# ADS modules

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**Abstract**

We study the class of *ADS* rings and modules introduced by Fuchs [F]. We give some connections between this notion and classical notions such as injectivity and quasi-continuity. A simple ring $R$ such that $R_R$ is ADS must be either right self-injective or indecomposable as a right $R$-module. Under certain conditions we can construct a unique ADS hull up to isomorphism. We introduce the concept of completely ADS modules and characterize completely ADS semiperfect right modules as direct sum of semisimple and local modules.

## 1 INTRODUCTION

The purpose of this note is to study the class of $ADS$ rings and modules. Fuchs [F] calls a right module $M$ right $ADS$ if for every decomposition $M = S \oplus T$ of $M$ and every complement $T'$ of $S$ we have $M = S \oplus T'$. Clearly any ring in which idempotents are central (in particular commutative rings or reduced rings) has the property that $R_R$ is $ADS$. Moreover, if $R$ is commutative then every cyclic $R$-module is $ADS$. We note that every right quasi-continuous module (also known as $\pi$-injective module) is right $ADS$, but not conversely. However, a right ADS module which is also CS is quasi-continuous. We provide equivalent conditions for a module to be $ADS$. A module need not have an $ADS$ hull in the usual sense but we show that, under some hypotheses, every nonsingular right module possesses a right $ADS$ hull which is unique up to isomorphism. We call a right module $M$ completely $ADS$ if each of its subfactors is $ADS$. We characterize completely $ADS$ semiperfect right modules as direct sums of semisimple and local modules. In particular we give an alternative proof of the characterizations of semiperfect $\pi c$-rings (rings whose cyclics are quasi-continuous).

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# 2 Definitions and Notations

Throughout every module will be a right module unless otherwise stated. All rings have identity and all modules are unital. A module $M$ is called *continuous* if it satisfies (C1): each complement in $M$ is a direct summand, and (C2): if a submodule $N$ of $M$ is isomorphic to a direct summand of $M$ then $N$ itself is a direct summand of $M$. A module $M$ is called *quasi-continuous* ($\pi$-injective) if it satisfies (C1) and (C3): the sum of two direct summands of $M$ with zero intersection is again a direct summand of $M$. Equivalently a module $M$ is quasi-continuous if and only if every projection $\pi_i : N_1 \oplus N_2 \longrightarrow N_i$, where $N_i$ $(i = 1, 2)$ are submodules of $M$, can be extended to $M$.

For two modules $A$ and $B$, we say that $A$ *is* $B$*-injective* if any homomorphism from a submodule $C$ of $B$ to $A$ can be extended to an homomorphism from $B$ to $A$. We note that if $A$ is $B$-injective and $A$ is contained in $B$ then $A$ is a direct summand of $B$. A module $M$ is called *semiperfect* if each of its homomorphic images has a projective cover. A submodule $N$ of a module $M$ is *small* in $M$ if for any proper submodule $P$ of $M$, $P + N \neq M$. We will write $N << M$. Let $A$ and $P$ be submodules of a module $M$. Then $P$ is called a supplement of $A$ if it is minimal with the property $A + P = M$.

A module $M$ is *discrete* if it satisfies ($\mathrm{D}_1$): for every submodule $A$ of $M$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subset A$ and $M_2 \cap A$ is small in $M$, and ($\mathrm{D}_2$): if $A$ is a submodule of $M$ such that $M/A$ is isomorphic to a direct direct summand of $M$, then $A$ is a direct summand of $M$. A module $M$ is called *quasi-discrete* if it satisfies $\mathrm{D}_1$ and $\mathrm{D}_3$: If $M_1$ and $M_2$ are summands of $M$ and $M = M_1 + M_2$ then $M_1 \cap M_2$ is a summand of $M$.

For any module $M$, $E(M)$ denotes the injective hull of $M$. We recall a useful result of Azumaya that for any two modules $M$ and $N$, if $M$ is $N$-injective then for any $R$-homomorphism $\sigma : E(N) \to E(M)$, $\sigma(N) \subseteq M$.

# 3 PROPERTIES OF ADS MODULES

We begin with a lemma which is useful in checking the ADS property of a module. This was proved by Burgess and Raphael [BR], however, for the sake of completeness, we provide the proof.

**Lemma 3.1.** *An* $R$*-module* $M$ *is ADS if and only if for each decomposition* $M = A \oplus B$*,* $A$ *and* $B$ *are mutually injective.*

*Proof.* Suppose $M$ is $ADS$. We prove $A$ is $B$-injective. Let $C$ be a submodule of $B$ and let $f : C \to A$ be an $R$-homomorphism. Set $X$=$\{c + f(c) \mid c \in C\}$.

Then $X \cap A = 0$. So $X$ is contained in a complement, say $K$, of A. Then by hypothesis, $M = A \oplus K$. The trick is to define an $R$ -homomorphism $g : B \to A$ which is a composition of the projection $\pi_K : M \to K$ along $A$ followed by the projection $\pi_A : M \to A$ along $B$ and restricting to $B$. By writing an element $c \in C$ as $c = (c + f(c)) - f(c)$, we see that $\pi_A \pi_K = f$ on $C$ and hence $\pi_A \pi_K$ is an extension of $f$.
Conversely, suppose for each decomposition $M = A \oplus B$, $A$ and $B$ are mutually injective. Let $C$ be a complement of $A$. Set $U = B \cap (A \oplus C)$ which is nonzero because $A \oplus C$ is essential in $M$. Let $\pi_A$ be the projection of $A \oplus C$ on to $A$ and $f : U \to A$ be the restriction of $\pi_A$ to $U$. This can be extended to $g : B \to A$, by assumption. Let $b \in B$ and let $D = (b - g(b))R + C$. We claim D$\cap A = 0$. Let $a \in A$ and let $a = br - g(b)r + c$ for some $c \in C$. This gives $br = a + g(b)r - c \in U$ and so $f(br) = a + g(b)r$ because $f$ is the identity on $A$ and 0 on $C$. This yields $a = 0$, proving our claim. Thus $D = C$ and hence $b - g(b) \in C$ for all $b \in B$. Therefore, $b = (b - g(b)) + g(b) \in C \oplus A$ and so $M = A \oplus B \sqsubseteq C \oplus A$, proving that $M = C \oplus A$. □

Our next proposition gives equivalent statements as to when a module is ADS analogous to characterization of quasi-continuous modules (Cf. [GJ]).

**Proposition 3.2.** *For an R-module M the following are equivalent:*

*(i) M is ADS,*

*(ii) For any direct summand $S_1$ and a submodule $S_2$ having zero intersection with $S_1$, the projection map $\pi_i : S_1 \oplus S_2 \longrightarrow S_i$ $(i = 1, 2)$ can be extended to an endomorphism (indeed a projection) of $M$,*

*(iii) If $M = M_1 \oplus M_2$ then $M_1$ and $M_2$ are mutually injective,*

*(iv) For any decomposition $M = A \oplus B$, the projection $\pi_B : M \longrightarrow B$ is an isomorphism when it is restricted to any complement $C$ of $A$ in $M$,*

*(v) For any decomposition $M = A \oplus B$ and any $b \in B$, $A$ is $bR$-injective,*

*(vi) For any direct summand $A \subseteq^{\oplus} M$ and any $c \in M$ such that $A \cap cR = 0$, $A$ is $cR$-injective.*

*Proof.* (i)⇒ (ii) Let $\hat{S}_2$ be a complement of $S_1$ containing $S_2$. Then by definition of ADS module, $M = S_1 \oplus \hat{S}_2$. Hence the canonical projections $\hat{\pi}_1 : S_1 \oplus \hat{S}_2 \longrightarrow S_1$ and $\hat{\pi}_2 : S_1 \oplus \hat{S}_2 \longrightarrow \hat{S}_2$ are clearly extensions of $\pi_1$ and $\pi_2$.
(ii)⇒ (i) Let $M = A \oplus B$ and let $C$ be a complement of $A$ in $M$. We must show that $M = A \oplus C$. By hypothesis, the projection $\pi : A \oplus C \longrightarrow C$ can be extended to an endomorphism $f : M \longrightarrow M$. We claim $f(M) \subset C$. Since $A \oplus C$ is essential in $M$, for any $0 \neq m \in M$, there exists an essential right ideal $E$ of $R$ such that $0 \neq mE \subset A \oplus C$. This gives $f(m)E = \pi(mE) \subset C$. Since $C$ is closed

in $M$, this yields $f(m) \in C$, proving our claim. We also remark that $f^2 = f$, $M = Ker(f) \oplus im(f)$ and $Ker(f) = \{m - f(m) \mid m \in M\}$. We now show that $Ker(f) = A$. For any $a \in A$, clearly $a = a - f(a) \in Ker(f)$, hence $A \subset Ker(f)$. Now let $0 \neq m - f(m) \in Ker(f)$. There exists $r \in R$ such that $0 \neq (m - f(m))r \in A \oplus C$. This implies $f[(m - f(m))r] = f(mr) - f(f(mr)) = f(mr) - f(mr) = 0$. Since $f$ extends $\pi$, this means that $0 \neq (m - f(m))r \in Ker(\pi) = A$. But $A$ being closed in $M$, we conclude $A = Ker(f)$, completing the proof.
(i)⇔(iii) This is Lemma 3.1 above.
(i)⇔(iv) Let $C$ be a complement of $A$. Then $\ker(\pi_B|_C) = 0$. Since $A \oplus C = (A \oplus C) \cap (A \oplus B) = ((A \oplus C) \cap B) + A$, we have $\pi_B(C) = \pi_B(A \oplus C) = \pi_B((A \oplus C) \cap B) = (A \oplus C) \cap B$. This gives $\pi_B(C) = B$ when $M$ is ADS. On the other hand if $\pi_B(C) = B$ then $M = A \oplus C$, hence $M$ is ADS.
(i)⇔(v) This is classical (Cf. Proposition 1.4 in [MM])
(i)⇒(vi) Consider $C$ a complement of $A$ containing $cR$. Since $M$ is ADS we have $M = A \oplus C$. Using $(v)$, this leads to $A$ being $cR$-injective.
(vi)⇒(i) This is clear since if $M = A \oplus B$, $(vi)$ implies that $A$ is $bR$-injective for all $b \in B$ and proposition 1.4 in [MM] yields that $A$ is $B$-injective. □

Let us mention the following necessary condition for a module to be ADS.

**Corollary 3.3.** *Let $M_R$ be an ADS module. For any direct summand $A \subseteq^{\oplus} M$ and any $(a, c, r) \in A \times M \times R$ such that $cR \cap A = 0$ and $ann(cr) \subseteq r.ann(a)$ there exists $a' \in A$ such that $a = a'r$. If $R$ is a right PID the converse is true.*

*Proof.* By Proposition 3.2(vi), we know that $A$ is $cR$-injective. Consider $\varphi \in Hom_R(crR, A)$ defined by $\varphi(cr) = a$. The condition on annihilators guarantees that $\varphi$ is well defined. By relative injectivity, this map can be extended to $\overline{\varphi} : cR \longrightarrow A$, and hence we get $a = \varphi(cr) = \overline{\varphi}(c)r$. We obtain the desired result by defining $a' = \overline{\varphi}(c)$.
If $R$ is a principal ideal domain then the submodules of $cR$ are of the form $crR$ for some $r \in R$. The condition mentioned in the statement of the corollary makes it possible to extend any map in $Hom_R(crR, A)$ to a map in $Hom_R(cR, A)$ for any direct summand $A \subseteq^{\oplus} M$. Invoking Proposition 3.2(vi), we can thus conclude that $M$ is ADS. □

It is known that the sum of two closed submodules of a quasi-continuous module is closed [GJ]. We prove that the direct sum of two closed submodules of an ADS module is again closed when one of them is a summand.

**Proposition 3.4.** *Let $A$ and $B$ be two closed submodules of an ADS module $M$ such that $A$ is a summand and $A \cap B = 0$, Then $A \oplus B$ is a closed submodule of $M$.*

*Proof.* Let $C$ be a complement of $A$ containing $B$. Since $M$ is ADS, we have $M = A \oplus C$. Let $x = a + c$ be in the closure of $A \oplus B$ in $M$, where $a \in A$ and $c \in C$. Since $a \in A \subseteq cl(A \oplus B)$, we have that $a \in cl(A \oplus B)$. Hence there exists an essential right ideal $E$ of $R$ such that $cE \subseteq (A \oplus B) \cap C = B$. The fact that $B$ is closed implies $c \in B$. Hence $x \in A \oplus B$, as desired. □

**Remark 3.5.** Let $A, B$ be closed submodules of an ADS module $M$ such and $A$ is a direct summand of $M$. If $A \cap B$ is a direct summand of $M$, then $A + B$ is closed. Indeed let $K$ be a complement of $A \cap B$. Since $M$ is ADS we have $M = (A \cap B) \oplus K$. Hence $A + B = A \oplus (K \cap B)$. The above proposition then yields the result.

The proposition that follows gives an interesting property of an ADS module. The original statement is due to Gratzer and Schmidt. (Cf. Theorem 9.6 in [F]). We first prove the following lemma.

**Lemma 3.6.** *Let $M = B \oplus C$ be a decomposition of $M$ with projections $\beta : M \to B$, $\gamma : M \to C$. Then $M = B \oplus C_1$ if and only if there exists $\theta \in End(M)$ such that $C_1 = (\gamma - \beta\theta\gamma)(M)$*

*Proof.* Suppose that $M = B \oplus C_1$ with projections $\beta_1$ on $B$ and $\gamma_1$ on $C_1$. We will show that $\beta_1 = \beta + \beta\theta\gamma$ and $\gamma_1 = \gamma - \beta\theta\gamma$ with $\theta = \gamma - \gamma_1$. We have $B < \ker(\theta)$, so $\theta = \theta\beta + \theta\gamma = \theta\gamma$.
If $m = b + c = b_1 + c_1$, where $b, b_1 \in B, c \in C, c_1 \in C_1$. then $\theta(m) = c - c_1 = b_1 - b \in B$. Thus $\beta\theta = \theta$. Hence $\gamma_1 = \gamma - \theta = \gamma - \beta\theta\gamma$. Also $\beta_1 = 1_A - \gamma_1 = \beta + \gamma - \gamma_1 = \beta + \beta\theta\gamma$.

Conversely, if $\beta_1, \gamma_1$ are defined as above, that is $\beta_1 = \beta + \beta\theta\gamma$ and $\gamma_1 = \gamma - \beta\theta\gamma$ for any $\theta \in End(M)$, then $\beta_1 + \gamma_1 = 1_A$, $\beta_1^2 = \beta_1$, $\gamma_1^2 = \gamma_1$, $\beta_1\gamma_1 = \gamma_1\beta_1 = 0$. Therefore, $M = \beta_1 M \oplus \gamma_1 M$. Since $\beta_1(M) \subset B$ and $\beta_1(b) = \beta(b) = b$ for $b \in B$, we have $M = B \oplus (\gamma - \beta\theta\gamma)(M)$, as required. □

Using the same notations as in the previous lemma we state the following corollary.

**Corollary 3.7.** *A module $M$ is ADS if and only if for any decomposition $M = B \oplus C$ the complements of $B$ in $M$ are all of the form $(\gamma - \beta\theta\gamma)(M)$ for some $\theta \in End(M)$.*

**Proposition 3.8.** *Let $M = B \oplus C$ be a decomposition of an ADS $R$-module $M$. Let $\beta$ and $\gamma$ be projections on $B$ and $C$ respectively. Then the intersection $D$ of all the complements of $B$ is the maximal fully invariant submodule of $M$ which has zero intersection with $B$.*

*Proof.* Let $\theta \in End(M)$. Then $C_1 = (\gamma - \beta\theta\gamma)M$ is again a complement of $B$. For $c \in D$ we have $(\gamma - \beta\theta\gamma)(c) = c$ and $\gamma c = c$, because $c \in C_1 \cap C$. Hence $\beta\theta c = 0$ and $\theta c \in C$. This holds for all complements $C$, so $\theta c \in D$, so $D$ is fully invariant in $M$ with $D \cap B = 0$. On the other hand, assume $X$ is fully invariant with $X \cap B = 0$. Since $M = B \oplus C$, and $\pi_B(X) \subseteq X$ and $\pi_C(X) \subseteq X$, this leads to $X = (X \cap B) \oplus (X \cap C) = X \cap C$. Hence $X < C$. Since $M$ is ADS this holds for any complement of $B$ in $M$, and hence $X \subseteq D$. □

It is known that an indecomposable regular ring which is right continuous is right self-injective (Cf. Corollary 13.20 in [G] ). The following theorem is a generalization of this result for simple rings without the assumption of regularity. We may add that an indecomposable two-sided continuous regular ring is simple (Cf. [G] Corollary 13.26).

**Theorem 3.9.** *Let $R$ be an ADS simple ring. Then either $R_R$ is indecomposable or $R$ is a right self-injective regular ring.*

*Proof.* Let $Q$ be the right maximal quotient of $R$ which is regular right self-injective. Since $R$ is right (left) nonsingular $E(R) = Q$. Suppose $R$ is not right indecomposable and let $e$ be a non trivial idempotent. Then since $R$ is ADS $eR$ is $(1-e)R$-injective (Cf. Lemma 3.1). Furthermore, since $Hom((1-e)Q, eQ) \cong eQ(1-e)$, $(eQ(1-e)(1-e)R \subseteq eR$. and so $R = R(1-e)R \subset Q(1-e)R$.This yields, $1 \in Q(1-e)R$ . Therefore $Q = Q(1-e)R$, and so $eQ = eR$. Similarly $(1-e)Q = (1-e)R$ hence $R = Q$, i.e. $R$ is a right self-injective regular ring. □

**Corollary 3.10.** *A simple regular right continuous ring is right self-injective.*

# 4 ADS HULLS

We now proceed to construct an *ADS* hull of a nonsingular module. Burgess and Raphael (Cf. [BR]) claimed that an example can be constructed of a finite dimensional module over a finite dimensional algebra which has no *ADS* hull. We show that, under some circumstances, such an ADS hull does exist.

**Lemma 4.1.** *Suppose $M$ is nonsingular. Then $M$ is ADS iff for every decomposition $E(M) = E_1 \oplus E_2$ where $E_1 \cap M$ is a direct summand of $M$, then $M = (E_1 \cap M) \oplus (E_2 \cap M)$.*

*Proof.* Suppose $M$ is ADS. We may write $M = (E_1 \cap M) \oplus K$ where $K$ is a complement of $E_1 \cap M$. Let $e_i : (E_1 \cap M) \oplus (E_2 \cap M) \longrightarrow E_i \cap M$ be the projection map. Then by Proposition 3.2(ii) there exists $e_i^* : M \longrightarrow M$ that extends $e_i$. Let $\pi_i : E_1 \oplus E_2 \longrightarrow E_i$ be the natural projection. Since $E(M)$ is injective we can further extends $e_i*$ to $e_i^{**} \in End(E(M))$. We claim $e_i^{**}$ is an idempotent in $End(E(M)$. Indeed let $x \neq 0$ be any element in $E(M)$ and $A$ an essential right ideal of $R$ such that $0 \neq xA \subseteq M$ We have $(e_i^{**})^2(x)A = (e_i^{**})^2(xA) = (e_i^*)^2(xA)) = e_i^*(xA) = e_i^{**}(xA) = e_i^{**}(x)A$. This yields the claim, since $M$ is nonsingular. Thus $e_i^{**}(E(M)) = \pi_i(E(M)) = E_i$. Now $M \subseteq_e E(M) = E_1 \oplus E_2$ implies $E_1 \cap M \subseteq_e (E_1 \oplus E_2) \cap E_1$. Similarly $E_2 \cap M \subseteq_e E_2$ and so $e_i^{**} = \pi_i$ on $E_1 \cap M \oplus E_2 \cap M \subseteq_e M \subseteq_e E(M)$. Since $M$ is nonsingular $e_i^{**} = \pi_i$ on $E(M)$. In particular, $\pi_i(M) \subseteq M$ and so $M = (\pi_1 + \pi_2)(M) \subseteq \pi_1(M) \oplus \pi_2(M) \subset (E_1 \cap M) \oplus (E_2 \cap M)$.

Conversely, let $M = A \oplus B$ and $C$ be a complement of $A$. We must show that $M = A \oplus C$. Since $A \oplus C <_e M$, we get $E(M) = E(A) \oplus E(C)$. Since both $A$ and $C$ are closed in $M$, we have $E(A) \cap M = A$ and $E(C) \cap M = C$. Since $A$ is a direct summand of $M$ we have, thanks to the hypothesis, $M = (E(A) \cap M) \oplus (E(C) \cap M) = A \oplus C$, as desired. □

**Theorem 4.2.** *Let $M$ be a right $R$-module. Then $M$ is ADS if and only if for every $e = e^2$, $f = f^2 \in End(E(M))$ with $eM \subset M$ and $fE(M) = eE(M)$, we have $fM \subset M$.*

*Proof.* Let us prove necessity: $(1-f)(E(M)) \cap M \subseteq_e (1-f)(E(M))$ and $f(E(M)) \cap M \subseteq_e f(E(M))$. Thus $((1-f)(E(M)) \cap M) \oplus (f(E(M)) \cap M) \subseteq_e M$. We claim $f(E(M)) \cap M = e(M)$. Note first that $e(E(M)) \cap M = f(E(M)) \cap M$. Clearly $eE(M) \cap M \subseteq eM$. Let $C = (1-f)(E(M)) \cap M$. Then $C \oplus eM \subseteq_e M$. Because $eM$ is closed $C$ is a complement of $eM$ in $M$ (Cf. Lemma 6.32 in Lam's book). Because $M$ is ADS we have $M = e(M) \oplus C$. Let $g$ be the projection of $eM$ along $C$, so that $g(M) = e(M)$. Now $g(M) = e(M) \subseteq f(E(M))$. This gives $eM = (M) = fg(M) = fe(M)$. Since $C$ is contained in $(1-f)(E(M))$, $f(C) = 0$. Then $fM = f(C \oplus eM) = eM \subseteq M$.
Conversely, let $M = eM \oplus (1-e)(M)$ and $C$ be a complement of $e(M)$ in $M$. We want to show $M = e(M) \oplus C$. Now, $C \oplus e(M) \subseteq_e M$ and so $E(C) \oplus E(eM) = E(M)$. Hence $E(C) \oplus eE(M) = E(M)$. Let $f$ be the projection on $eE(M)$ along $E(C)$. We have $f(E(M)) = e(E(M))$ and $E(C) = (1-f)(E(M))$. By hypothesis we have $f(M) \subseteq M$. Let $m$ be in $M$. Then $m \in M = E(C) \oplus f(E(M))$, say $m = c + f(m)$, where $c \in E(C)$. $c = m - f(m) \in E(C) \cap M = C$, because $C$ is closed. We conclude that $M = C \oplus e(M)$. □

We may recall that any endomorphism $f \in End_R(M)$ of a nonsingular module $M$ can be uniquely extended to an endomorphism $f^*$ of its injective hull $E(M)$. Let us mention moreover that if $f = f^2$ then $f^* = (f^*)^2$. Under these notations we obtain the following corollary.

**Corollary 4.3.** *Let $M$ be a right nonsingular $R$-module. $M$ is ADS if and only if for every $e = e^2 \in End(M)$ and $f = f^2 \in End(E(M))$ with $fE(M) = e^*E(M)$, we have $fM \subset M$.*

We are now ready to show, that under some circumstances, an ADS hull can be constructed for a nonsingular module. For a nonsingular right $R$-module $M$, we continue to let $e^*$ denote the unique extension of $e^2 = e \in End(M)$ to the injective hull $E(M)$ of $M$.

**Theorem 4.4.** *Let $M_R$ be a nonsingular right $R$-module. Let $\overline{M}$ denote the intersection of all the ADS submodules of $E(M)$ containing $M$. Suppose that for any $e^2 = e \in End(\overline{M})$ and for any ADS submodule $N$ of $E(M)$ containing $M$ we have $e^*(N) \subset N$. Then, $\overline{M}$ is, up to isomorphism, the unique ADS hull of $M$.*

*Proof.* Let $\Omega$ be the set of ADS submodules $N$ such that $M < N < E(M)$. Then $\overline{M} = \bigcap_{N\in\Omega} N$. We claim that $\overline{M}$ is ADS. Clearly $E(\overline{M}) = E(M)$. Let $e = e^2 \in End_R(\overline{M}), f^2 = f \in End(E(M))$ such that $e(\overline{M}) \subseteq \overline{M}$ and $f(E(M)) = e^*(E(M))$. Since $M$ is nonsingular and $e(\overline{M}) \subseteq \overline{M}$, we have $e(N) \subseteq N$ for every $N \in \Omega$. So, for every $N \in \Omega$, $f(N) \subset N$ because $N$ is ADS. Let $x \in \overline{M}$. Then $x \in N$ for every $N \in \Omega$. Hence $f(x) \in N$ for every $N \in \Omega$. Therefore $f(x) \in \bigcap_{N\in\Omega} N = \overline{M}$, that is $f(\overline{M}) \subseteq \overline{M}$, proving our claim. □

**Remarks 4.5.** Let us remark that the condition stated in the above theorem is in particular fulfilled if we consider the ADS hull of a nonsingular ring. Indeed in this case we consider the ADS rings between $R$ and $Q := E(R)$ and projections are identified with idempotents of the rings. Of course, these idempotents remain idempotents in overrings.

# 5 COMPLETELY *ADS* MODULES

**Theorem 5.1.** *Let $M = \oplus_{i\in I} M_i$ be a decomposition of a module $M$ into a direct sum of indecomposable modules $M_i$. Suppose $M$ is completely ADS. Then*

*(i) For every $(i, j) \in I^2$, $i \neq j$, $M_i$ is $M_j$-injective,*

*(ii) If $(i, j) \in I^2$, $i \neq j$ are such that $Hom_R(M_i, M_j) \neq 0$, then $M_j$ is simple.*

*(iii) $M = S \oplus T$ where $S$ is semisimple and $T = \oplus_{j\in J\subset I} M_j$ is a direct sum of indecomposable modules. Moreover, for any $\theta \in End(M)$ we have $\theta(S) \subset S$ and for $j \in J$, $\theta(M_j) \subseteq M_j \oplus S$.*

*Proof.* Since the ADS property is inherited by direct summands, statement $(i)$ is an obvious consequence of Lemma 3.1.
$(ii)$ For convenience, let us write $i = 1$, $j = 2$ and suppose that $0 \neq \sigma \in Hom_R(M_1, M_2)$. We have $\sigma(M_1) \oplus M_2 \oplus \ldots \cong M_1/\ker(\sigma) \oplus M_2 \oplus \cdots = M/\ker(\sigma)$ is ADS, by assumption. Hence $\sigma(M_1)$ is $M_2$-injective and, since $\sigma(M_1) \subseteq M_2$, we get that $\sigma(M_1)$ is a direct summand of $M_2$. But $M_2$ is indecomposable, hence

$\sigma(M_1) = M_2$. We conclude that $M_2 \oplus M_2 = \sigma(M_1) \oplus M_2$ is $ADS$. This means that $M_2$ is $M_2$-injective i.e. $M_2$ is quasi-injective.

Let us now show that for any $0 \neq m_2 \in M_2$, $m_2R = M_2$. Since $\sigma(M_1) = M_2$, there exists $m_1 \in M_1$ such that $\sigma(m_1) = m_2$. We remark that $\sigma(m_1R) \oplus M_2 = \frac{m_1R}{\ker\sigma\cap m_1R} \oplus M_2 = \frac{m_1R\oplus M_2}{\ker\sigma\cap m_1R}$ is a submodule of $\frac{M}{\ker\sigma\cap m_1R}$. Since $M$ is completely ADS, we conclude that $\sigma(m_1R) \oplus M_2$ is ADS. As earlier in this proof, relative injectivity and indecomposability lead to $\sigma(m_1R) = M_2$. Hence $m_2R = M_2$, as desired.

$(iii)$ Let $I_1$ consist of those $i \in I$ such that there exists $j \in I$, $j \neq i$ with $Hom_R(M_j, M_i) \neq 0$. We define $S := \oplus_{i\in I_1} M_i$ and $T := \oplus_{j\in J} M_j$ where $J := I \setminus I_1$. Statement $(ii)$ above implies that $M = S \oplus T$ where $S$ is semisimple and $T$ is a sum of indecomposable modules. Moreover if $j \in J$, then for any $i \in I$, $i \neq j$, we have $Hom_R(M_i, M_j) = 0$. It is clear that, for any $\theta \in End(M)$ we must have $\theta(S) \subset S$. For $j \in J$ and $x \in M_j$ let us write $\theta(x) = y + z$, where $z \in S$ and $y \in T$. Since, for $l \in J$, $l \neq j$, $Hom_R(M_j, M_l) = 0$, we have $\pi_l\theta(x) = 0$, where $\pi_l : M \to M_l$ is the natural epimorphism. Thus $\pi_l(y) = 0$. This shows that $y \in M_j$, as required. □

Oshiro's theorem states that any quasi-discrete module is a direct sum of indecomposable modules (Cf. [MM] Theorem 4.15). Hence the above theorem 5.1 applies to completely ADS quasi-discrete modules. In general for a quasi-discrete module we have the following theorem:

**Theorem 5.2.** *Let $M$ be a completely ADS quasi-discrete module. Then $M$ can be written as $M = S \oplus M_1 \oplus M_2$, where $S$ is semisimple, $M_1$ is a direct sum of local modules and $M_2$ is equal to its own radical.*

*Proof.* Corollary 4.18 and Proposition 4.17 in [MM] imply that $M = N \oplus M_2$ where $N$ has a small radical and $M_2$ is equal to its own radical. Theorem 5.1 applied to $N$ yields the conclusion. □

We now apply the previous theorem to the case of semiperfect modules.

**Theorem 5.3.** *Let $M$ be a semiperfect module with a completely ADS projective cover $P$. Then $M$ can be presented as $M = S \oplus T$ where $S$ is semisimple and $T$ is a sum of local modules. Moreover any partial sum in this decomposition contains a supplement of the remaining terms.*

*Proof.* Clearly $P$ is semiperfect and projective (Cf. Theorem 11.1.5 in [K]). Combining the statements in 42.5 in [W] and corollary 4.54 in [MM], we get that $P$ is discrete and is a direct sum of local modules. The remark preceding the present theorem then implies that we can write $P = S' \oplus T'$ where $S'$ is semisimple and $T'$ is a direct sum of indecomposable local modules. Let $\sigma$ be an onto homomorphism from $P$ to $M$ with small kernel $K$. We thus have $M = \sigma(S') + \sigma(T')$. Since homomorphic images of $M$ have projective covers, Lemma 4.40 [MM] shows that $\sigma(T')$

contains a supplement $X$ of $\sigma(S')$. In particular, we have $\sigma(S') \cap X << X$. Since $\sigma(S')$ is semisimple we conclude that $\sigma(S') \cap X = 0$ and hence $M = \sigma(S') \oplus \sigma(T')$. Since homomorphic images of a local module are still local, we conclude that the terms appearing in $\sigma(T')$ are local modules. The last statement is a direct consequence of the Lemma 4.40 [MM]. □

Let us mention that local rings which are not uniform give examples of semiperfect completely ADS modules which are not CS and hence not quasi-continuous.

The following corollary characterizes semiperfect $\pi c$-rings providing a new proof of Theorem 2.4 in [GJ].

**Theorem 5.4.** *Let $R$ be a semiperfect ring such that every cyclic module is quasi-continuous. Then $R = \oplus_{i \in I} A_i$ where each $A_i$, $i \in I$ is simple artinian or a valuation ring.*

*Proof.* Since $R$ is semiperfect $R = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ a direct sum of indecomposable right ideals. In view of the fact that quasi-continuous modules are ADS, Theorem 5.1 gives a decomposition $R = e_1R \oplus e_2R \oplus \cdots \oplus e_kR \oplus \cdots \oplus e_nR$ where $e_iR$ are simple right ideals for $1 \leq i \leq k$ and $e_jR$ are local right ideals for $k < j \leq n$. Let $\sigma$ be an homomorphism from $e_sR$ to $e_tR$ for some $1 \leq s, t \leq n$. Then $e_sR/\ker\sigma$ embeds in $e_tR$. Since $R/\ker(\sigma)$ is quasi-continuous, $e_sR/\ker\sigma$ is $e_tR$-injective and hence $e_sR/\ker(\sigma)$ splits in $e_tR$. This shows that either $e_sR/\ker(\sigma) \cong e_tR$ or $\ker(\sigma) = e_sR$, that is $\sigma = 0$. Since $e_tR$ is projective, if $e_sR/\ker(\sigma) \cong e_tR$, then $\ker(\sigma)$ splits in $e_sR$, thus $\ker(\sigma) = 0$. In short we get that if $\sigma \neq 0$ then $e_sR \cong e_tR$, the latter isomorphism implies $e_sR$ and $e_tR$ are minimal right ideals (Cf. Lemma 2.3 in [GJ]). By grouping the right ideals $e_iR$ according to their isomorphism classes, we get $R = A_1 \oplus A_2 \oplus \cdots \oplus A_l$, $l \leq n$, where each $A_i$ is either a simple artinian ring or a local ring. We claim that if $A_i$ is a local ring then it is a valuation ring. We thus have to show that any pair of two nonzero submodules $C, D$ of the ring $A_i$ are comparable. Let us consider the right submodules $\frac{C}{C \cap D}$ and $\frac{D}{C \cap D}$ of $\frac{R}{C \cap D}$. Since $A_i/(C \cap D)$ is a local quasi-continuous it is uniform, but $C/(C \cap D) \bigcap D/(C \cap D) = 0$. Therefore $C/(C \cap D)$ or $D/(C \cap D) = 0$ hence $C$ and $D$ are indeed comparable. □

Let us conclude this paper with some questions:

1. It is known that if $R_R$ and ${}_RR$ are both CS then $R$ is Dedekind finite. What could be the analogue of this for ADS modules?

2. Does a directly finite ADS module have the internal cancellation property? (Cf. Thm 2.33 in [MM], for the quasi-continuous case).

3. What can be said of a module which is ADS and has the $C_2$ property?

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