MONOTONICITY OF NONNEGATIVE MATRICES

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Abstract. We present a nonnegative rank factorization of a nonnegative matrix A for the case in which one or both of $A^{(1)}A$ and $AA^{(1)}$ are nonnegative. This gives, in particular, a known result for characterizing nonnegative matrices when $A^{\dagger}A$ or $AA^{\dagger}$ is nonnegative. We applied this characterization to the derivation of known results based on the characterization of nonnegative monotone matrices.

[1]A matrix $A = (a_{ij})$ is nonnegative if $a_{ij} \geq 0$ for all i, j, and the nonnegativity is expressed as $A \geq 0$. If there exists a matrix X such that X satisfies the following equations, for $\lambda \sqsubseteq \{1,2,3,4,5\}$: (1)$AXA = A$, (2)$XAX = X$, (3)$AX = (AX)^T$, (4)$XA = (XA)^T$, and (5) $AX = XA$, then X is called a $\lambda-$inverse of A,, also known as a generalized inverse of A. A λ-inverse of A is denoted $A^{(\lambda)}$. If $A^{(\lambda)} \geqslant 0$, then A is referred to as λ-monotone. For $\lambda = \{1,2,3,4\}$, X is the Moore–Penrose inverse of A. If $\lambda = \{1,2,5\}$, then X indicates the group inverse of A. Whereas the Moore–Penrose inverse always exists and is unique, the group inverse exists if and only if the index of A is 1 and unique. The Moore–Penrose and group inverses of A are denoted by $A^{\dagger}$ and $A^{\#}$, respectively. For $\lambda = 1$, the matrix $X = A^{(1)}$ is known as the 1-inverse of A. For an example of the applications of 1-inverses in interval linear programming, see Ben–Israel and Greville [1]. Related work has bee motivated by the utility of characterizing a nonnegative matrix A such that a linear system $Ax = B$ has a nonnegative solution or a best approximate nonnegative solution when the output matrix B is also nonnegative. Several sufficiency conditions have been demonstrated under a variety of hypotheses. For a linear system $Ax = b$, $x = A^{(1,3)}b$ is a best approximate solution to the minimum norm, or $x = A^{(1)}b$ is a solution provided that the system $Ax = b$ is consistent. Along these lines, some authors have studied the conditions under which $A^{(\lambda)}$ is nonnegative. For example, for $\lambda = \{1,2,3,4\}$, see ([1], Theorem 5.2), for $\lambda = \{1,5\}$, see ([5], Theorem 1), and for $\lambda = 1$, see ([6], Theorem 2). Under a weaker hypotheses, Jain–Tynan [4] considered nonnegative matrices A such that $A^{(1,3)}A$ is nonnegative or $A^{(1,4)}A$ is nonnegative. An $n \times n$ nonnegative matrix is monomial if each row and each column has exactly one nonzero entry. Unless otherwise stated, by "vector" we mean a "column vector".

The purpose of this paper is to improve the known results presented in [4]. This work characterizes nonnegative matrices A such that $A^{(1)}A$ or $AA^{(1)}$ is nonnegative. As a consequence, some known results are obtained for the cases in which $A^{(1)}$ is

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[1]S. K. Jain would like to dedicate this paper to the honor of Professor P. Lee, National University of Taiwan, upon his retirement.

nonnegative or $A^{(1,3)}$ is nonnegative. A new characterization is presented for the case in which the matrix A has a monotone group inverse.

The reader is referred to [1], [2], and [7] for definitions and results relating to generalized inverses.

1. Preliminaries

We first state the following key result due to Flor [3], which characterizes nonnegative idempotent matrices.

Lemma 1. *If E is any nonnegative idempotent matrix of rank d, then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the zeros in the matrices are zero blocks of appropriate size, $C, D \geq 0$, $J = XY^T$,

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_d \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_d \end{bmatrix},$$

x_i and y_i are positive vectors with $y_i^T x_i = 1$, and y_i^T is the transpose of y_i.

Lemma 2. *([2], p.68) Let A be a nonnegative $r \times n$ ($n \times r$) matrix of rank r. Then A has a nonnegative right (left) inverse if and only if it has a monomial submatrix of rank r.*

2. Main Results

Theorem 3. *Let A be a nonnegative $n \times n$ matrix of rank d. Then the following are equivalent:*

(i) There exists an $A^{(1)}$ such that $A^{(1)}A \geqq 0$ ($AA^{(1)} \geqq 0$).
(ii) There exists a permutation matrix P (Q) such that $PAP^T = FG$, ($QAQ^T = F_1G_1$),

where $F = \begin{bmatrix} ((a^{11})_{ij}) \\ ((a^{21})_{ij}) \\ ((a^{31})_{ij}) \\ ((a^{41})_{ij}) \end{bmatrix}$ is an $n \times d$ full column rank nonnegative matrix, $((a^{11})_{ij})$ are nonnegative $d \times d$ block matrices, where blocks are column vectors,

$$G = \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix}, Y^T = \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & y_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_d^T \end{bmatrix}, \ y_i^T \text{ are positive vectors,}$$

and D is a nonnegative matrix.

$$(F_1 = \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix}, G_1 = \begin{bmatrix} ((b^{11})_{ij}) & ((b^{12})_{ij}) & ((b^{13})_{ij}) & ((b^{14})_{ij}) \end{bmatrix},$$

where $((b^{1k})_{ij})$ are nonnegative $d \times d$ block (row vector) matrices,

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_d \end{bmatrix},$$

x_i are positive vectors, and C is a nonnegative matrix.)

(*iii*) $A^{(1,2)}A \geqq 0$ ($AA^{(1,2)} \geqq 0$).

Proof. Let A be an $n \times n$ nonnegative matrix of rank d such that $A^{(1)}A \geq 0$, for some $A^{(1)}$. Since $A^{(1)}A$ is idempotent, by Flor (Lemma 1) there exists a permutation matrix P such that

$$PA^{(1)}AP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J, X, Y, B, and C are as defined in the Lemma 1. Note that rank $A^{(1)}A$ = rank A = rank $J = r$. We next partition PAP^T in conformity with the partition of $PA^{(1)}AP^T$ and let

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

Since $PAP^TPA^{(1)}AP^T = PAP^T$,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

This implies that the third and fourth columns of PAP^T are zero columns, the second column is a right D multiple of the first column, and A_{i1}, $i = 1, 2, 3, 4$ satisfies the equation $UJ = U$ in the variable U. Thus, the rank of A is the rank of the first column of the above block partitioned matrix A. To solve the equation

$UJ = U$, we partition U, which is in conformity with the partitioning of J, as in Lemma 1, and write $U = [U'_{ij}]$ accordingly as a $d \times d$ block matrix. By multiplying U by J and comparing its entries with the corresponding entries of J, we obtain the result that each block submatrix U_{ij} is of rank ≤ 1 and, indeed, it is of the form $U_{ij} = u_{ij}y^T$, where u_{ij} is a nonnegative vector of length d. This yields $U = [u_{ij}]Y^T$. Restricting U to the submatrix A_{i1}, we may write $A_{i1} = ((a^{(k1)})_{ij})Y^T$, $k = 1, 2, 3, 4$, where $(a^{(k1)})_{ij}$ are nonnegative column vectors of length d. Therefore,

$$\begin{aligned} PAP^T &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \\ &= \begin{bmatrix} ((a^{11})_{ij})Y^T & ((a^{11})_{ij})Y^T D & 0 & 0 \\ ((a^{21})_{ij})Y^T & ((a^{21})_{ij})Y^T D & 0 & 0 \\ ((a^{31})_{ij})Y^T & ((a^{31})_{ij})Y^T D & 0 & 0 \\ ((a^{41})_{ij})Y^T & ((a^{41})_{ij})Y^T D & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} ((a^{11})_{ij}) \\ ((a^{21})_{ij}) \\ ((a^{31})_{ij}) \\ ((a^{41})_{ij}) \end{bmatrix} \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix} \\ &= FG, \end{aligned}$$

a nonnegative full rank factorization of PAP^T, as desired. Given the condition $AA^{(1)} \geqq 0$, we can obtain a similar factorization. This proves that (i) implies (ii).

Note that simply by interchanging the columns of F and the rows of G, $A = (P^T F)(GP) = F'G'$ (say) is a nonnegative rank factorization of A.

Next, we show that $(ii) \Longrightarrow (iii)$. By (ii), $A = F'G'$ is a full rank factorization of A. Recall that a full row rank matrix posses a right inverse, and a full column rank matrix possesses a left inverse.

Also, observe that $P^{-1}G_r = \begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a right inverse G'_r of G' (Note that Y is a $d \times d$ diagonal block matrix). Choosing $A^{(1,2)} = G'_r F'_l$, where F'_l is some left inverse of F', we have $A^{(1,2)}A = G'_r F'_l F'G' = G'_r G' \geqq 0$, as desired.

$(iii) \Longrightarrow (i)$ is obvious. ∎

The following result, which is an immediate consequence of the above theorem, is well known.

Corollary 4. *The class of nonnegative $\{1\}$-monotone matrices is the same as the class of nonnegative $\{1,2\}$-monotone matrices.*

As in ([4], Example 3), $AA^{(1)}$ may be nonnegative, but $AA^{(1)}$ need not be nonnegative. The proof of the theorem for the case in which both $AA^{\dagger}$ and $A^{\dagger}A$ are nonnegative in ([4], Theorem 7) is quite technical. Below is provided a very short argument and proof of a more general result.

Theorem 5. *Let A be a nonnegative $n \times n$ matrix of rank d. Then the following are equivalent:*

(i) There exists an $A^{(1)}$ such that $A^{(1)}A \geqq 0$ and $AA^{(1)} \geqq 0$.

(ii) A has full rank nonnegative rank factorizations of the type $A = F'G'$ and $A = F''G''$, where G' has a nonnegative right inverse G'_r and F'' has a nonnegative left inverse $F_l^{''}$. Furthermore, $G^{''} = UG'$, where U is a nonnegative invertible matrix, $G' = \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix} P$ is as in Theorem 3 above, and $F' = F^{''}V$, where V is a nonnegative invertible matrix and $F' = P^T \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix}$. In other words, for two factorizations of the type stated, F' and $F^{''}$ are conjugate, and the same property holds for G' and $G^{''}$.

Proof. $(i) \Rightarrow (ii)$. The first part of the statement follows from Theorems 3 and 4. The last part of the statement is addressed by first considering $A = F'G' = F^{''}G$. Then $G'' = F''F'G' = UG'$, where $U = F^{''}F'$ is a nonnegative $d \times d$ matrix of rank d and, hence, is invertible. Similarly, $F' = F^{''}V$, where V is invertible.

$(ii) \Rightarrow (i)$ is clear. ∎

Remark 6. *The above theorem can be invoked to yield the known characterizations of nonnegative $\lambda-$monotone matrices for the subsets λ of $\{1,2,3,4,5\}$. Jain–Snyder [6] provided a description of $\lambda-$monotone matrices for $\lambda = [1]$ and for the case in which $A^{(1)}$ is a polynomial in A. The above theorems provide, as a consequence, an explicit characterization of nonnegative matrices A such that $A^{(1,3)} \geqslant 0$ (see Berman–Plemmons [2], Theorem 6.2, p.123).*

Theorem 7. *Let A be a nonnegative matrix. Then the following are equivalent.*

(i) $A^{(1,3)} \geqslant 0$.

(ii) $A = P^T \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} W \begin{bmatrix} Y^T & 0 & 0 & 0 \end{bmatrix} P$, where X and Y are $n \times d$ matrices, as in Lemma 1, W is a nonsingular $d \times d$ monomial matrix, and P is a permutation matrix.

(iii) $A^\dagger \geqq 0$.

(iv) $A^{(1,4)} \geqq 0$.

Proof. $(i) \Longrightarrow (ii)$. Let $A^{(1,3)} \geqslant 0$. Then by choosing $A^{(1)} = A^{(1,3)}$, we have $A^{(1)}A \geqq 0$ and $AA^{(1)} \geqq 0$. Both $A^{(1)}A$ and $AA^{(1)}$ are symmetric, which implies that $C = 0$ and $D = 0$ in part (ii) of the statement of Theorem 3. Thus, by invoking Theorem 3, A has full rank nonnegative rank factorizations of the types $A = F'G'$ and $A = F''G''$, where G' has a nonnegative right inverse G'_r and $F^{''}$ has a nonnegative left inverse $F_l^{''}$.

Furthermore, $G^{''} = UG'$, where U is a nonnegative invertible matrix and $G' = \begin{bmatrix} Y^T & 0 & 0 & 0 \end{bmatrix} P$. Also, $F' = F''V$, where V is a nonnegative invertible matrix and $F'' = P^T \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then

$$A = F''G'' = P^T \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} UG' = P^T \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} U \begin{bmatrix} Y^T & 0 & 0 & 0 \end{bmatrix} P.$$

Also,

$$A = F'G' = F''V \begin{bmatrix} Y^T & 0 & 0 & 0 \end{bmatrix} P = P^T \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} V \begin{bmatrix} Y^T & 0 & 0 & 0 \end{bmatrix} P.$$

By equating the two expressions for A and using the properties of X and Y, we obtain $U = V$. Furthermore,

$$A^{(1,3)} = P^T \begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1} \begin{bmatrix} X^T & 0 & 0 & 0 \end{bmatrix} P \geqq 0,$$

which implies that $U^{-1} \geqq 0$. This shows, by Lemma 2, that U is monomial. This proves $(i) \Longrightarrow (ii)$.

$(ii) \Longrightarrow (iii)$. In the proof of $(i) \Longrightarrow (ii)$, the formula given for $A^{(1,3)}$ also holds for $A^\dagger$. Hence, $A^\dagger \geqq 0$.

$(iii) \Longrightarrow (iv)$ is obvious. The statement (iv) yields the statement (ii) exactly in the same manner as the proof of the statement $(i) \Longrightarrow (ii)$. Since we have already shown $(ii) \Longrightarrow (iii) \Longrightarrow (i)$, it follows that $(iv) \Longrightarrow (i)$. This completes the proof. ∎

The characterization of nonnegative matrices having nonnegative group inverses was considered by Jain–Kwak–Goel [5]. Although some authors have provided equivalent conditions for the monotonicities of various generalized inverses, the conditions required for the monotonicity of a monotone group have not been studied except in [5]. Stochastic matrices having nonnegative group inverses are considered in [5]. It is interesting that an application of Theorem 3 provides a new equivalent statement for the monotonicity of the group inverse.

Theorem 8. *Let A be a nonnegative matrix of index 1. Then the following statements are equivalent:*

(i) $A^\# \geqslant 0$.

(ii) There exists a permutation matrix P such that

$$PAP^T = FG = \begin{bmatrix} ((a^{11})_{ij}) \\ 0 \\ ((a^{31})_{ij}) \\ 0 \end{bmatrix} \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix}$$

is a full rank nonnegative factorization, where $((a^{11})_{ij})$ is a nonnegative monomial $d \times d$ block matrix (the block entries of which are the columns $(a^{11})_{ij}$), the block submatrix $((a^{31})_{ij})$ is a constant multiple of the block submatrix $((a^{11})_{ij})$, and

$$Y^T = \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & y_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_d^T \end{bmatrix}, \; y_i \text{ are positive vectors.}$$

(iii) There exists a permutation matrix P such that

$$PAP^T = F_1G_1 = \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \begin{bmatrix} ((b^{11})_{ij}) & ((b^{12})_{ij}) & 0 & 0 \end{bmatrix}$$

is a full rank nonnegative factorization, where $((b^{11})_{ij})$ is a nonnegative monomial $d \times d$ block matrix, the block entries of which are row vectors $((b^{11})_{ij})$, the submatrix $((b^{12})_{ij})$ is a constant multiple C of the block matrix $((b^{11})_{ij})$, and

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_d \end{bmatrix}, \; x_i \text{ are positive vectors.}$$

Proof. We will prove $(i) \Longleftrightarrow (ii)$. Assume (i). Since $A^{\#}A = AA^{\#} \geqslant 0$, by Theorem 3, there exists a permutation matrix P such that

$$PAP^T = FG = (F_1G_1)$$

where

$$F = \begin{bmatrix} ((a^{11})_{ij}) \\ ((a^{21})_{ij}) \\ ((a^{31})_{ij}) \\ ((a^{41})_{ij}) \end{bmatrix}$$

is an $n \times d$ full column rank nonnegative matrix, $((a^{11})_{ij})$ are nonnegative $d \times d$ block matrices (the $i-j$th block entry of the block matrix $((a^{11})_{ij})$ is column $(a^{11})_{ij}$), and

$$G = \begin{bmatrix} Y^T & Y^TD & 0 & 0 \end{bmatrix}, Y^T = \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & y_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_d^T \end{bmatrix}.$$

Note that

$$F_1 = \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix}$$

and $G_1 = \begin{bmatrix} ((b^{11})_{ij}) & ((b^{12})_{ij}) & ((b^{13})_{ij}) & ((b^{14})_{ij}) \end{bmatrix}$, where $((b^{1k})_{ij})$ are nonnegative $d \times d$ block matrices (the $i-j$th block entry of the block matrix $((b^{1k})_{ij})$ is the row vector $((b^{1k})_{ij})$ and $X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_d \end{bmatrix}$. By comparing the two factorizations, we obtain $((a^{21})_{ij}) = 0$, $((a^{41})_{ij}) = 0$, $((b^{(13)})_{ij}) = ((b^{(14)})_{ij}) = 0$. Thus,

$$A = \begin{bmatrix} ((a^{11})_{ij}) \\ 0 \\ ((a^{31})_{ij}) \\ 0 \end{bmatrix} \begin{bmatrix} Y^T & Y^TD & 0 & 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \begin{bmatrix} ((b^{11})_{ij}) & ((b^{12})_{ij}) & 0 & 0 \end{bmatrix}.$$

This implies that $((a^{11})_{ij})Y^T = X((b^{11})_{ij})$. Observe that after multiplying the factors of A, it follows that the $(1,1)$-block entry determines the rank of A, since other block entries are multiple of the $(1,1)$-block entry. Furthermore, since $Y^TY = I$, rank$(((a^{11})_{ij}))$ = d. Similarly, $X^TX = I$ yields rank $(X((b^{11})_{ij}))$ = rank$((b^{11})_{ij})$= d. Now, $GF = Y^T((a^{11})_{ij})$ is an invertible $d \times d$ matrix according to the Cline Theorem ([1], Theorem 2, p.163). Also, by the Cline formula, $A^{\#} = F(GF)^{-2}G \geq 0$. Since G has a nonnegative right inverse $\begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix}$, it follows that $F(GF)^{-2} \geq 0$. This implies that $GF(GF)^{-2} \geq 0$, and so $(GF)^{-1} \geq 0$. Therefore, by Lemma 2, GF is a monomial matrix. Now, $GF = Y^T((a^{11})_{ij})$, where $Y^T = \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & y_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_d^T \end{bmatrix}$. This shows that $((a^{11})_{ij})$ is a block monomial matrix because y_i are positive vectors. This proves (ii). Let us now assume (ii). We have $GF = Y^T((a^{11})_{ij})$. Because $((a^{11})_{ij})$ is a block monomial matrix and $Y^T = \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & y_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_d^T \end{bmatrix}$, y_i are positive vectors, it follows that $Y^T((a^{11})_{ij})$ is a $d \times d$ monomial nonnegative matrix. By applying the Cline formula, $A^{\#} = F(GF)^{-2}G$, we obtain $A^{\#} \geq 0$, proving (i). The proof of $(i) \Leftrightarrow (iii)$ is similar. This completes the proof. ∎

We conclude with an illustration of Theorem 8.

We know that $A^{\#}$ is a polynomial in A. Let us choose A to be a 4×4 matrix with rank $A = 2$. Following the form of the full rank factorization provided in Theorem 8 above, let

$$F = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}, G = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} \end{bmatrix}.$$

Then

$$A = FG = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 6 & 9 \\ 8 & 12 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \right].$$

$$GF = \begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}, (GF)^{-1} = \frac{1}{21} \begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}, (GF)^{-2} = \frac{1}{21} I.$$

$$A^{\#} = F(GF)^{-2}G = \frac{1}{21} FG = \frac{1}{21} A.$$

We note an interesting fact: $(GF)^{-1} = p(GF)$, where $p(t)$ is a polynomial over a field, if and only if $A^{\#} = p(A)$. In this example, $p(t) = \frac{1}{21}t$.

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