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WHEN IS EVERY MODULE WITH ESSENTIAL SOCLE A DIRECT SUMS OF QUASI-INJECTIVES?

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ABSTRACT. In the present paper we study the structure of rings, over which essential extensions of semisimple modules are direct sums of quasi injectives. In the special case of commutative rings these rings are precisely artinian PIR and so every module over such rings is a direct sum of cyclics as characterized by Köther and Cohen-Kaplansky.

1. Introduction

All rings considered in this paper have unity and all modules are right unital. Given a module M, let $\operatorname{Soc}(M)$ and $\mathcal{E}(M)$ denote respectively the socle and the injective hull of M. The notation $N \subseteq_e M$ means that N is an essential submodule of M. Recall that M is quasi-injective iff M is invariant under $\operatorname{End}(\mathcal{E}(M))$. Next, recall that M is called q.f.d. if every factor module of M has finite uniform dimension. A ring R is said to be right q.f.d. if R_R is q.f.d.

It was proved in [1] that a ring R is right noetherian if and only if essential extensions of direct sums of injective R-modules are again direct sums of injective R-modules. The purpose of the present paper is to investigate the structure of rings R satisfying the following condition:

(1) essential extensions of semisimple R-modules are direct sums of quasi-injectives.

We show that if in addition R is a right q.f.d. ring, then R is right noetherian (Theorem 2.2).

Making use of some interesting topological arguments involving the spectrum of a ring, we show that a commutative ring R satisfies (1) if and only if it is an artinian PIR (Theorem 2.6). By invoking well-known characterisations given by Köthe [6] and Cohen-Kaplansky [3], it then follows that such rings are exactly commutative rings over which every module is a direct sum of cyclic modules.

2. Results

A family $\{N_i \mid i \in \mathcal{I}\}$ of submodules of M_R is said to be independent, if

$$N_j \cap \sum_{i \in \mathcal{I}, i \neq j} N_i = 0 \quad \text{for all } j \in \mathcal{I}.$$

Lemma 2.1. *Let R be a right q.f.d ring and let $\{S_i \mid i \in \mathcal{I}\}$ be a family of simple R-modules. Let $G = \oplus_{i \in \mathcal{I}} \mathcal{E}(S_i)$, let $E = \mathcal{E}(G)$ and let $\Lambda = \operatorname{End}(E_R)$. Then $E = \Lambda G$. In other words, the quasi-injective hull of G coincides with the injective hull E of G.*

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Proof. Given $x \in E$, it is enough to show that $x \in \Lambda G$. Since R_R is q.f.d., xR has finite uniform dimension. As $\text{Soc}(E) = \text{Soc}(G) = \oplus_{i\in\mathcal{I}} S_i \subseteq_e E$, we conclude that $\text{Soc}(xR) = \text{Soc}(E) \cap xR$ is a finitely generated essential submodule of xR. Write $\text{Soc}(xR) = \oplus_{j=1}^n T_j$, where each T_j is a simple module. Clearly there exist pairwise distinct indexes $i_1, i_2, \dots, i_n \in \mathcal{I}$ such that $T_j \cong S_{i_j}$ for all $j = 1, 2, \dots, n$. Let U be an essential closure of xR in E. Obviously U is an injective module. Next, let U_j be an essential closure of T_j in U, $j = 1, 2, \dots, n$. Then each U_j is injective and $U = \oplus_{j=1}^n U_j$ because $\oplus_{j=1}^n T_j \subseteq_e xR$. Obviously each $U_j \cong \mathcal{E}(S_{i_j})$ and so there exists $f \in \Lambda$ such that $f(\oplus_{j=1}^n \mathcal{E}(S_{i_j})) = \oplus_{j=1}^n U_j = U$. We see that $xR \subseteq U \subseteq f(G)$ and the proof is thereby complete.

Theorem 2.2. *The following statements are equivalent for a ring R:*

(a) *R is right q.f.d. and for any given family $\{S_i \mid i = 1, 2, \dots\}$ of simple modules, any essential extension of $\oplus_{i=1}^\infty \mathcal{E}(S_i)$ is the direct sum of quasi-injective modules;*
(b) *R is right noetherian.*

Proof. In view of Bass-Papp theorem [8], the implication (b)$\Longrightarrow$(b) is obvious.

(a)$\Longrightarrow$(b) Let $\{S_i \mid i = 1, 2, \dots\}$ be a family of simple modules, let $G = \oplus_{i=1}^\infty \mathcal{E}(S_i)$ and let $E = \mathcal{E}(G)$. According to [1, Theorem 1.3], a ring R is right noetherian if and only if essential extensions of direct sums countably many of injective hulls of simple modules are direct sums of injectives. Therefore it is enough to show that any submodule $G \subseteq M \subseteq E$ is a direct sum of injective modules. Let $G \subseteq M \subseteq E$. Since $\oplus_{i=1}^\infty S_i = \text{Soc}(G) \subseteq_e G \subseteq_e M$, it follows from our assumption that $M = \oplus_{k\in\mathcal{K}} U_k$ where each U_k is a nonzero quasi-injective module.

Clearly $\text{Soc}(U_k) = \text{Soc}(G) \cap U_k \subseteq_e U_k$ for all $k \in \mathcal{K}$. We claim that each U_k is an essential extension of a direct sum of injective hulls of simple modules. Since $\text{Soc}(U_k) \subseteq_e U_k$, it is enough to show that together with any simple submodule T the module U_k contains its injective hull. As $T \subseteq \oplus_{i=1}^\infty S_i$, there exists a positive integer n such that $T \subseteq \oplus_{i=1}^n S_i$. Therefore T is contained in the injective submodule $\oplus_{i=1}^n \mathcal{E}(S_i) \subseteq G \subseteq M$ and so there exists an injective submodule U of $\oplus_{i=1}^n \mathcal{E}(S_i)$ containing T as an essential submodule. Let $\pi_k : M \to U_k$ be the canonical projection. Then $\pi_k|_T$ is an identity map because $T \subseteq U_k$. Therefore $\pi_k|_U$ is a monomorphism. We see that $\pi_k(U)$ is an injective submodule of U_k containing T as an essential submodule and our claim is proved. It now follows from Lemma 2.1 that each U_k is an injective module and the proof is thereby complete.

Through the rest of the paper R denotes a commutative ring. Let $\mathcal{I}$ be an infinite set. A family $\mathcal{F}$ of subsets of $\mathcal{I}$ is said to be *an ultrafilter on $\mathcal{I}$* if

(a) $\mathcal{I} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;
(b) $U \cap V \in \mathcal{F}$ for all $U, V \in \mathcal{F}$;
(c) if $Y \in \mathcal{F}$ and $Y \subset Z \subseteq \mathcal{I}$, then $Z \in \mathcal{F}$;
(e) for any $Y \subseteq \mathcal{I}$, either $Y \in \mathcal{F}$, or $\mathcal{I} \setminus Y \in \mathcal{F}$.

Denoting by $\text{Ult}(\mathcal{I})$ the set of all ultrafilters on $\mathcal{I}$, we note that $|\text{Ult}(\mathcal{I})| = 2^{2^{|\mathcal{I}|}}$ by [4, Theorem 3.6.11]. Let $\{F_i \mid i \in \mathcal{I}\}$ be a family of fields, let $R = \prod_{i\in\mathcal{I}} F_i$ and let $\text{Spec}(R)$ be the set of all prime ideals of R. Given a subset $X \subseteq \mathcal{I}$, we consider a characteristic function $e_X : \mathcal{I} \to \{0, 1\}$ of X as an idempotent in R. Let $P \in \text{Spec}(R)$. Set

$$\mathcal{F}_P = \{X \subseteq \mathcal{I} \mid e_X \notin P\}.$$

It is well-known and easy to check that $\mathcal{F}_P$ is an ultrafilter on $\mathcal{I}$ and the correspondence $P \mapsto \mathcal{F}_P$ determines a bijection $\mathrm{Spec}(R) \to \mathrm{Ult}(\mathcal{I})$. Therefore

$$|\mathrm{Spec}(R)| = 2^{2^{|\mathcal{I}|}}. \tag{2}$$

Let $\mathcal{I} = \{1, 2, \dots\}$. It is well-known that there exists family $\{\mathcal{I}_\alpha \mid \alpha \in [0,1]\}$ of infinite subsets of $\mathcal{I}$ such that

$$|\mathcal{I}_\alpha \cap \mathcal{I}_\beta| < \infty \quad \text{for all } \alpha \neq \beta \in [0,1]. \tag{3}$$

Indeed, for any number $\alpha \in [0,1]$ there exists a sequence $r_{n_1}, r_{n_2}, \dots, r_{n_k}, \dots$ of distinct rational numbers such that $\lim_{k\to\infty} r_{n_k} = \alpha$. Set $\mathcal{I}_\alpha = \{n_1, n_2, \dots\}$. Obviously (3) is satisfied.

Lemma 2.3. *Let R be a commutative ring such that every factor ring of R with essential socle is a direct sum of quasi-injective R-modules. Then R_R is q.f.d.*

Proof. We claim that any factor ring A of R with essential socle is self-injective. Indeed, by our assumption $A = \oplus_{k\in\mathcal{K}} U_k$ where each U_k is a quasi-injective R-module. Since A is a cyclic R-module, $|\mathcal{K}| < \infty$. As each U_k is a homomorphic image of R_R, it is also a ring. Therefore each U_k is a self-injective ring and so A, being a finite direct sum of self-injective rings, is also self-injective.

Note that every factor ring of R satisfies the assumption of the lemma.

Assume to the contrary that R_R is not q.f.d. Then some factor ring A of R has infinite uniform dimension. Therefore A contains an independent family $\{C_i \mid i = 1, 2, \dots\}$ of nonzero cyclic submodules. Let M_i be a maximal submodule of C_i, $i = 1, 2, \dots$. Set $M = \sum_{i=1}^\infty M_i = \oplus_{i=1}^\infty M_i$ and note that $(\oplus_{i=1}^\infty C_i)/M \cong \oplus_{i=1}^\infty C_i/M_i$. Setting $B = A/M$ and $S_i = C_i/M_i$, $i = 1, 2, \dots$, we see that the ring B contains an independent family $\{S_i \mid i = 1, 2, \dots\}$ of simple submodules. Factoring out the complement of $\oplus_{i=1}^\infty S_i$, we reduce the proof to the case when $\oplus_{i=1}^\infty S_i \subseteq_e B$. By the above result B is a self-injective ring. Set $S = \mathrm{Soc}(B) = \oplus_{i=1}^\infty S_i$. Since B is self-injective, the canonical homomorphism of rings $B \to \mathrm{End}(S_R)$ is surjective and so $\mathrm{End}(S_R)$ is commutative. Therefore S is square free and so $\mathrm{End}(S_R) = \prod_{i=1}^\infty F_i$ where each $F_i = \mathrm{End}(S_i)$ is a field. Therefore we may assume without loss of generality that $R = \prod_{i=1}^\infty F_i$ where each F_i is a field. It now follows from (2) that

$$|\mathrm{Spec}(R)| = 2^c \tag{4}$$

where $c = |[0,1]|$.

Set $\mathcal{I} = \{1, 2, \dots\}$ and let $\{\mathcal{I}_\alpha \mid \alpha \in [0,1]\}$ be as in (3). Next, let $e_\alpha : \mathcal{I} \to \{0, 1\}$ be the characteristic function of $\mathcal{I}_\alpha$ which we consider as an element of $R = \prod_{i=1}^\infty F_i$. It now follows from (3) that $e_\alpha e_\beta \in \mathrm{Soc}(R) = \oplus_{i=1}^\infty F_i$ for all $\alpha \neq \beta \in [0,1]$. Let $Q = R/\mathrm{Soc}(R)$ and let $u_\alpha = e_\alpha + \mathrm{Soc}(R) \in Q$, $\alpha \in [0,1]$. Obviously $\{u_\alpha \mid \alpha \in [0,1]\}$ is a family of nonzero pairwise orthogonal idempotents of Q and so $\{u_\alpha Q \mid \alpha \in [0,1]\}$ is an independent family of ideals of Q. Arguing as above we get that there exists a family of fields $\{G_\alpha \mid \alpha \subset [0,1]\}$ such that the ring $G = \prod_{\alpha\in[0,1]} G_\alpha$ is a homomorphic image of Q (and so that of R). Therefore $|\mathrm{Spec}(G)| \leq |\mathrm{Spec}(R)| = 2^c$ by (4). On the other hand, $|\mathrm{Spec}(G)| = 2^{2^c}$ by (2), a contradiction. Thus R_R is q.f.d.

Making use of Lemma 2.3 and Theorem 2.2 together, we obtain the following result.

Theorem 2.4. *Let R be a commutative ring such that essential extensions of semisimple R-modules are direct sums of quasi-injectives. Then R is noetherian.*

Let m be a maximal ideal of a commutative ring R, let $S = R \setminus m$ and let $R_m = S^{-1}R$. It is well-known that R_m is a local ring and a flat R-module and, moreover, for any right R_m-modules M and N,

$$\mathrm{Hom}(M_R, N_R) = \mathrm{Hom}(M_{R_m}, N_{R_m}) \tag{5}$$

(see [2]). Since R_m is a flat R-module, any injective right R_m-module is an injective R-module canonically [8, Corollary 3.6]. It now follows from (5) that an R_m-module is quasi-injective if and only if it is quasi-injective as R-module. It is easy to see that an R_m-module is simple if and only if it is simple as an R-module. Therefore for any R_m-module M we have that

$$\mathrm{Soc}(M_{R_m}) = \mathrm{Soc}(M_R). \tag{6}$$

If N is a submodule of M, then obviously $N_{R_m} \subseteq_e M_{R_m}$ if and only if $N_R \subseteq_e M_R$.

Remark 2.5. *Let R be a commutative ring with maximal ideal m. Suppose that any right R-module with essential socle is a direct sum of quasi-injective R-modules. Then every right R_m-module with essential socle is a direct sum of quasi-injective R_m modules.*

Proof. Let M be a right R_m-module with essential socle. By (6), $\mathrm{Soc}(M_R) \subseteq_e M_R$ and so $M_R = \oplus_{k\in\mathcal{K}} U_k$ where each U_k is a quasi-injective R-module. Given $k \in \mathcal{K}$, $x \in U_k$ and $s \in R \setminus m$, we have $xs^{-1} \in M$ and so there exist $k_1, k_2, \ldots, k_n \in \mathcal{K}$ with $k_1 = k$ and $x_{k_i} \in U_{k_i}$ such that $xs^{-1} = x_{k_1} + \ldots + x_{k_n}$. Therefore $x = x_{k_1}s + \ldots + x_{k_n}s$ forcing $x_{k_2} = \ldots = x_{k_n} = 0$ and $xs^{-1} = x_{k_1} \in U_k$. Hence each U_k is an R_m-module. By the above discussion, each U_k is a quasi-injective R_m-module.

Theorem 2.6. *For a commutative ring R the following conditions are equivalent:*

(a) *Essential extensions of semisimple modules are direct sums of quasi-injectives.*
(b) *Every R-module is a direct sum of cyclic R-modules.*
(c) *R is an artinian principal ideal ring.*

Proof . (a)$\Longrightarrow$(c) Let A be a factor ring of R and let M be an A-module with essential socle. Clearly $\mathrm{Soc}(M_R) = \mathrm{Soc}(M_A)$ and so $\mathrm{Soc}(M_R) \subseteq_e M_R$. Therefore $M_R = \oplus_{k\in\mathcal{K}} U_k$ where each U_k is a quasi-injective R-module. Obviously U_k is a quasi-injective A-module and so every factor ring of R satisfies the assumption in (a).

By Theorem 2.4, R is a noetherian ring. Assume that R is local. We claim that

$$R \text{ is an artinian principal ideal ring.} \tag{7}$$

Let m be the maximal ideal of R. Then $m^2 \subset m$ by Nakayama's lemma. As it was shown in the proof of Lemma 2.3, R/m^2 is a self-injective ring because $m/m^2 = \mathrm{Soc}(R/m^2) \subseteq_e R/m^2$. On the other hand, R/m^2 is local and so R/m^2 is an indecomposable R/m^2-module. Therefore m/m^2 is a simple module. We see that there exists an element $a \in R$ such that $m = aR + m^2$ and so Nakayama's lemma yields that $m = aR$. If $a^n = 0$ for some positive integer n, then $m^n = 0$ and so R is artinian by Akizuki-Cohen theorem [7, Corollary 23.12]. Note that in this case every ideal of R contains some power of m.

Suppose that any nonzero ideal of R contains some power of m. We claim that R is a principal ideal ring and any nonzero ideal of R is equal to some power of m. Indeed, let U be a nonzero ideal of R and let $k = \max\{\ell \mid m^\ell \subseteq U\}$. Assume that $m^k \neq U$. Since R/m^k is a local artinian ring (and so its socle is essential), it is self-injective and hence the socle of R/m^k is just a minimal ideal of R/m^k. Since $(m^{k-1}/m^k)m = 0$, we conclude that m^{k-1}/m^k is the only minimal ideal of the ring R/m^k. As U/m^k is a nonzero ideal of the artinian ring R/m^k, it contains a minimal ideal of R/m^k. That is to say $U/m^k \supseteq m^{k-1}/m^k$ and so $m^{k-1} \subsetneq U$, a contradiction proving our claim. We now see that to prove (7), it is enough to show that a is nilpotent.

Assume to the contrary that a is not nilpotent. Then there exists a prime ideal p of R maximal with respect to the property $p \cap \{a, a^2, \dots, a^n, \dots\} = \emptyset$. Factoring out p, we reduce the proof to the case when R is a local domain with maximal ideal $m = aR$ such that every nonzero ideal U of R contains some power of m. By the above result R is a principal ideal ring and any ideal of R is of the form $m^k = a^k R$.

Let $F = \{r/a^n \mid r \in R, n = 0, 1, \dots\}$ be the field of quotients of R. It is well-known that $M = F/R$ is an injective R-module. Let $x_n = a^{-n} + R \subset M$. It is easy to see that

(8) every proper submodule of M has the form $x_n R$ for some $n \geq 0$.

Obviously $x_n R = \{x \in M \mid xa^n = 0\}$ and so $x_n R$ is invariant under any endomorphism of M. That is to say,

(9) $x_n R$ is a quasi-injective R-module.

Set $U = \prod_{n=1}^{\infty} x_n R$, $V = \oplus_{n=1}^{\infty} x_n R \subseteq U$ and

$$W = \{u \in U \mid ua^n = 0 \quad \text{for some } n \geq 1\}.$$

Clearly W is a submodule of U containing V. If $w \in W$ and $wa^n = 0$, then

$$w = (x_1 r_1, x_2 r_2, \dots, x_{n-1} r_{n-1}, x_n r_n, x_n r_{n+1}, \dots, x_n r_{n+k}, \dots)$$

for some $r_i \in R$. Since $x_{n+k} = x_n a^k$, $n, k = 1, 2, \dots$, we conclude that for every $\ell \geq 1$ there exists $v_\ell \in V$ and $w_\ell \in W$ such that $w = v_\ell + w_\ell a^\ell$. Therefore W/V is a divisible R-module. On the other hand, U (and so W) contains no nonzero divisible submodules. Therefore, if $W = W_1 \oplus W_2$ with $V \subseteq W_1$, then $W_2 = 0$. Indeed, $W/V = W_1/V \oplus W_2$ and so W_2 is a divisible module forcing $W_2 = 0$. In order to get a contradiction to our assumption on a, it is enough to show that there exist nonzero submodules W_1 and W_2 of W such that $W = W_1 \oplus W_2$ and $V \subseteq W_1$.

Since W is a torsion R-module, any cyclic submodule of W is artinian and so $\mathrm{Soc}(W) \subseteq_e W$. By our assumption $W = \sum_{i \in \mathcal{I}} Q_i$ where each Q_i is a nonzero quasi-injective module. Obviously, $\mathrm{Soc}(Q_i) \subseteq_e Q_i$, $i \in \mathcal{I}$.

Given $i \in \mathcal{I}$, we claim that there exists a positive integer n_i such that Q_i is isomorphic to a direct sum of copies of $x_{n_i} R$. Let S_1 and S_2 be two simple submodules of Q_i and let P_1 and P_2 be their essential closures in Q_i respectively. Since $S_1 \simeq x_1 R \simeq S_2$ and $\mathcal{E}(x_1 R) = M$, we conclude that each P_i is isomorphic to a submodule of M. As W (and so Q_i) contains no nonzero divisible modules, these submodules of M are proper. Therefore (8) implies that $P_1 \cong x_k R$ and $P_2 \cong x_\ell R$ for some positive integers k and ℓ. Assume that $k \neq \ell$, say $k > \ell$. Then $x_\ell R \subset_e x_k R$. Let P be the submodule of P_1 isomorphic to $x_\ell R$. Clearly $P \subset_e P_1$. Let $\alpha : P \to P_2$ be an isomorphism. Since Q_i is quasi-injective, α can be extended an endomorphism of $\alpha : Q_i \to Q_i$. As $P \subset_e P_1$, $\alpha|_{P_1}$ is a monomorphism and so $P_2 = \alpha(P) \subset_e \alpha(P_1)$ contradicting the choice of P_2. Therefore $k = \ell$ and so an essential closure in Q_i of any simple submodule is isomorphic to $x_{n_i} R$ where $n_i = k$.

Since $\mathrm{Soc}(Q_i) \subseteq_e Q_i$, Q_i contains an essential submodule Q which is isomorphic to a direct sum of copies of $x_{n_i}R$.

Take any $0 \neq x \in Q_i$. Then $xa^n = 0$ and $xa^{n-1} \neq 0$ for some positive integer n. Since R is a principal ideal ring, xR is isomorphic to a finite direct sum of modules of the form x_tR with $t \leq n$. As an essential closure in Q_i of any of these modules is isomorphic to $x_{n_i}R$, we conclude that $n \leq n_i$ and so $xa^{n_i} = 0$ forcing $Q_ia^{n_i} = 0$. Therefore we can consider Q_i as an R/m^{n_i}-module. Note that R/m^{n_i}-modules $x_{n_i}R$ and R/m^{n_i} are isomorphic and so (9) yields that the R/m^{n_i}-module $x_{n_i}R$ is injective. Taking into account that R/m^{n_i} is a noetherian ring, we infer from Bass-Papp theorem that direct sums of injective R/m^{n_i}-modules are injective; in particular, Q is an injective R/m^{n_i}-module. As $Q \subseteq_e Q_i$, we get $Q = Q_i$, proving our claim. We conclude that

$$W = \oplus_{j\in\mathcal{J}} T_j \text{ where each } T_j \cong x_{k_j}R. \tag{10}$$

We now claim that $|\mathcal{J}| \geq c$ where $c = |[0,1]|$. Since $|\mathcal{J}|$ is equal to the uniform dimension of $\mathrm{Soc}(W)$, it is enough to find an independent family $\{S_\alpha \mid \alpha \in [0,1]\}$ of simple submodules of W. Let $\{\mathcal{I}_\alpha \mid \alpha \in [0,1]\}$ be a family of infinite subset of $\{1,2,\dots\}$ as in (3). We define elements $z_\alpha \in U$, $\alpha \in [0,1]$, as follows: $z_\alpha = (z_{\alpha 1}, z_{\alpha 2}, \dots) \in U$ where

$$z_{\alpha i} = \begin{cases} x_1 & \text{if } i \in \mathcal{I}_\alpha, \\ 0 & \text{if } i \notin \mathcal{I}_\alpha. \end{cases}$$

Obviously each $z_\alpha \in W$. Let $A \subseteq [0,1]$ be a finite nonempty subset. Given $\beta \in [0,1] \setminus A$. $|\mathcal{I}_\beta \cap \mathcal{I}_\alpha| < \infty$ for all $\alpha \in A$. Recalling that $|\mathcal{I}_\beta| = \infty$, we conclude that $\mathcal{I}_\beta \not\subseteq \cup_{\alpha\in A}\mathcal{I}_\alpha$ and so $z_\beta R \cap \sum_{\alpha\in A} z_\alpha R = 0$. We see that $\{z_\alpha R \mid \alpha \in [0,1]\}$ is an independent family of simple submodules of W proving that $|\mathcal{J}| \geq c$.

Recall that $V = \oplus_{n=1}^{\infty} x_nR$. Therefore there exists a countable subset $\mathcal{K} \subseteq \mathcal{J}$ such that $V \subseteq W_1 = \oplus_{k\in\mathcal{K}} T_k$. Set $W_2 = \oplus_{j\in\mathcal{J}\setminus\mathcal{K}} T_j$. By (10), $W = W_1 \oplus W_2$. Since $|\mathcal{J}| \geq c$, $W_2 \neq 0$, a contradiction. Thus a is nilpotent and so R is a local artinian principal ideal ring.

Consider now the general case. Let m be any maximal ideal of R. By Remark 2.5, the ring R_m satisfies the assumption in (a) and so R_m is a local artinian principal ideal ring. It now follows from Akizuki-Cohen theorem that there exists a positive integer n and local artinian rings R_i such that $R = \prod_{i=1}^{n} R_i$. Since each R_i satisfies the assumption in (a), we conclude that each R_i is a local artinian principal ideal ring, proving (c).

(c)$\Longrightarrow$(a) By [5, Theorem 25.4.6A] every factor ring A of R is self-injective. Therefore every cyclic R-module is quasi-injective. It now follows from [5, Theorem 25.4.6A, Theorem 25.4.2] that every R-module is a direct sum of quasi-injective modules.

The equivalence of (b) and (c) is well-know (see [5, Theorem 25.4.6A, Theorem 25.4.2]).

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