ALGEBRAS AND SEMIGROUPS OF LOCALLY SUBEXPONENTIAL GROWTH

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Abstract. We prove that a countable dimensional associative algebra (resp. a countable semigroup) of locally subexponential growth is M_∞-embeddable as a left ideal in a finitely generated algebra (resp. semigroup) of subexponential growth. Moreover, we provide bounds for the growth of the finitely generated algebra (resp. semigroup). The proof is based on a new construction of matrix wreath product of algebras.

1. Introduction

G. Higman, H. Neumann and B. H. Neumann [6] proved that every countable group embeds in a finitely generated group. The papers [9], [10], [11], [13] show that some important properties can be inherited by these embeddings. In the recent remarkable paper [2], L. Bartholdi and A. Erschler proved that a countable group of locally subexponential growth embeds in a finitely generated group of subexponential growth.

Following the paper [6], A. I. Malcev [7] showed that every countable dimensional algebra over a field embeds into a finitely generated algebra.

Let A be an associative algebra over a ground field F. Let X be a countable set. Consider the algebra $M_\infty(A)$ of $X \times X$ matrices over A having finitely many nonzero entries. Clearly, there are many ways the algebra A embeds into $M_\infty(A)$.

Key words and phrases. growth function, associative algebra, wreath product.

We say that an algebra A is M_∞-embeddable in an algebra B if there exists an embedding $\varphi : M_\infty(A) \to B$. The algebra A is M_∞-embeddable in B as a (left, right) ideal if the image of φ is a (left, right) ideal of B.

The construction of a wreath product in [1] implied the following refinement of the theorem of Malcev: every countable dimensional algebra is M_∞-embeddable in a finitely generated algebra as an ideal.

In this paper, we

(1) prove the analog of Bartholdi-Erschler theorem for algebras: every countable dimensional associative algebra of locally subexponential growth is M_∞-embeddable in a 2-generated algebra of subexponential growth as a left ideal;
(2) provide estimates for the growth of the finitely generated algebra above;
(3) consider the case of a countable dimensional algebra of Gelfand-Kirillov dimension $\leq d$ and M_∞-embed it in a 2-generated algebra of Gelfand-Kirillov dimension $\leq d+2$ as a left ideal;
(4) establish the similar results for semigroups.

J. Bell, L. Small and A. Smoktunowicz [3] embedded an arbitrary, countable dimensional algebra of Gelfand-Kirillov dimension d in a 2-generated algebra of Gelfand-Kirillov dimension $\leq d+2$.

2. Definitions and Main Results

Let A be an associative algebra over a ground field F that is generated by a finite dimensional subspace V. Let V^n denote the span of all products $v_1 \cdots v_k$, where $v_i \in V$, $k \leq n$. Then $V^1 \subseteq V^2 \subseteq \cdots$ and $\bigcup\limits_{n\geq 1} V^n = A$. The function $g(V,n) = \dim_F V^n$ is called the growth function of A.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and the set of positive integers, respectively. Given two functions $f, g : \mathbb{N} \to [1,\infty)$, we say that $f \preceq g$ (f is asymptotically less than or equal to g) if there exists a constant $c \in \mathbb{N}$ such that $f(n) \leq cg(cn)$ for all $n \in \mathbb{N}$. If $f \preceq g$ and $g \preceq f$, then f and g are said to beasymptotically equivalent, i.e., $f \sim g$.

We say that a function f is weakly asymptotically less than or equal to g if for arbitrary $\alpha > 0$ we have $f \preceq gn^{\alpha}$ (denoted $f \preceq_w g$).

If V, W are finite dimensional generating subspaces of A, then $g(V,n) \sim g(W,n)$. We will denote the class of equivalence of $g(V,n)$ as g_A.

A function $f : \mathbb{N} \to [1, \infty)$ is said to be subexponential if for an arbitrary $\alpha > 0$

$$\lim_{n\to\infty} \frac{f(n)}{e^{\alpha n}} = 0.$$

For a growth function $f(n)$ of an algebra, it is equivalent to $f(n) \precnapprox e^n$ and to $\lim\limits_{n\to\infty} \sqrt[n]{f(n)} = 1$.

If a function $f(n)$ is subexponential but $n^{\alpha} \precnapprox f(n)$ for any $\alpha > 0$, then $f(n)$ is said to be intermediate. In the seminal paper [5], R. I. Grigorchuk constructed the first example of a group with an intermediate growth function. Finitely generated associative algebras with intermediate growth functions are more abundant (see [12]).

A not necessarily finitely generated algebra A is of locally subexponential growth if every finitely generated subalgebra B of A has a subexponential growth function.

We say that the growth of A is locally (weakly) bounded by a function $f(n)$ if for an arbitrary finitely generated subalgebra of A, its growth function is $\preceq f(n)$ (resp. $\preceq_w f(n)$).

A function $h(n)$ is said to be superlinear if $\dfrac{h(n)}{n} \to \infty$ as $n \to \infty$.

The main result of this paper is:

Theorem 1. *Let $f(n)$ be an increasing function. Let A be a countable dimensional associative algebra whose growth is locally weakly bounded by $f(n)$. Let $h(n)$ be a superlinear function. Then the algebra A is M_∞-embeddable as a left ideal in a 2-generated algebra whose growth is weakly bounded by $f(h(n))n^2$.*

We then use Theorem 1 to derive an analog of the Bartholdi-Erschler theorem (see [2]).

Theorem 2. *A countable dimensional associative algebra of locally subexponential growth is M_∞-embeddable in a 2-generated algebra of subexponential growth as a left ideal.*

A finitely generated algebra A has polynomially bounded growth if there exists $\alpha > 0$ such that $g_A \preceq n^\alpha$. Then

$$GK\dim(A) = \inf\{\alpha > 0 | g_A \preceq n^\alpha\}$$

is called the Gelfand-Kirillov dimension of A. If the growth of A is not polynomially bounded, then we let $GK\dim(A) = \infty$. If the algebra A is not finitely generated then the Gelfand-Kirillov dimension of A is defined as the supremum of Gelfand-Kirillov dimensions of all finitely generated subalgebras of A.

J. Bell, L. Small, A. Smoktunowicz [3] proved that every countable dimensional algebra of Gelfand-Kirillov dimension $\leq n$ is embeddable in a 2-generated algebra of Gelfand-Kirillov dimension $\leq n + 2$.

We use Theorem 1 to prove

Theorem 3. *Every countable dimensional algebra of Gelfand-Kirillov dimension $\leq n$ is M_∞-embeddable in a 2-generated algebra of Gelfand-Kirillov dimension $\leq n + 2$ as a left ideal.*

The proof of Theorem 1 is based on a new construction of the matrix wreath product $A \wr F[t^{-1}, t]$. We view it as an analog of the wreath product of a group G with an infinite cyclic group $\mathbb{Z}$ that played an essential role in the Bartholdi-Erschler proof [2].

The construction is similar to that of [1], though not quite the same.

Analogs of Theorems 1, 2, 3 are true also for semigroups. Recall that T. Evans [4] proved that every countable semigroup is embeddable in a finite 2-generated semigroup.

We will formulate analogs of Theorems 1, 2, 3 for semigroups for the sake of completeness, omitting some definitions that are similar to those for algebras.

Let P be a semigroup. Consider the Rees type semigroup

$$M_\infty(P) = \bigcup_{i,j\in\mathbb{Z}} e_{ij}(P),$$

$e_{ij}(a)e_{kq}(b) = \delta_{jk}e_{iq}(ab)$; $a, b \in P$. We say that a semigroup P is M_∞-embeddable in a semigroup S if there is an embedding $\varphi : M_\infty(P) \to S$. We say that P is M_∞-embeddable in S as a (left) ideal if $\varphi(M_\infty(P))$ is a (left) ideal of S.

Theorem 1′. *Let $f(n)$ be an increasing function. Let P be a countable semigroup whose growth is locally weakly bounded by $f(n)$. Let $h(n)$ be a superlinear function. Then the semigroup P is M_∞-embeddable as a left ideal in a finitely generated semigroup whose growth is weakly bounded by $f(h(n))n^2$.*

Theorem 2′. *A countable semigroup of locally subexponential growth is M_∞-embeddable in a finitely generated semigroup of subexponential growth as a left ideal.*

Theorem 3′. *Every countable semigroup of Gelfand-Kirillov dimension $\leq d$ is M_∞-embeddable in a finitely generated semigroup of Gelfand-Kirillov dimension $\leq d + 2$ as a left ideal.*

3. Matrix Wreath Products

As above, let $\mathbb{Z}$ be the ring of integers. For an associative F-algebra A, consider the algebra $\widetilde{M}_\infty(A)$ of infinite $\mathbb{Z} \times \mathbb{Z}$ matrices over A having finitely many nonzero entries in each column. The subalgebra of $\widetilde{M}_\infty(A)$ that consists of matrices having finitely many nonzero entries is denoted as $M_\infty(A)$. Clearly, $M_\infty(A)$ is a left ideal of $\widetilde{M}_\infty(A)$.

For an element $a \in A$ and integers $i, j \in \mathbb{Z}$, let $e_{ij}(a)$ denote the matrix having a in the position (i, j) and zeros everywhere else. For a matrix $X \in \widetilde{M}_\infty(A)$, the entry at the position (i, j) is denoted as $X_{i,j}$.

The vector space $\widetilde{M}_\infty(A)$ is a bimodule over the algebra $F[t^{-1}, t]$ via the operations: if $X \in \widetilde{M}_\infty(A)$ then $(t^kX)_{i,j} = X_{i-k,j}$ for all $i, j, k \in \mathbb{Z}$. In other words left multiplication by t^k moves all rows of X up by k steps. Similarly, $(Xt^k)_{i,j} = X_{i,j+k}$, so multiplication by t^k on the right moves all columns of X left by k steps.

Consider the semidirect sum

$$A \wr F[t^{-1}, t] = F[t^{-1}, t] + \widetilde{M}_\infty(A)$$

and its subalgebra

$$A \,\bar{\wr}\, F[t^{-1}, t] = F[t^{-1}, t] + M_\infty(A).$$

These algebras are analogs of the unrestricted and restricted wreath products of groups with $\mathbb{Z}$.

Let A be a countable dimensional algebra with 1. We say that a matrix $X \in \widetilde{M}_\infty(A)$ is a generating matrix if the entries of X generate A as an algebra.

Let $X \in \widetilde{M}_\infty(A)$ be a generating matrix. Consider the subalgebra of $A \wr F[t^{-1}, t]$ generated by $t^{-1}, t, e_{00}(1), X$,

$$S = \langle t^{-1}, t, e_{00}(1), X \rangle.$$

Lemma 4. *The algebra $M_\infty(A)$ is a left ideal of S.*

Proof. Suppose that entries $X_{i_1,j_1}, X_{i_2,j_2}, \cdots$ generate A. We have $e_{ij}(1) = t^i e_{00}(1) t^{-j}$ and

$$e_{00}(X_{i_k,j_k}) = e_{0i_k}(1) X e_{j_k 0}(1) = e_{00}(1) t^{-i_k} X t^{j_k} e_{00}(1) \in S.$$

This implies that $e_{00}(A) \subseteq S$ and therefore $e_{ij}(A) = t^i e_{00}(A) t^{-j} \subseteq S$. We proved that $M_\infty(A) = \sum_{i,j \in \mathbb{Z}} e_{ij}(A) \subseteq S$.

Since $M_\infty(A)$ is a left ideal in the algebra $A \wr F[t^{-1}, t]$ the assertion of the lemma follows. □

For a fixed $n \in \mathbb{Z}$ by n^{th} diagonal we mean all integers pairs (i, j) such that $i - j = n$.

Lemma 5. *If a generating matrix X has finitely many nonzero diagonals, then $M_\infty(A)$ is a two-sided ideal in S.*

Proof. If a matrix X has finitely many nonzero diagonals, then $M_\infty(A) X \subseteq M_\infty(A)$, which implies the claim. □

We say that a sequence $c = (a_1, a_2, a_3, \cdots)$ of elements of the algebra A is a generating sequence if the elements $a_1, a_2, \cdots$ generate A.

For the sequence c, consider the matrix $c_{0,N} = \sum_{j=1}^{\infty} e_{0j}(a_j) \in \widetilde{M}_\infty(A)$. This matrix has elements a_j at the positions $(0, j)$, $j \geq 1$, and zeros everywhere else.

Consider the subalgebra

$$A^{(c)} = \langle t, t^{-1}, e_{00}(1), c_{0,N} \rangle$$

of the matrix wreath product $A \wr F[t^{-1}, t]$. As shown in Lemma 4, the countable dimensional algebra A is M_∞-embeddable in the finitely generated algebra $A^{(c)}$ as a left ideal.

When speaking about algebras $A^{(c)}$ we always consider the generating subspace $V = span(t, t^{-1}, e_{00}(1), c_{0,N})$ and denote $g(V, n) = g(n)$.

For a generating sequence $c = (a_1, a_2, \cdots)$, let W_n be the subspace of A spanned by all products $a_{i_1} \cdots a_{i_r}$ such that $i_1 + \cdots + i_r \leq n$.

Denote

$$M_{[-n,n]\times[-n,n]}(W_n) = \sum_{-n \leq i,j \leq n} e_{ij}(W_n),$$

$$M_{[-n,n]\times 0}(W_n) = \sum_{i=-n}^{n} e_{i0}(W_n).$$

Lemma 6. *(1)* $e_{00}(W_n) \subseteq V^{2n+1}$;
(2)

$$V^n \subseteq M_{[-n,n]\times[-n,n]}(W_n) + \sum_{\substack{i\geq 1, -n\leq j\leq n,\\ i+|j|\leq n}} M_{[-n,n]\times 0}(W_i) c_{0N} t^j + \sum_{j=-n}^{n} F t^j.$$

Proof. If $i_1 + \cdots + i_r \leq n$, then $e_{00}(a_{i_1} \cdots a_{i_r}) = c_{0N} t^{i_1} c_{0N} t^{i_r} e_{00}(1) \in V^{2n+1}$, which proves part (1).

Let us start the proof of part (2) with the inclusion

$$e_{00}(1) V^n e_{00}(1) \subseteq e_{00}(W_n).$$

Let w be the product of length $\leq n$ in $t^{-1}, t, e_{00}(1), c_{0N}$. If w does not involve c_{0N}, then $we_{00}(1) \in \sum_{i=-n}^{n} Fe_{i0}(1)$ and therefore $e_{00}(1)we_{00}(1) \in Fe_{00}(1)$.

Suppose now that w involves c_{0N}, $w = w'c_{0N}w''$, the subproduct w'' does not involve c_{0N}. We have $c_{0N} = e_{00}(1)c_{0N}$. Hence $e_{00}(1)we_{00}(1) = e_{00}(1)w'e_{00}(1)c_{0N}w''e_{00}(1)$. Let d_1 be the length of the product w', and let d_2 be the length of the product w'' with $d_1 + d_2 \leq n - 1$. By the induction assumption on the length of the product, we have $e_{00}(1)w'e_{00}(1) \in e_{00}(w_{d_1})$. As we have mentioned above $w''e_{00}(1) \in \sum_{i=-d_2}^{d_2} Fe_{i0}(1)$. It is straightforward that

$$c_{0N}(\sum_{i=-d_2}^{d_2} Fe_{i0}(1)) \in e_{00}(\sum_{i=1}^{d_2} Fa_i) \subseteq e_{00}(W_{d_2}).$$

Now, $e_{00}(1)we_{00}(1) \in e_{00}(W_{d_1})e_{00}(W_{d_2}) \subseteq e_{00}(W_n)$, which proves the claimed inclusion.

Let us denote the right hand side of the inclusion of Lemma 6 (2) as $RHS(n)$. We claim that $(Ft + Ft^{-1} + Fe_{00}(1))RHS(n-1) \subseteq RHS(n)$ and $RHS(n-1)(Ft + Ft^{-1} + Fe_{00}(1)) \subseteq RHS(n)$.

Let us check, for example, that $M_{[-n+1,n-1]\times 0}(W_i)c_{0,N}t^j e_{00}(1) \subseteq RHS(n)$ provided that $i + |j| \leq n - 1$. Indeed, $t^j e_{00}(1) = e_{j0}(1)$,

$$c_{0N}e_{j0}(1) = \begin{cases} 0 & \text{if } j \leq 0, \\ e_{00}(a_j) & \text{if } j \geq 1. \end{cases}$$

Now,

$$M_{[-n+1,n-1]\times 0}(W_i)e_{00}(a_j) \subseteq M_{[-n+1,n-1]\times 0}(W_i a_j) \subseteq M_{[-n,n]\times 0}(W_{n-1}).$$

Hence, to check that a product of length $\leq n$ in $t^{-1}, t, e_{00}(1), c_{0N}$ lies in $RHS(n)$, we may assume that the product starts and ends with c_{0N}. Now,

$$c_{0N}V^{n-2}c_{0N} = e_{00}(1)c_{0N}V^{n-2}c_{00}(1)c_{0N} \subseteq e_{00}(1)V^{n-1}e_{00}(1)c_{0N}$$

$$\subseteq e_{00}(W_{n-1})c_{0N} \subseteq RHS(n),$$

which completes the proof of the lemma. □

Denote $w(n) = \dim_F W_n$.

Corollary 7. $w(n) \leq g(2n+1)$, $g(n) \leq 2(2n+1)^2 w(n) + 2n + 1$.

4. Growth of the Algebras $A^{(c)}$

Now we are ready to prove Theorem 1. Let $f(n)$ be an increasing function, i.e., $f(n) \leq f(n+1)$ for all n and $f(n) \to \infty$ as $n \to \infty$. Let A be a countable dimensional algebra whose growth is locally weakly bounded by $f(n)$. Let $h(n)$ be a superlinear function.

Let elements $b_1, b_2, \cdots$ generate the algebra A. Choose a sequence $\epsilon_k > 0$ such that $\lim\limits_{k\to\infty} \epsilon_k = 0$. Denote $V_k = span_F(b_1, \cdots, b_k)$. By the assumption, there exist constants $c_k \geq 1$, $k \geq 1$, such that

$$dim_F V_k^n \leq c_k f(c_k n)(c_k n)^{\epsilon_k}$$

for all $n \geq 1$.

Increasing ϵ_k and c_k we can assume that

$$dim_F V_k^n \leq f(c_k n) n^{\epsilon_k} \tag{1}$$

Indeed, choose a sequence ϵ'_k, $k \geq 1$, such that $0 < \epsilon_k < \epsilon'_k$, $\lim\limits_{k\to\infty} \epsilon'_k = 0$.

There exists $\mu_k \geq 1$ such that

$$n^{\epsilon'_k - \epsilon_k} > c_k^{\epsilon_k + 1}$$

for all $n > \mu_k$.

The function $f(n)$ is an increasing function. Hence, there exists c'_k such that

$$f(c'_k) \geq c_k f(c_k i)(c_k i)^{\epsilon_k},$$

$i = 1, \cdots, \mu_k$.

Now we have

$$dim_F V_k^n \leq f(c'_k n) n^{\epsilon'_k}$$

for all $n \geq 1$.

From now on, we will assume (1) for arbitrary $k \geq 1$, $n \geq 1$.

Choose an increasing sequence $n_1 < n_2 < \cdots$ such that $c_k n \leq h(n)$ for all $n \geq n_k$.

Define a generating sequence $c = (a_1, a_2, \cdots)$ as follows: $a_i = b_k$ if $i = n_k$; $a_i = 0$ if i does not belong to the sequence $n_1, n_2, \cdots$.

We will show that the growth function of $A^{(c)}$ is weakly bounded by $f(h(n))n^2$. Choose $\alpha > 0$.

For an integer $n \geq n_1$, fix k such that $n_k \leq n < n_{k+1}$. Then

$$W_n = span(a_{i_1} \cdots a_{i_r} | i_1 + \cdots + i_r \leq n; a_{i_1}, \cdots, a_{i_r} \in \{b_1, \cdots, b_k\}) \subseteq V_k^n$$

Hence, $w(n) \leq f(c_k n) n^{\epsilon_k}$. From $n_k \leq n$ it follows that $c_k n \leq h(n)$. If n is sufficiently large, then we also have $\epsilon_k < \alpha$. Then

$$w(n) \leq f(c_k n) n^{\epsilon_k} \leq f(h(n)) n^{\alpha}.$$

By Lemma 6 (2) we have $g(n) \leq w(n) n^2$. Therefore $g(n) \leq f(h(n)) n^{\alpha+2}$.

We have M_∞-embedded the algebra A as a left ideal in a finitely generated algebra $B = A^{(c)}$ of growth $\leq f(h(n)) n^{\alpha+2}$.

V. Markov [8] showed that for a sufficiently large n, the matrix algebra $M_n(B)$ is 2-generated. Clearly, $M_n(B)$ has the same growth as B. Since $M_n(M_\infty(A)) \cong M_\infty(A)$, it follows that the algebra A is M_∞-embedded in $M_n(B)$ as a left ideal. This completes the proof of Theorem 1.

In order to prove Theorem 2, we will need two elementary lemmas.

Lemma 8. *Let $g_k(n)$, $k \geq 1$, be an increasing sequence of subexponential functions $g_k : N \to N$, $g_k(n) \leq g_{k+1}(n)$ for all k, n. Then there exists a subexponential function $f : N \to N$ and a sequence $1 \leq n_1 < n_2 < \cdots$, such that $g_k(n) \leq f(n)$ for all $n \geq n_k$.*

Proof. Choose $k \geq 1$. From $\lim_{n\to\infty} \frac{g_k(n)}{e^{\frac{1}{k}n}} = 0$, it follows that there exists n_k such that $\frac{g_k(n)}{e^{\frac{1}{k}n}} \leq \frac{1}{k}$ for all $n \geq n_k$. Without loss of generality, we will assume that $n_1 < n_2 < \cdots$. For an integer $n \geq n_1$, let $n_k \leq n < n_{k+1}$. Define $f(n) = g_k(n)$.

We claim that $f(n)$ is a subexponential function. Indeed, let $s \geq 1$. Our aim is to show that $\lim\limits_{n\to\infty} \frac{f(n)}{e^{\frac{1}{s}n}} = 0$.

Let $n \geq n_s$. Let k be a maximal integer such that $n_k \geq n$, so $n_k \geq n < n_{k+1}$, $s \leq k$. We have

$$\frac{f(n)}{e^{\frac{1}{s}n}} = \frac{g_k(n)}{e^{\frac{1}{s}n}} \leq \frac{g_k(n)}{e^{\frac{1}{k}n}} \leq \frac{1}{k}.$$

This implies $\lim\limits_{n\to\infty} \frac{f(n)}{e^{\frac{1}{s}n}} = 0$ as claimed. Choose $\ell \geq 1$. For all $n \geq n_\ell$, we have $n_k \leq n < n_{k+1}$, where $\ell \leq k$. Hence, $g_\ell(n) \leq g_k(n) = f(n)$. This completes the proof of the lemma. □

Lemma 9. *Let $f(n)$ be a subexponential function. Then there exists a superlinear function $h(n)$ such that $f(h(n))$ is still subexponential.*

Proof. For an arbitrary $k \geq 1$, we have $\lim\limits_{n\to\infty} \frac{f(kn)}{e^{\frac{1}{k^2}kn}} = 0$. Hence there exists an increasing sequence $n_1 < n_2 < \cdots$ such that $f(kn) < \frac{1}{k}e^{\frac{n}{k}}$ for all $n \geq n_k$.

For an arbitrary $n \geq n_1$, choose $k \geq 1$ such that $n_k \leq n < n_{k+1}$. Let $\mu(n) = k$. Then $h(n) = n\mu(n)$ is a superlinear function since $\mu(n) \leq \mu(n+1)$ and $\mu(n) \to \infty$ as $n \to \infty$. Choose $\alpha > 0$. For a sufficiently large n, we have $k = \mu(n) > \frac{1}{\alpha}$. Then

$$f(n\mu(n)) = f(kn) < \frac{1}{k}e^{\frac{1}{k}n} < \frac{1}{k}e^{\alpha n}.$$

Hence $\lim\limits_{n\to\infty} \frac{f(h(n))}{e^{\alpha n}} = 0$, which completes the proof of the lemma. □

Proof of Theorem 2. Let A be a countable dimensional associative algebra that is locally of subexponential growth. By Lemma 8, there exists a subexponential function $f(n)$ such that the growth of A is locally asymptotically bounded by $f(n)$. By Lemma 9, there exists a superlinear function $h(n)$ such that $f(h(n))$ is still a subexponential function. By Theorem 1 for an arbitrary $\alpha > 0$, we can M_∞-embed the algebra A as a left ideal in a 2-generated algebra of growth $\leq f(h(n))n^\alpha$. A product of two subexponential functions is a subexponential function. Hence, the function $f(h(n))n^\alpha$ is subexponential. This finishes the proof of Theorem 2. □

Proof of Theorem 3. Let A be a countable dimensional associative algebra of Gelfand-Kirillov dimension d. Then the growth of A is weakly asymptotically bounded by n^d. The function $h(n) = n \ln n$ is superlinear. By Theorem 1, the algebra A is M_∞-embeddable as a left ideal is a 2-generated algebra B whose growth is weakly asymptotically bounded by $(n \ln n)^d n^2$, in other words, the growth of B is asymptotically bounded by $n^{d+2+\alpha}(\ln n)^d$ for any $\alpha > 0$. This implies $GK \dim B \leq d + 2$ and completes the proof of Theorem 3. □

Now let us discuss the similar theorems for semigroups: Theorems $1'$, $2'$, $3'$.

Let P be a semigroup with 1. Let F be an arbitrary field. Consider the semigroup algebra $F[P] \wr F[t^{-1}, t]$. Let $c = (a_1, a_2, \cdots)$ be a sequence of elements $a_i \in P \cup \{0\}$ that generate the semigroup $P \cup \{0\}$.

Consider the algebra $F[P]^{(c)}$ and the semigroup $P^{(c)}$ generated by $t, t^{-1}, e_{11}(1), c_{0,N}$. Arguing as in the proof of Lemma 4, we see that $M_\infty(P)$ is a left ideal of the semigroup $P^{(c)}$.

Starting with an arbitrary generating sequence $b_1, b_2, \cdots$ of the semigroup P and diluting it with zeros as in the proof of Theorem 1, we get a generating sequence $c = (a_1, a_2, \cdots)$ of the semigroup $P \cup \{0\}$ such that the semigroup $P^{(c)}$ has the needed growth properties. The proof just follow from the proofs of Theorem 1, 2, 3.

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