) = 0$ implies that u is central and there is nothing to prove. Therefore we may assume without loss of generality that $(1-u)Ru \neq 0$.

We set $Q = \sum_{i=1}^{n} u_i R u_i$ and $I = \sum_{i \neq j} u_i R u_j$. Since $u = u_1 + u_2 + \ldots + u_n$, we have

$$uRu = Q + I. \tag{9}$$

We claim that

$$\begin{aligned} &Q \text{ is the direct sum of local rings } u_i R u_i,\ i = 1, 2, \ldots, n; \\ &I \text{ is a nilpotent ideal of the ring } uRu \text{ and } J(uRu) = \textstyle\sum_{i=1}^{n} J(u_i R u_i) + I; \\ &(uRu)/J(uRu) = \oplus_{i=1}^{n} (u_i R u_i)/J(u_i R u_i). \end{aligned} \tag{10}$$

Indeed, since $u_i u_j = 0$ for all $i \neq j$ by (4), Q is the direct sum of rings $u_i R u_i$, $i = 1, 2, \ldots, n$. It follows from both (6) and [15, Proposition 2.7] that $u_i R$ is an indecomposable continuous R-module. Since, $u_i R u_i = \mathrm{End}(u_i R_R)$, [15, Proposition 3.5] yields that $(u_i R u_i)/J(u_i R u_i)$ is von Neumann regular. As $u_i R_R$ is indecomposable, [15, Lemma 3.8] now yields that $(u_i R u_i)/J(u_i R u_i)$ has no nontrivial idempotents and so $u_i R u_i$ is a local ring. Therefore Q is the direct sum of local rings $u_i R u_i$, $i = 1, 2, \ldots, n$.

By Lemma 2.5, each $u_i R u_j$ (with $i \neq j$) is an ideal of the ring R. Since $(u_R u_j)^2 = 0$, we see that I, being a sum of nilpotent ideals, is also an nilpotent ideal of R.

Obviously $I \subseteq J(uRu)$, $Q + I = Q \oplus I$ and so the factor ring $(uRu)/I$ is canonically isomorphic to $Q = \oplus_{i=1}^{n} u_i R u_i$. The result now follows from the obvious fact that $J(Q) = \oplus_{i=1}^{n} J(u_i R u_i)$. Therefore (10) is proved.

Recall that each $T_i \subseteq u_i R \subseteq uR = uRu$ by (8) and so $T_i u = T_i$. Therefore

$$\text{each } T_i \text{ is a simple right } uRu\text{-module} \tag{11}$$

by both (5) and Lemma 2.3(5).

By (10), $(uRu)/J(uRu)$ is the direct sum of skew fields $(u_i R u_i)/J(u_i R u_i)$, and so there are exactly n pairwise non-isomorphic simple right uRu-modules. On the other hand we know that each T_i is a simple right uRu-module. Since $\{T_i\}_R \not\cong \{T_j\}_R$ for $i \neq j$, it follows from Lemma 2.3(7) that $\{T_i\}_{uRu} \not\cong \{T_j\}_{uRu}$ for $i \neq j$ and whence

$$\text{every simple right } uRu\text{-module is isomorphic to some } T_i. \tag{12}$$

Recall that $(1-u)Ru \neq 0$ by our assumption. According to Lemma 2.5, $(1-u)Ru = \sum_{\omega \in \Omega_{1-u,u}} S_\omega$. Clearly each S_ω is a right uRu-module and so it is a simple right uRu-module by both (5) and Lemma 2.3(5). According to (12), $\{S_\omega\}_{uRu} \cong \{T_i\}_{uRu}$ for some $1 \leq i \leq n$, and whence Lemma 2.3(7) implies that $\{S_\omega\}_R \cong \{T_i\}_R$. Finally, $S_\omega \cap T_i \subseteq (1-u)R \cap uR = 0$, contradicting square freeness of R. Thus $(1-u)Ru = 0$ and the proof is completed. □

Given $\omega \in \Omega$, it follows from Lemma 2.6 that $e_\omega R(1-e_\omega) \neq 0$ and so Lemma 2.1 yields that

$$e_\omega R e_\omega \text{ is a skew field and } S_\omega = e_\omega R(1-e_\omega) \text{ for all } \omega \in \Omega. \tag{13}$$

In particular, $S_\omega^2 = 0$, which explains our attention to nilpotent minimal right ideals of R.

Lemma 2.7 *Let $\omega \in \Omega$. Then the factor ring $\overline{R} = R/r(R; S_\omega)$ is a skew field, $\dim(\{S_\omega\}_{\overline{R}}) = 1$ and $\overline{R}_R \cong \{S_\omega\}_R$. Morover,*

$$\ell(R; S_\omega) = (1-e_\omega)Re_\omega + e_\omega R(1-e_\omega) + (1-e_\omega)R(1-e_\omega)$$

and $R/\ell(R; S_\omega) \cong e_\omega R e_\omega$ is a skew field. In particular, S_ω is a completely reducible left R-module.

Proof. Set $e = e_\omega$ and $S = S_\omega$. Since S is a simple right R-module by (5), $r(R; S)$ is a primitive right ideal of R and so $\overline{R}$ is a skew field by (3). We see that S is both a right vector space over $\overline{R}$ and a simple right module. Therefore $\dim(S_{\overline{R}}) = 1$. Take any $0 \neq x \in S$. Then the map $\varphi : R \to S$, $r \mapsto xr$, is an epimorphism. Obviously $\ker(\varphi) \supseteq r(R; S)$. Since $\overline{R}$ is a skew field, $\ker(\varphi) = r(R; S)$ and so φ induces an isomorphism of R-modules $\overline{R}$ and S.

Since $S \subseteq eR$, $eS = S$. It now follows from both Lemma 2.5 and (13) that $(1-e)ReS = (1-e)ReR(1-e) = 0$ and so

$$P = (1-e)Re + eR(1-e) + (1-e)R(1-e) \subseteq \ell(R; S).$$

Clearly $P + eRe = R$. By (13), eRe is a skew field and so $eRe \cap \ell(R; S) = 0$; otherwise $\ell(R; S) = R$ which is impossible. Since $R = eRe + P$ and $P \subseteq \ell(R; S)$, we conclude from the modular law that $\ell(R; S) = P$. Therefore $R/P \cong eRe$ and so R/P is a skew field. It is now clear that S is a completely reducible left R/P-module. That is to say S is completely reducible left R-module. This completes the proof. □

Lemma 2.8 *Let $\omega \in \Omega$. Suppose that there exists $\gamma \in \Gamma \setminus \{\omega\}$ with $e_\omega R e_\gamma \neq 0$. Further, let w be an idempotent of R such that $e_\omega w = w e_\omega = e_\gamma w = w e_\gamma = 0$. Then*

(1) $S_\omega = e_\omega R e_\gamma$ and $e_\omega R w = 0$; in particular, $e_\omega R e_\alpha = 0$ for all $\alpha \in \Gamma$ with $\alpha \notin \{\omega, \gamma\}$.

(2) $\gamma \in \Omega \setminus \{\omega\}$.

(3) $e_\gamma R e_\gamma$ is a skew field, $\{S_\omega\}_R \cong \{(e_\gamma R)/S_\gamma\}_R$ and $\dim(\{S_\omega\}_{e_\gamma R e_\gamma}) = 1$.

Proof. (1) Set $e = e_\omega$ and $S = S_\omega$. Since $ee_\gamma = 0 = e_\gamma e$ by (4), Lemma 2.5 implies that eRe_γ is an ideal of R and $eRe_\gamma \subseteq \mathrm{Soc}(eR) = S$ forcing $eRe_\gamma = S$. Therefore

$$Se_\gamma = S \quad \text{and} \quad S(1 - e_\gamma) = 0. \tag{14}$$

It follows at once from (14) that $Sw = 0$. Suppose that $eRw \neq 0$. Then Lemma 2.5 yields that $eRw \subseteq \mathrm{Soc}(eR) = S$ and so $eRw = S$ contradicting $Sw = 0$. Therefore $eRw = 0$. Given $\alpha \in \Gamma \setminus \{\omega, \gamma\}$, $e_\omega e_\alpha = e_\alpha e_\omega = e_\gamma e_\alpha = e_\alpha e_\gamma = 0$ by (4) and so $eRe_\alpha = 0$ by the above result (with $w = e_\alpha$).

(2) Indeed, assume to the contrary that $\gamma \in \Gamma \setminus \Omega$. Then e_γ is a central idempotent by the definition of Ω. Next, since $\gamma \neq \omega$, $ee_\gamma = 0$ by (4) and so $S = eRe_\gamma = ee_\gamma R = 0$, a contradiction. Thus $\gamma \in \Omega \setminus \{\omega\}$.

(3) Since $\gamma \in \Omega$, (13) implies that $e_\gamma R e_\gamma$ is a skew field, $S_\gamma = e_\gamma R(1 - e_\gamma)$ and $e_\gamma R = e_\gamma R e_\gamma + S_\gamma$. By (14), $SS_\gamma = Se_\gamma R(1 - e_\gamma) \subseteq S(1 - e_\gamma) = 0$. Next, pick $0 \neq s \in S$. Since $se_\gamma = s$,

$$S = sR = se_\gamma R = s(e_\gamma R e_\gamma + S_\gamma) = se_\gamma R e_\gamma$$

and so we conclude that $\dim(S_{e_\gamma R e_\gamma}) = 1$ and $S_R \cong \{(e_\gamma R)/S_\gamma\}_R$ via $x \mapsto sx$, $x \in e_\gamma R$. The proof is completed. □

Let $\omega \in \Omega$ be as in Lemma 2.8. It follows from Lemma 2.8(1) that there exists a uniquely determined element $\gamma \in \Omega$ satisfying (1)–(3). We denote γ by $\sigma(\omega)$. If $\omega \in \Omega$ and $e_\omega R e_\alpha = 0$ for all $\alpha \in \Gamma \setminus \{\omega\}$, then we shall say that $\sigma(\omega)$ is not defined. We now set

$$\Omega' = \{\omega \in \Omega \mid \sigma(\omega) \text{ is defined}\} \quad \text{and} \quad \Omega'' = \Omega \setminus \Omega'.$$

Lemma 2.9 *The map $\sigma : \Omega' \to \Omega$ is injective.*

Proof. Assume that there exist $\alpha, \beta \in \Omega'$ with $\alpha \neq \beta$ and $\sigma(\alpha) = \sigma(\beta)$. Set $\gamma = \sigma(\alpha)$. By Lemma 2.8(3), both $\{S_\alpha\}_R$ and $\{S_\beta\}_R$ are isomorphic to $\{(e_\gamma R)/S_\gamma\}_R$ and so $\{S_\alpha\}_R \cong \{S_\beta\}_R$ contradicting square freeness of R. Therefore σ is injective and the lemma is proved. □

Lemma 2.10 *Suppose that $|\Omega| = \infty$. Then there exist an infinite subset $\Delta \subseteq \Omega$ and mutually orthogonal idempotents u_δ, $\delta \in \Delta$, of R such that $S_\delta u_\delta \neq 0$ and $(u_\delta R) \cap S_\delta = 0$ for all $\delta \in \Delta$.*

Proof. Let $\omega \in \Omega$. By (5), S_ω is a simple right R-module and so each $P_\omega = r(R; S_\omega)$ is a primitive right ideal of R. Setting $P = \cap_{\omega \in \Omega} P_\omega$, we see that $J(R) \subseteq P$. Therefore (2) yields that both $\overline{R} = R/J(R)$ and R/P are strongly regular rings. Since $\{S_\alpha\}_R \ncong \{S_\beta\}_R$ for all $\alpha \neq \beta \in \Omega$, Lemma 2.7 implies that $P_\alpha \neq P_\beta$ (otherwise $S_\alpha \cong R/P_\alpha = R/P_\beta \cong S_\beta$). Therefore $\{P_\omega/P \mid \omega \in \Omega\}$ is an infinite set of distinct primitive ideals of R/P and so R/P is not a semisimple Artinian ring. Therefore it contains an infinite family $\{v_i \mid i = 1, 2, \ldots\}$ of nonzero pairwise orthogonal idempotents by [3, Corollary 2.16]. Since $v_i \neq 0$ and $P = \cap_{\omega\in\Omega}P_\omega$, there exists $\omega_i \in \Omega$ with $v_i \notin P_{\omega_i}/P$. Obviously each P_{ω_i}/P is a primitive (and so prime) ideal of the ring R/P. Recalling that R/P is strongly regular, we see that each v_i is central and so $v_iv_j = 0$ forces $v_j \in P_{\omega_i}/P$ for all $j \neq i$. In particular, $\omega_i \neq \omega_j$ for all $i \neq j$ and whence the set $\Delta = \{\omega_i \mid i = 1, 2, \ldots\}$ is infinite. For the sake of uniformity of notation we set $v_{\omega_i} = v_i$ for all $i = 1, 2 \ldots$ and note that $v_\delta \notin P_\delta/P$, $\delta \in \Delta$.

Since $\overline{R}$ is von Neumann regular and $R/P = \overline{R}/\overline{P}$ (where $\overline{P} = P/J(R)$), it follows from [3, Proposition 2.18] that there exists a family $\{w_\delta \mid \delta \in \Delta\}$ nonzero pairwise orthogonal idempotents of the ring $\overline{R}$ such that $w_\delta + \overline{P} = v_\delta \in R/P$ for all $\delta \in \Delta$. As $v_\delta \notin P_\delta/P$, $w_\delta \notin \overline{P_\delta}$ for all $\delta \in \Delta$, where $\overline{P_\delta} = P_\delta/J(R)$.

Next, (13) implies that $S_\delta = e_\delta R(1 - e_\delta)$ for all $\delta \in \Delta$. In particular, $1 - e_\delta \notin P_\delta$. Let $r \mapsto \overline{r}$, $r \in R$, be the canonical projection $R \to \overline{R}$. We see that $\overline{1 - e_\delta} \notin \overline{P_\delta}$. Since $\overline{R}$ is strongly regular and $\overline{P_\delta}$ is a prime ideal of $\overline{R}$, we conclude that $z_\delta = w_\delta\overline{1 - e_\delta} \notin \overline{P_\delta}$ for all $\delta \in \Delta$ and $\{z_\delta \mid \delta \in \Delta\}$ is a set of pairwise orthogonal idempotents.

It follows from [16, Theorem 4.9] that there exists a family of nonzero pairwise orthogonal idempotents $\{u_\delta \mid \delta \in \Delta\}$ such that $\overline{u_\delta} = z_\delta$ for all $\delta \in \Delta$. Therefore each $u_\delta \notin P_\delta$ and so $S_\delta u_\delta \neq 0$. Next, $\overline{u_\delta e_\delta} = \overline{u_\delta}\,\overline{e_\delta} = z_\delta\overline{e_\delta} = 0$ and so $u_\delta e_\delta \in J(R)$. Assume that $(u_\delta R) \cap S_\delta \neq 0$. Then $S_\delta \subseteq u_\delta R$. In particular, $u_\delta S_\delta = S_\delta$. Since $e_\delta S_\delta = S_\delta$, $(u_\delta e_\delta)S_\delta = S_\delta$. On the other hand, S_δ is completely reducible left R-module by Lemma 2.7 and so $J(R)S_\delta = 0$ forcing $(u_\delta e_\delta)S_\delta = 0$, a contradiction. Thus $(u_\delta R) \cap S_\delta = 0$. This completes the proof. □

We now need the following result which is a special case of [9, Proposition 2.8]. Recall that a family $\{A_i \mid i \in I\}$ of right ideal of R is said to independent if $\sum_{i\in I} A_i = \oplus_{\in I} A_i$.

Lemma 2.11 *Suppose that $\{A_i \mid i \in I\}$ is an independent family of right ideals of R. If for each $i \in I$ there exists a right ideal B_i in R that is a homomorphic image of A_i with $A_i \cap B_i = 0$, then I is finite.*

Lemma 2.12 *The set Ω is finite.*

Proof. Suppose that $|\Omega| = \infty$. Let $\Delta \subseteq \Omega$ and $\{u_\delta \mid \delta \in \Delta\}$ be as in Lemma 2.10. Set $A_\delta = u_\delta R$, $\delta \in \Delta$. Given $\delta \in \Delta$, $u_\delta \notin r(R; S_\delta)$ and so $su_\delta \neq 0$ for some $0 \neq s \in S_\delta$. Therefore $S_\delta = su_\delta R = sA_\delta$ and so $\{S_\delta\}_R$ is a homomorphic image of $\{A_\delta\}_R$ for all $\delta \in \Delta$. Next, $A_\delta \cap S_\delta = 0$ by Lemma 2.10. Therefore Lemma 2.11 yields that $|\Delta| < \infty$, a contradiction. Thus Ω is finite and the proof is completed. □

We have gathered enough information in order to construct the ring R_1. To this end, we introduce the following concept. A sequence $\{\omega_1, \omega_2, \ldots, \omega_n\}$ of elements of the set Ω' is called *a cycle* if $\sigma(\omega_n) = \omega_1$ and $\sigma(\omega_i) = \omega_{i+1}$ for all

$i = 1, 2, \ldots, n-1$. It follows from Lemma 2.8(2) that any cycle in Ω contains more than one element. We continue with the following lemma.

Lemma 2.13 *Let $\{\omega_1, \omega_2, \ldots, \omega_n\}$ be a cycle and let $u = \sum_{i=1}^{n} e_{\omega_i}$. Then u is a central idempotent of R and uR is a non-local indecomposable right continuous right π-ring.*

Proof. Given $1 \leq i \leq n$,

$$e_{\omega_i}(1-u) = (1-u)e_{\omega_i} = e_{\sigma(\omega_i)}(1-u) = (1-u)e_{\sigma(\omega_i)} = 0$$

by (4) and whence Lemma 2.8(1) (with $w = 1-u$, $\omega = \omega_i$ and $\gamma = \sigma(\omega_i)$) implies that $e_{\omega_i}R(1-u) = 0$. Since $uR = \oplus_{i=1}^{n} e_{\omega_i}R$, we get $uR(1-u) = 0$ and so Lemma 2.6 implies that u is central. Clearly uR is a right continuous right π-ring.

Assume that uR is not an indecomposable ring. Then there exists a nonzero central idempotent v of the ring uR with $v \neq u$. Clearly v is a central idempotent of the ring R. It now follows from (7) that either $ve_{\omega_i} = e_{\omega_i}$ or $ve_{\omega_i} = 0$. Set $I = \{1 \leq i \leq n \mid ve_{\omega_i} = e_{\omega_i}\}$ and $J = \{1 \leq i \leq n \mid ve_{\omega_i} = 0\}$. Since $vu = v$ and $u = \sum_{i=1}^{n} e_{\omega_i}$, we now get $v = \sum_{i\in I} e_{\omega_i}$ and $u - v = \sum_{j\in J} e_{\omega_j}$. Clearly there exists $1 \leq k \leq n$ such that either $k \in I$ and $k+1 \in J$ or $k \in J$ and $k+1 \in I$. Say, $k \in I$ and $k+1 \in J$. By Lemma 2.8(3) there exists an epimorphism $e_{\omega_{k+1}}R \to S_{\omega_k}$. Therefore there exists a nonzero homomorphism $(u-v)R \to vR$ which is impossible because $u - v$ and v are orthogonal central idempotents. Hence uR is an indecomposable ring.

Finally, as we noted just before the lemma, any cycle in Ω contains more than one element. Therefore $n \geq 2$. It now follows from both (9) and (10) that the ring $uR = uRu$ is not local. The proof is completed. □

According to Lemma 2.12 the set Ω is finite and so $|\Omega'| < \infty$. Therefore, the set Ω' contains only finitely many cycles. It follows from the injectivity of the map σ (see Lemma 2.9), that distinct cycles are disjoint. Gathering together the direct summands of the ring R corresponding to the cycles, we obtain R_1, the direct summand of the ring R which is the direct sum of finitely many of indecomposable right continuous right π-rings. Therefore we may assume that

the set Ω' contains no cycles.

If $\Omega' = \Omega$, then $\sigma : \Omega \to \Omega$ is a permutation by Lemma 2.9 and so Ω is a union of disjoint cycles, a contradiction. Therefore

$$\Omega' \neq \Omega. \tag{15}$$

We set $e = \sum_{\omega\in\Omega} e_\omega$ and note that e is an idempotent and

$$eR = \oplus_{\omega\in\Omega} e_\omega R \supseteq \oplus_{\omega\in\Omega} S_\omega \tag{16}$$

by (4). We claim that

$$(1-e)Re = 0. \tag{17}$$

Indeed, assume that $(1-e)Re \neq 0$. Making use of Lemma 2.5 we see that $(1-e)Re = \sum_{\omega \in \Omega_{1-e,e}} S_\omega$ for some nonempty subset $\Omega_{1-e,e} \subseteq \Omega$ and so (16) implies that $eR \cap (1-e)R \supseteq (1-e)Re \neq 0$, a contradiction, which proves (17). Lemma 2.3(1) and (17) together imply that

$$(1-e)R \text{ is a ring with identity } 1-e. \tag{18}$$

We now claim that

$$\text{every idempotent } u \in (1-e)R \text{ is a central element of the ring } (1-e)R. \tag{19}$$

Indeed, $u\{(1-e)R\}(1-u) \subseteq uR(1-u)$ and so Lemmas 2.5 and 2.4 together imply that either $u\{(1-e)R\}(1-u) = 0$ or it contains some S_ω, $\omega \in \Omega$. The latter case is ruled out because it would imply that

$$S_\omega \subseteq [u\{(1-e)R\}(1-u)] \cap eR \subseteq (1-e)R \cap eR = 0.$$

Therefore $u\{(1-e)R\}(1-u) = 0$. Since $(1-u)\{(1-e)R\}u = (1-e)(1-u)Ru$, the same argument shows that $(1-u)\{(1-e)R\}u = 0$ and thus u is a central idempotent of the ring $(1-e)R$.

It follows from Lemma 2.3(8) that

$$(1-e)R \text{ is a right contnuous right } \pi\text{-ring}. \tag{20}$$

Recall that $\Omega'' = \Omega \setminus \Omega'$. According to (15), $\Omega'' \neq \emptyset$. Given $\omega \in \Omega''$, we have that $e_\omega R e_\gamma = 0$ for all $\gamma \in \Gamma$ by the definition of the set Ω'. In particular, $e_\omega Re = 0$ and so $S_\omega e = 0$. Therefore $S_\omega(1-e) = S_\omega$ and whence S_ω is a right $(1-e)R$-module by Lemma 2.3(2). We claim that

$$S_\omega \text{ is a simple injective right } (1-e)R\text{-module for all } \omega \in \Omega''. \tag{21}$$

Indeed, according to Lemma 2.3(5), S_ω is a simple right $(1-e)R$-module. Since $S_\omega \subseteq eR$, we see that $(1-e)R + S_\omega = (1-e)R \oplus S_\omega$. Being a right ideal of the right π-ring R, $(1-e)R \oplus S_\omega$ is a quasi-continuous right R-module and so S_ω is an injective $(1-e)R$-module by Lemma 2.3(9). Therefore (21) is proved.

Lemma 2.14 *There exists a family $\{p_\omega \mid \omega \in \Omega'' = \Omega \setminus \Omega'\}$ of nonzero pairwise orthogonal idempotents of the ring $(1-e)R$ such that*

(1) $p_\omega R$ is a right continuous right π-ring with identity p_ω all of whose idempotents are central, and S_ω is a simple injective right $p_\omega R$-module for all $\omega \in \Omega''$; moreover, $r(p_\omega R; S_\omega)$ is an essential right ideal of the ring $p_\omega R$, $p_\omega R / r(p_\omega R; S_\omega)$ is a skew field and the $p_\omega R$-module S_ω is not embedable into $p_\omega R$.

(2) $e_\alpha p_\omega = 0 = p_\omega e_\alpha$ *and* $p_\omega R e_\alpha = 0$ *for all* $\alpha \in \Omega$, $\omega \in \Omega''$.

(3) $e_\alpha R p_\omega = 0$ *for all* $\alpha \in \Omega \setminus \{\omega\}$ *and* $\omega \in \Omega''$.

Proof. Set $u = 1 - e$ and $T = uR$. It follows from (17), (19) and (20) together that u is the identity of the right continuous right π-ring T all of whose idempotents are central. Further, let $\omega \in \Omega''$. By (21), S_ω is a simple right T-module and so $P_\omega = r(T; S_\omega)$ is a right primitive ideal of T. Now (3) yields that T/P_ω is a skew field. Setting $P = \cap_{\omega\in\Omega''} P_\omega$ and making use of Chinese Reminder theorem we get that $T/P \cong \prod_{\omega\in\Omega''} T/P_\omega$ and so the ring T/P contains pairwise orthogonal idempotents u_ω, $\omega \in \Omega''$, such that $P_\omega/P = (1 - u_\omega)(T/P)$. That is to say

$$1 - u_\omega \in P_\omega/P \text{ for all } \omega \in \Omega''. \tag{22}$$

Recall that $\overline{T} = T/J(T)$ is a strongly regular ring by (2) and $J(T) \subseteq P$. Therefore it follows from [3, Proposition 2.18] that the ring $\overline{T}$ contains pairwise orthogonal idempotents v_ω, $\omega \in \Omega''$ such that $v_\omega + \overline{P} = u_\omega$ in $\overline{T}/\overline{P} = T/P$, where $\overline{P} = P/J(T)$. Next, [16, Theorem 4.9] implies there exists a family of nonzero pairwise orthogonal idempotents $\{p_\omega \mid \omega \in \Omega''\} \subseteq T$ such that $p_\omega + J(T) = v_\omega$ for all $\omega \in \Omega''$. Therefore $p_\omega + P = u_\omega$ in T/P. Now (22) yields that

$$S_\omega p_\omega = S_\omega \text{ for all } \omega \in \Omega''. \tag{23}$$

Since all idempotents of T are central, we conclude that $p_\omega R = p_\omega T$ is a right continuous right π-ring with identity p_ω all of whose idempotents are central. Moreover, (21) and (23) together yield that S_ω is a simple injective right $p_\omega R$-module. Let $Q = r(p_\omega R; S_\omega)$. Assume that the right $p_\omega R$-module S_ω is embedable into $p_\omega R$. Say, $S_\omega \cong K \subseteq p_\omega R$. Since p_ω is a central idempotent of the ring T, we conclude that $p_\omega R(1 - p_\omega) = 0$ and so Lemma 2.3 implies that K is a right R-module, and right R-modules S_ω and K are isomorphic. As $S_\omega \cap K \subseteq e_\omega R \cap (1-e)R = 0$, we get a contradiction with square freeness of R.

Assume that Q is not an essential right ideal of the ring $p_\omega R$. Then there exists a nonzero right ideal K of $p_\omega R$ such that $Q \cap K = 0$. Clearly, Q is a right primitive ideal of $p_\omega R$ and so (3) implies that Q is a maximal right ideal of $p_\omega R$ (and $p_\omega R/Q$ is a skew field). Therefore $p_\omega R = Q \oplus K$. Clearly right $p_\omega R$-modules S_ω and K are isomorphic, which is impossible by the above result. Therefore Q is an essential right ideal of the ring $p_\omega R$ and so the first statement of the lemma is proved.

Recall that $u = 1 - e$ and $e = \sum_{\beta\in\Omega} e_\beta$. Since $p_\omega \in T$ and u is the identity of the ring T, we conclude that $e_\alpha p_\omega = 0 = p_\omega e_\alpha$ for all $\alpha \in \Omega$ and $\omega \in \Omega''$. Further, $p_\omega R e_\alpha \subseteq (1-e)Re$. Since $(1-e)Re = 0$ by (17), we see that $p_\omega R e_\alpha = 0$ and so the second statement of the lemma is proved.

Now let $\alpha \in \Omega \setminus \{\omega\}$. Suppose that $e_\alpha R p_\omega \neq 0$. Recall that $e_\alpha R = e_\alpha R e_\alpha + S_\alpha$ by (13). Since $e_\alpha p_\omega = 0$, we conclude that

$$S_\alpha p_\omega \neq 0.$$

If $\sigma(\alpha)$ is defined, then $S_\alpha e_{\sigma(\alpha)} = S_\alpha$ by Lemma 2.8(1). Since $e_{\sigma(\alpha)} p_\omega = 0$, we get a contradiction $S_\alpha p_\omega = 0$. Therefore $\sigma(\alpha)$ is not defined forcing $\alpha \in \Omega''$. Since $\alpha \neq \omega$, $p_\alpha p_\omega = 0$. On the other hand $S_\alpha p_\alpha = S_\alpha$ by the first statement of the lemma, forcing a contradiction $S_\alpha p_\omega = 0$. Thus $e_\alpha R p_\omega = 0$ and the lemma is proved. □

A sequence $\{\omega_1, \omega_2, \ldots, \omega_n\} \subseteq \Omega$ is called *a chain* if $\omega_i \in \Omega'$ and $\sigma(\omega_i) = \omega_{i+1}$ for all $i = 1, 2, \ldots, n-1$. A chain is called *maximal* if it is not a proper subset of any other chain. Clearly a chain $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is maximal if and only if both $\omega_n \in \Omega''$ and $\omega_1 \notin \sigma(\Omega')$. Recall that $\sigma : \Omega' \to \Omega$ is an injective map by Lemma 2.9, Ω' does not contain cycles by our assumption and $|\Omega| < \infty$ by Lemma 2.12. Therefore Ω is a disjoint union of maximal chains.

Lemma 2.15 *Let $\{\omega_1, \omega_2, \ldots, \omega_n\} \subseteq \Omega$ be a maximal chain and let $w = p_{\omega_n} + \sum_{i=1}^n e_{\omega_i}$. Then w is a central idempotent of the ring R and the ring wR is of the form $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$.*

Proof. Set $w' = \sum_{i=1}^n e_{\omega_i}$ and note that $w = p_{\omega_n} + w'$. It follows from both (4) and Lemma 2.14(2) that w is sum of pairwise orthogonal idempotents and so w is an idempotent. Assume that $wR(1-w) \neq 0$. Then by Lemma 2.5, $wR(1-w) = \sum_{\alpha \in \Omega_{w,1-w}} S_\alpha$. Given $\alpha \in \Omega_{w,1-w}$, the inclusion $S_\alpha \subseteq e_\alpha R$ implies that $e_\alpha S_\alpha = S_\alpha$. According to Lemma 2.14(3), $p_{\omega_n} e_\alpha = 0$ and so $p_{\omega_n} S_\alpha = 0$ as well. Since $S_\alpha \subseteq wR(1-w)$, $wS_\alpha = S_\alpha$ (and $S_\alpha(1-w) = S_\alpha$) forcing $w'S_\alpha = S_\alpha$. Taking into account (4), we conclude that $\alpha = \omega_i$ for some $1 \leq i \leq n$. If $i < n$, then Lemma 2.8(1) yields

$$S_\alpha = S_\alpha e_{\sigma(\omega_i)} = S_\alpha (1-w) e_{\sigma(\omega_i)} = 0,$$

a contradiction. Therefore $i = n$. Note, that $S_\alpha p_{\omega_n} = S_\alpha$ by the choice of p_ω's (see Lemma 2.14). Therefore

$$S_\alpha = S_\alpha p_{\omega_n} = S_\alpha (1-w) p_{\omega_n} = 0,$$

a contradiction. Hence $wR(1-w) = 0$.

Next, suppose that $(1-w)Rw \neq 0$. Again applying Lemma 2.5, we see that $(1-w)Rw = \sum_{\alpha \in \Omega_{1-w,w}} S_\alpha$. Since $e_{\omega_i}(1-w) = 0$, we conclude that $e_{\omega_i} S_\alpha = 0$ for all $1 \leq i \leq n$ and $\alpha \in \Omega_{1-w,w}$. Recalling that $e_\alpha S_\alpha = S_\alpha$, we infer that

$$\alpha \notin \{\omega_1, \omega_2, \ldots, \omega_n\}.$$

Next, both the injectivity of σ and the maximality of the chain $\{\omega_1, \omega_2, \ldots, \omega_n\}$ imply that $\sigma(\alpha) \notin \{\omega_1, \omega_2, \ldots, \omega_n\}$ or $\sigma(\alpha)$ is not defined. Hence Lemma 2.8 implies that $e_\alpha R e_{\omega_i} = 0$ for all $1 \leq i \leq n$ and so $S_\alpha w' = 0$. On the other hand, $S_\alpha \subseteq (1-w)Rw$ and so $S_\alpha w = S_\alpha$, forcing

$$S_\alpha p_{\omega_n} = S_\alpha.$$

According to Lemma 2.14(3), $e_\alpha R p_{\omega_n} = 0$ forcing $S_\alpha p_{\omega_n} = 0$, a contradiction. Therefore w is a central idempotent.

Set $v_i = e_{\omega_i}$, $i = 1, 2, \ldots, n$, $v_{n+1} = p_{\omega_n}$ and note that Lemmas 2.8 and 2.14 together imply that $\{v_i \mid 1 \leq i \leq n+1\}$ is a family of pairwise orthogonal idempotents such that

$$v_i R v_j \neq 0 \quad \text{if and only if } i = j, \text{ or } i \leq n \text{ and } j = i+1. \tag{24}$$

Set $\Delta = v_{n+1} R v_{n+1}$, $D_i = v_i R v_i$ and $V_i = v_i R v_{i+1}$ for all $i = 1, 2, \ldots, n$. According to Lemma 2.14(1), Δ is a right continuous right π-ring with identity v_{n+1} all of whose idempotents are central, and V_n is a simple injective right Δ-module; moreover, $r(\Delta; V_n)$ is an essential ideal of the ring Δ, the factor ring $\Delta / r(\Delta; V_n)$ is a skew field, and the right Δ-module V_n is not embedable into Δ. By Lemma 2.7, $\overline{R} = R/r(R; V_n)$ is a skew field and $\dim(\{V_n\}_{\overline{R}}) = 1$. Note that $r(\Delta; V_n) = r(R; V_n) \cap \Delta$ and $\Delta = v_{n+1} R v_{n+1} = v_{n+1} R$ is a right ideal of R. Since $V_n v_{n+1} = V_n$, we conclude that the image $\overline{\Delta} = \Delta / r(\Delta; V_n)$ of Δ in $\overline{R}$ is a nonzero right ideal in $\overline{R}$ and so $\overline{\Delta} = \overline{R}$ forcing $\dim(\{V_n\}_{\overline{\Delta}}) = 1$.

Next, Lemma 2.7 implies that each D_i is a skew field, while Lemma 2.8(3) yields that

$$\dim(\{V_i\}_{D_{i+1}}) = 1 \text{ for all } i = 1, 2, \ldots, n-1.$$

It follows from (24) that

$$wR = \oplus_{i,j=1}^{n+1} v_i R v_j = D_1 \oplus D_2 \oplus \ldots \oplus D_n \oplus \Delta \oplus V_1 \oplus V_2 \oplus \ldots \oplus V_n. \tag{25}$$

Set $G = G_{n+1}(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$. Given

$$d_1 \in D_1, d_2 \in D_2, \ldots, d_n \in D_n, \delta \in \Delta, \ v_1 \in V_1, v_2 \in V_2, \ldots, v_n \in V_n,$$

we denote by $[d_1, \ldots, d_n, \delta, v_1, \ldots, v_n]$ the matrix

$$\begin{pmatrix} d_1 & v_1 & & & & \\ & d_2 & v_2 & & & \\ & & d_3 & v_3 & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & \cdot & d_n & v_n \\ & & & & & \delta \end{pmatrix} \in G.$$

One can easily check that

$$\begin{aligned} &[d_1, \ldots, d_n, \delta, v_1, \ldots, v_n][d_1', \ldots, d_n', \delta', v_1', \ldots, v_n'] = \\ &\quad [d_1 d_1', \ldots, d_n d_n', \delta\delta', d_1 v_1' + v_1 d_2', \ldots, d_{n-1} v_{n-1}' + v_{n-1} d_n', v_n \delta' + d_n v_n'] \end{aligned} \tag{26}$$

We now define a map $f : wR \to G$ by the rule

$$f(d_1 + \ldots + d_n + \delta + v_1 + \ldots + v_n) = [d_1, \ldots, d_n, \delta, v_1, \ldots, v_n]$$

for all $d_1 \in D_1, \ldots, d_n \in D_n, \delta \in \Delta, v_1 \in V_1, \ldots, v_n \in V_n$. It follows from (25) that f is a bijective additive map. Taking into account (26) we see that f is an isomorphism of rings. This completes the proof. □

Gathering together all direct summands of the ring R determined by maximal chains, we reduce the proof to the case when $\Omega = \emptyset$ and this case has already been considered. Thus the ring R has the decomposition described in Theorem 1.2. Since the direct sum of finitely many of right continuous right π-rings is again a right continuous right π-ring, the rest of the Theorem 1.2 follows from the following result.

Proposition 2.16 *Let $R = G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$. Then R is a right continuous right π-ring.*

Proof. Let $1 \le i \le n+1$ and let e_i be the matrix whose (i,i)-entry is equal to 1 and all the other ones are equal to 0. It is easy to see that e_jRe_{j+1} is a minimal right ideal of R while e_jR/e_jRe_{j+1} is a simple module for all $j = 1, 2, \ldots, n$. Moreover, e_jRe_{j+1} is the only proper nonzero submodule of e_jR. Note that

$$e_iRe_j \neq 0 \text{ if and only if } i = j, \text{ or } i \le n \text{ and } j = i+1. \tag{27}$$

Given a right ideal K of the ring Δ, we set $\widehat{K}$ to be the set of all matrices whose $(n+1, n+1)$-entries are from K and all the other ones are equal to 0. Clearly $\widehat{K}$ is a right ideal of R. Moreover, if U is a right ideal of R, then

$$\text{there exists a right ideal } K \text{ of } \Delta \text{ such that } Ue_{n+1} = \widehat{K}. \tag{28}$$

Given $1 \le i \le n$ and a right ideal K of Δ, we claim that

$$e_iR \text{ and } \widehat{K} \ (e_iRe_{i+1} \text{ and } \widehat{K}) \text{ are are mutually injective.} \tag{29}$$

Indeed, let U be a submodule of $\widehat{K}$. By (28), $U = \widehat{V}$ for some right ideal V of Δ contained in K. First assume that $i < n$. Since $Ue_{n+1} = U$ and $\widehat{K}e_{n+1} = \widehat{K}$ while $e_iRe_{n+1} = 0$ by (27),

$$\mathrm{Hom}(\{e_iRe_{i+1}\}_R, \widehat{K}_R) = 0 \quad \text{and} \quad \mathrm{Home}(U_R, e_iR_R) = 0.$$

As e_iRe_{i+1} is the only nonzero proper submodule of e_iR, (29) is proved in this case. Assume now that $i = n$. Let $f \in \mathrm{Hom}(\{e_nRe_{n+1}\}_R, \widehat{K})$. Then f induces a homomorphism of Δ-modules $f' : V_n \to K \subseteq \Delta$. Since right Δ-modules V_n and $\Delta/r(\Delta; V_n)$ are isomorphic and $\Delta/r(\Delta; V_n)$ is not embedable into Δ by our assumption, we conclude that $f' = 0$ forcing $f = 0$ and $\mathrm{Hom}(\{e_nRe_{n+1}\}_R, \widehat{K}) = 0$. As e_nRe_{n+1} is the only proper nonzero submodule of e_nR, we see that $\widehat{K}$ is e_nR-injective. Now let $g : U_R \to e_nR_R$ where U is a submodule of the right R-module $\widehat{K}$. Then

$$g(U) = g(Ue_{n+1}) = g(U)e_{n+1} \subseteq e_nRe_{n+1}.$$

Since V_n is an injective right Δ-module, e_nRe_{n+1} is an injective $e_{n+1}R$-module. Therefore there exists $h : \widehat{K}_{e_{n+1}R} \to (e_nRe_{n+1})_{e_{n+1}R}$ such that $h|_U = g$. As $e_{n+1}R(1-e_{n+1}) = 0$, Lemma 2.3(6) implies that $h : \widehat{K}_R \to (e_nRe_{n+1})_R$. Therefore e_nR is $\widehat{K}$-injective.

Given $1 \le i, j \le n$ with $i \neq j$, we claim that

$$e_iR \text{ and } e_jR \ (e_iRe_{i+1} \text{ and } e_jR) \text{ are are mutually injective.} \qquad (30)$$

Indeed, recall that e_jRe_{j+1} is the only proper nonzero submodule of e_jR. Let $f : (e_jRe_{j+1})_R \to e_iR$ (or $f : (e_jRe_{j+1})_R \to (e_iRe_{i+1})_R$). Since $j \neq i$, (27) yields that $e_iRe_{j+1} \neq 0$ if and only if $j = i-1$. In this case $e_iRe_{j+1} = e_iRe_i$. Therefore $e_iRe_{j+1} \subseteq e_iRe_i$ in both cases. We now have $f(e_jRe_{j+1}) = f(e_jRe_{j+1})e_{j+1} \subseteq e_iRe_{j+1} \subseteq e_iRe_i$. In order to show that $f = 0$, it is now enough to show that e_iRe_i contains no nonzero right ideals of R. Since e_iRe_i contains no nonzero right ideals of R, we conclude that $f(e_jRe_{j+1}) = 0$. Therefore $f = 0$ and so (30) is proved.

Recall that Δ_Δ is a continuous module all of whose submodules are quasi-continuous. It now follows from Lemma 2.3(6) that $\widehat{\Delta}_R = e_{n+1}R$ is a continuous module all of whose submodules are quasi-continuous. Next, both e_iR and e_iRe_{i+1} are continuous R-modules because they are uniform of finite length. Since $R = \oplus_{i=1}^{n+1} e_iR$ and e_iR and e_jR are mutually injective by (29) and (30), [15, Theorem 3.16] yields that R_R is continuous.

Let U be a right ideal of R. In order to complete the proof, it is enough to show that U_R is quasi-continuous. In view of [15, Proposition 2.7], we may assume without loss of generality that $U \subseteq_e R$. Then $e_iRe_{i+1} \subseteq U$ for all $i = 1, 2, \ldots, n$. Set $W = \sum_{i=1}^n e_iRe_{i+1}$ and note that $W \subseteq U$. Since the factor ring R/W is isomorphic to the ring $(\oplus_{i=1}^n D_n) \oplus \Delta$ and U/W is a right ideal of R/W, we conclude that there exist a partition I, J of the set $\{1, 2, \ldots, n\}$ and a right ideal K of Δ such that

$$U = (\oplus_{i\in I} e_iR) \oplus (\oplus_{j\in J} e_jRe_{j+1}) \oplus \widehat{K}.$$

Now [15, Theorem 2.3], (29) and (30) together imply that U_R is quasi-continuous, completing the proof. $\square$

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