Journal of Algebra and Its Applications
Vol. 20, No. 12 (2021) 2150238 (7 pages)

DOI: 10.1142/S0219498821502388

Embeddings in matrix wreath products of algebras

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Received 10 November 2021
Accepted 15 November 2021
Published 30 November 2021

Communicated by A. Leroy

We use matrix wreath products to show that (1) every countable dimensional nonsingular algebra is embeddable in a finitely generated nonsingular algebra, (2) for every infinite dimensional finitely generated PI-algebra A there exists an epimorphism $\widehat{A} \xrightarrow{\varphi} A$, where $(\ker\varphi)^3 = (0)$ and the algebra $\widehat{A}$ is not representable by matrices over a commutative algebra. If the algebra A is commutative, then $\widehat{A}$ satisfies the ACC on two-sided ideals as in the recent examples of Greenfeld and Rowen.

Keywords: Matrix wreath products; nonsingular algebras; PI-algebras.

Mathematics Subject Classification 2020: 16Pxx, 16Rxx, 16Sxx

1. Introduction

In [3] (see also [1, 2]) we introduced a construction of a matrix wreath product of two associative algebras. This construction turned out instrumental in embedding of algebras in finitely generated algebras with various properties. In this paper, we present further applications of matrix wreath products to nonsingular and PI-algebras.

Let F be a field and let A, B be associative F-algebras. Let $\mathrm{Lin}(B, A \otimes_F B)$ be the vector space of all linear transformations from B to $A \otimes_F B$. Given a basis

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$b_i, i \in I$, of the algebra B we can identify $\mathrm{Lin}(B, A \otimes_F B)$ with the space $M_I(A)$ of all column finite $I \times I$ matrices over A. This defines a structure of an associative algebra on $\mathrm{Lin}(B, A \otimes_F B)$.

Consider the direct sum of vector spaces

$$A \wr B = B + \mathrm{Lin}(B, A \otimes_F B).$$

For an arbitrary element $b \in B$ and an arbitrary transformation $f : B \to B \otimes_F A$, define the linear transformations $fb, bf \in \mathrm{Lin}(B, A \otimes_F B)$ as follows:

$$(fb)(b') = f(bb'), \quad b' \in B,$$
$$(bf)(b') = (1 \otimes b)f(b'), \quad b' \in B.$$

In [3] it was shown that this product makes $A \wr B$ an associative algebra. For indices $i, j \in I$ and an element $a \in A$ let $e_{ij}(a) \in M_I(A)$ denote the matrix having the element a at the position (ij) and zeros at other positions.

A linear transformation $\gamma : B \to A$ is called generic if the image $\gamma(B)$ generates A as an algebra. Fix an element $b \in B$ and consider the linear transformation $\gamma_b : B \to A \otimes_F B, b' \to \gamma(b') \otimes b,\ b' \in B$. If the element b lies in a selected basis of B then γ_b corresponds to a row matrix in $M_I(A)$. Of particular interest are the subalgebras $\langle B, \gamma_b \rangle$ and $\langle B, \gamma_b, e_{ij}(1) \rangle$, where $i, j \in I$, if $A \ni 1$.

Let $\mathbb{Z}$ denote the set of integers. As above consider the algebra $M_{\mathbb{Z}}(A)$ of all $\mathbb{Z} \times \mathbb{Z}$ matrices over A with finitely many nonzero entries in each column. Fixing a basis $t^i, i \in Z$, in the algebra $F[t, t^{-1}]$ of Laurent polynomials we identify the wreath product $A \wr F[t, t^{-1}] = F[t, t^{-1}] + \mathrm{Lin}(F[t, t^{-1}], A \otimes_F F[t, t^{-1}])$ with the semidirect sum $F[t, t^{-1}] + M_{\mathbb{Z}}(A)$. For integers $i, j \in \mathbb{Z}$ and an element $a \in A$ the matrix $e_{ij}(a)$ corresponds to the linear transformation $F[t, t^{-1}] \to A \otimes F[t, t^{-1}]$ that maps a basis element $t^k, k \in \mathbb{Z}$, to the element $a \otimes \delta_{jk} t^i$. For an arbitrary basis element t^k we have

$$(te_{ij}(a))(t^k) = (1 \otimes t)(\delta_{ij} a \otimes t^i) = \delta_{ij} a \otimes t^{i+1}.$$

This implies that

$$te_{ij}(a) = e_{i+1,j}(a). \tag{1}$$

Similarly,

$$(e_{ij}(a)t)(t^k) = e_{ij}(a)(t^{k+1}) = \delta_{j,k+1} a \otimes t^i = \delta_{j-1,k} a \otimes t^i.$$

This implies that

$$e_{ij}(a)t = e_{i,j-1}(a). \tag{2}$$

Theorem 1.1. (1) *If the algebra $A \ni 1$ is nonsingular and $\gamma : F[t, t^{-1}] \to A$ is a generic linear transformation then the algebra $\langle B, \gamma_1, e_{00}(1) \rangle$ is nonsingular as well*;

(2) *An arbitrary countable dimensional nonsingular algebra is embeddable in a finitely generated nonsingular algebra.*

The next application of matrix wreath products concerns PI-theory. The celebrated Amitsur–Levitzki theorem says that an algebra of matrices over a commutative algebra is PI. In 1971, Small [11] constructed an example of a finitely generated PI-algebra that is not representable by matrices over a commutative algebra. For more examples see [4, 5].

Theorem 1.2. (1) *For an arbitrary infinite dimensional finitely generated algebra B there exists a nonrepresentable finitely generated algebra $\widetilde{B}$ such that $B \cong \widetilde{B}/N, N^3 = (0)$.*

Consider the 2-dimensional algebra $F[a \mid a^3 = 0] = Fa + Fa^2, a^3 = 0$. The algebra $\widetilde{B}$ of Theorem 1.2 is a subalgebra $\langle B, \gamma_t \rangle$ of the wreath product $F[a \mid a^3 = 0] \wr B$ for some linear transformation $\gamma : B \to F[a \mid a^3 = 0]$ and an element $t \in B$.

Recently, Greenfeld and Rowen [4] constructed an example of a nonrepresentable finitely generated PI algebra with ascending chain condition on two-sided ideals.

Theorem 1.2. (2) *If the algebra B is commutative then the algebra $\widetilde{B}$ satisfies the ascending chain condition on two-sided ideals.*

2. Nonsingular Algebras

Let F be a field and let A be an associative F-algebra. Recall that a nonzero right ideal of the algebra A is called *essential* if it has nonzero intersection with any other nonzero right ideal of A. The right singular ideal $Z(A)$ consists of all elements $a \in A$ such that $r_A(a) = \{x \in A \mid ax = 0\}$ is an essential right ideal of A. An algebra A is called right *nonsingular* if $Z(A) = (0)$, see [8]. Let $S_{\mathbb{Z}}(A)$ be the right ideal of the algebra $M_{\mathbb{Z}}(A)$ that consists of $\mathbb{Z} \times \mathbb{Z}$-matrices having finitely many nonzero rows.

Lemma 2.1. *The element t has zero centralizer in $S_{\mathbb{Z}}(A)$.*

Proof. An arbitrary matrix $a \in M_{\mathbb{Z}}(a)$ can be uniquely represented as a (possibly infinite) sum

$$a = \sum e_{ij}(a_{ij}),$$

where $i, j \in \mathbb{Z}; a_{ij} \in A$. The formulas (1) and (2) imply

$$ta = \sum e_{ij}(a_{i-1,j}),$$

$$at = \sum e_{ij}(a_{i,j+1}).$$

If $ta = at$ then $a_{i-1,j} = a_{i,j+1}$ for all $i, j \in \mathbb{Z}$. In other words the matrix a is constant on each diagonal.

Suppose that $a \in S_{\mathbb{Z}}(A)$. Since every matrix in $S_{\mathbb{Z}}(A)$ has zeros on each diagonal it follows that $a = 0$. This completes the proof of the lemma. □

Let $M_\infty(A)$ denote the subalgebra of $M_{\mathbb{Z}}(A)$ that consists of $\mathbb{Z} \times \mathbb{Z}$-matrices having finitely many nonzero entries.

Let $\gamma : F[t, t^{-1}] \to A$ be a generic linear transformation. From [3, Lemma 3.10] it follows that the algebra $\langle F[t, t^{-1}], \gamma_1, e_{00}(1)\rangle$ contains $M_\infty(A)$. On the other hand it is easy to see that the algebra $\langle F[t, t^{-1}], \gamma_1, e_{00}(1)\rangle$ is contained in $F[t, t^{-1}] + S_{\mathbb{Z}}(A)$.

Denote $C = \langle F[t, t^{-1}], \gamma_1, e_{00}(1)\rangle$.

Lemma 2.2. *Every nonzero ideal of the algebra C has nonzero intersection with $e_{00}(A)$.*

Proof. We have $C = F[t, t^{-1}] + (C \cap M_{\mathbb{Z}}(A))$. We will show that every nonzero ideal of $C \cap M_{\mathbb{Z}}(A)$ has nonzero intersection with $e_{00}(A)$. Indeed, let $0 \neq a \in C \cap M_{\mathbb{Z}}(A), a = \sum_{i,j\in\mathbb{Z}} e_{ij}(a_{ij}), a_{ij} \in A$. Let $a_{ij} \neq 0$. By Lemma 3.10 of [3]

$$e_{00}(a_{ij}) = e_{0i}(1)ae_{j0}(1) \in C \cap e_{00}(A).$$

Now it is sufficient to show that every nonzero ideal of the algebra C has nonzero intersection with $M_{\mathbb{Z}}(A)$.

Let $J \lhd C, 0 \neq a \in J, a = f(t) + a'$, where $f(t) \in F[t, t^{-1}], a' \in M_{\mathbb{Z}}(A)$. If $a' = 0$ then $a = f(t) \neq 0$ and therefore

$$(0) \neq f(t)M_\infty(A) \subseteq J \cap M_{\mathbb{Z}}(A).$$

Suppose that $a' \neq 0$. We have

$$t^{-1}at = f(t) + t^{-1}a't.$$

By Lemma 2.1 $t^{-1}a't \neq a'$. Hence,

$$0 \neq a - t^{-1}at = a' - t^{-1}a't \in J \cap M_{\mathbb{Z}}(A).$$

This completes the proof of the lemma. □

Proof of Theorem 1.1(1). Suppose that the algebra A is nonsingular, but $Z(C) \neq (0)$. Then by Lemma 2.2 $Z(C) \ni e_{00}(a), 0 \neq a \in A$.

Since $Z(A) = (0)$ it follows that there exists a nonzero right ideal $\rho \lhd_r A$ such that for an arbitrary nonzero element $x \in \rho$ we have $ax \neq 0$.

Consider the right ideal

$$\widetilde{\rho} = \{f \in \mathrm{Lin}(F[t, t^{-1}], A \otimes F[t, t^{-1}]) \mid f(F[t, t^{-1}]) \subseteq \rho \otimes 1\}$$

of the algebra $A \wr F[t, t^{-1}]$. It corresponds to the row right ideal of $M_{\mathbb{Z}}(A)$ indexed by the identity $1 \in F[t, t^{-1}]$. The intersection $\widetilde{\rho} \cap C$ is a nonzero right ideal of the algebra C.

For an element $x \in \widetilde{\rho}$ the row $e_{00}(a)x$ is obtained from the row x by multiplication of all entries of the row x by the element a on the left. Hence, if $x \neq 0$ then $e_{00}(a)x \neq 0$. This contradicts the inclusion $e_{00}(a) \in Z(C)$ and completes the proof of Theorem 1.1(1). □

Proof of Theorem 1.1(2). Let A be a countable dimensional nonsingular algebra. Let $a_i, i \in \mathbb{Z}$, be a basis of the algebra A. Since the unital hull $\widehat{A} = A + F \cdot 1$ of the algebra A is nonsingular, without loss of generality we will assume that $A \ni 1$.

Consider the linear transformation $\gamma : F[t, t^{-1}] \to A, \gamma(t^i) = a_i, i \in \mathbb{Z}$. By Theorem 1.1(1) the finitely generated algebra $C = \langle F[t, t^{-1}], \gamma_1, e_{00}(1)\rangle$ is nonsingular.

By [3, Lemma 3.10] the algebra $M_\infty(A)$ is embeddable in the algebra C as a left ideal. The algebra A can be embedded in $M_\infty(A)$ in numerous ways. This completes the proof of Theorem 1.1(2). □

3. Nonrepresentable Finitely Generated PI-Algebras

Let A, B be associative algebras and let B_1 be a subalgebra of the algebra B. Choose a subspace $V \subseteq B$ such that $B = B_1 \dot{+} V$ is a direct sum of subspaces. The subspace $L' = \{f \in \mathrm{Lin}(B, A \otimes B) \,|\, f(B_1) \subseteq A \otimes_F B_1, f(v) = (0)\}$ can be identified with $\mathrm{Lin}(B_1, A \otimes_F B_1)$ and $B_1 L' \subseteq L', L' B_1 \subseteq L'$. Hence, $A \wr B_1$ is a subalgebra of the wreath product $A \wr B$.

A linear transformation $\gamma_1 : B_1 \to A$ can be extended to a linear transformation $\gamma : B \to A, \gamma(V) = (0)$. Let $b \in B_1$. The embedding $A \wr B_1 \to A \wr B$ gives rise to the embedding $\langle B_1, \gamma_{1b}\rangle \to \langle B, \gamma_b\rangle$.

Consider the 2-dimensional nilpotent algebra $A = Fa + Fa^2, a^3 = 0$. Let $F[t]$ be the algebra of polynomials in one variable without a constant term.

Define the linear transformation $\gamma : F[t] \to A$ as follows: $\gamma(t^{2^i}) = a, i \geq 1$. For an integer j that is not a power of 2 we define $\gamma(t^j) = 0$. Clearly, γ is a generic linear transformation.

Let R be an associative algebra. For a subset $X \subseteq R$ its left annihilator is defined as $\ell(x) = \{a \in R \,|\, aX = (0)\}$.

The algebra R satisfies the descending chain condition (dcc) on left annihilators if for an arbitrary ascending chain of subsets $X_1 \subseteq X_2 \subseteq \cdots$ of R the descending chain of annihilators $\ell(X_1) \supseteq \ell(X_2) \supseteq \cdots$ stabilizes.

If a finitely generated algebra R is embeddable into a matrix algebra over a commutative algebra then it is embeddable in a matrix algebra over a field (Malcev, [10]). Hence, the algebra R satisfies the descending chain condition on left annihilators.

Lemma 3.1. *The subalgebra $\langle F[t], \gamma_t\rangle$ of the wreath product $A \wr F[t]$ does not satisfy the dcc on left annihilators.*

Proof. We will show that the left annihilator of the set $\{t^i \gamma_t, 0 \leq i \leq n+1\}$ is strictly smaller than the left annihilator of $\{t^i \gamma_t, 0 \leq i \leq n\}$. For an arbitrary integer $k \geq 1$ we have

$$\gamma_t t^k \gamma_t = \gamma(t^{k+1})\gamma_t. \tag{3}$$

Since $a^2 \neq 0$ it follows that $\gamma t^k \gamma_t = 0$ if and only if $\gamma(t^{k+1}) = 0$, that is, if $k+1$ is not a power of 2.

There exists $k \geq 1$ such that $k+1, \ldots, k+n+1$ are not powers of 2, but $k+n+2$ is a power of 2. Then the element $\gamma_t t^k$ lies in the left annihilator of $\{t^i\gamma_t, 0 \leq i \leq n\}$, but not in the left annihilator of $\{t^i\gamma_t, 0 \leq i \leq n+1\}$. This completes the proof of the lemma. □

Proof of Theorem 1.2(1). Let B be a finitely generated infinite dimensional PI-algebra over a ground field F. Then the algebra B contains a transcendental element t. Indeed, if all elements of the algebra B are algebraic over F then the algebra B is finite dimensional (see [6, 7, 9, 12]).

Let $\gamma : F[t] \to A = Fa + Fa^2$ be the linear transformation defined above. Let $B = F[t] \dot{+} V$ be a direct sum of vector spaces. Extend γ to a linear transformation $B \to A$ via $\gamma(V) = (0)$.

We showed that the algebra $\langle F[t], \gamma_t\rangle$ embeds into the algebra $\widetilde{B} = \langle B, \gamma_t\rangle$ and that the algebra $\langle F[t], \gamma_t\rangle$ does not satisfy the dcc on left annihilators (Lemma 3.1). Hence, the algebra $\widetilde{B}$ does not satisfy dcc on left annihilators. Hence, the algebra $\widetilde{B}$ is not representable by matrices over a commutative algebra.

The kernel N of the natural homomorphism $\widetilde{B} \to B$ lies in $M_{\mathbb{Z}}(A)$, hence $N^3 = (0)$. This completes the proof of Theorem 1.2(1). □

Proof of Theorem 1.2(2). Choose a basis $\{b_i, i \in \mathbb{Z}\}$ of the algebra B. Suppose that the algebra B is commutative. By (3) an arbitrary product $\gamma_t b \gamma_t$, $b \in B$, is equal either to 0 or to $a\gamma_t$. This implies that the intersection $\widetilde{B} \cap M_{\mathbb{Z}}(Fa^2)$ is generated by $a\gamma_t$ as a B-bimodule. Since the algebra B is finitely generated and commutative it follows that $\widetilde{B} \cap M_{\mathbb{Z}}(Fa^2)$ is a Noetherian B-bimodule.

The B-bimodule $\widetilde{B} \cap M_{\mathbb{Z}}(A)/\widetilde{B} \cap M_{\mathbb{Z}}(Fa^2)$ is generated by the element γ_t, hence it is again Noetherian.

Finally, the B-bimodule $\widetilde{B}/\widetilde{B} \cap M_{\mathbb{Z}}(A)$ is isomorphic to the algebra B viewed as a B-bimodule, hence it is Noetherian.

We proved that the algebra $\widetilde{B}$ satisfies the ascending chain condition on B-submodules. Therefore, it satisfies the ascending chain condition on two-sided ideals. This completes the proof of Theorem 1.2(2). □

Acknowledgment

The first and second authors were funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, Grant No. (RG-61-130-42).

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