# ON SEMIGROUPS AND SEMIRINGS OF NONNEGATIVE MATRICES

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ABSTRACT. We characterize a finite cyclic semigroup of nonnegative matrices by generalizing a result known for the special case in which the semigroup is a group. We also characterize semirings $S$ of nonnegative matrices $A$ with diag $A \geqslant I$ whose multiplicative semigroup $S - \{0\}$ is cyclic.

## 1. PRELIMINARIES AND DEFINITIONS

For any square matrix $A$, let $A^D$ and $A^{\#}$ respectively denote the Drazin inverse and the group inverse of $A$. The group inverse of $A$ exists if and only if the index of $A$ is 1, in which case $A^D = A^{\#}$. The Greville formula states that if $p(x)$ is a polynomial over any field satisfied by $A$, then by rewriting $p(x) = cx^l(1 - xq(x))$, we obtain $A^D = A^l q(A)^{l+1}$. In addition, $l \geqq \text{index}A$.

All matrices in this paper are square nonnegative matrices. For definitions and terminology, refer to Ben-Israel and Greville [1] and Berman and Plemmons [2].

Our results are based on a theorem due to Jain, Kwak, and Goel [5] for nonnegative matrices $A$ that have a nonnegative group inverse $A^{\#}$. For convenience, we state the result below.

**Theorem 1.1.** *Let $A$ be a nonnegative square matrix such that $A^{\#} = \sum \alpha_i A^{m_i}$, where $\alpha_i$ are positive numbers and $m_i$ are positive integers. Then, there exists a permutation matrix $P$ such that $PAP^T$ is a direct sum of matrices of types $(I)$ and $(II)$ where*

*$(I)$ : $\beta xy^T$, $x, y$ are positive vectors with $y^T x = 1$ and $\beta$ is a positive root of $\sum \alpha_i x^{m_i+1} = 1$,*

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$$(II): \begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & .... & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & .... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{d1}x_dy_1^T & 0 & 0 & .... & 0 \end{bmatrix},$$

*where $x_i, y_i$ are positive vectors of the same order as $y_i^T x_i = 1$; $x_i$ and $x_j$ are not necessarily of the same order; $\beta_{12}, \beta_{23}, ..., \beta_{d1}$ are positive numbers and $d \geqq 2d \mid (m_i + 1)$ for some $m_i$ such that the product $\beta_{12}\beta_{23}...\beta_{d1}$ is a positive common root of at most $d$ of the following equations:*

$$\begin{aligned} \sum_{d|(m_i+1)} \alpha_i x^{(m_i+1)/d} &= 1, \\ \sum_{d|(m_i+1-k)} \alpha_i x^{(m_i+1-k)/d} &= 0, \ k = 1, 2, ..., (d-1). \end{aligned}$$

*The summation in each of the above equations runs over all $m_i$ for which $d \mid (m_i + 1 - k), k = 1, 2, ..., (d-1)$, with the convention that if there is no $m_i$ for which $d \mid (m_i + 1 - k)$, then the corresponding equation is ignored.*

## 2. Main Results

We begin by characterizing any finite multiplicative semigroup generated by a nonnegative matrix. This question has remained open for some time (c.f. [4], p. 44 and p. 46). Lewin [6] characterized nonnegative matrices that generate a finite cyclic group.

**Theorem 2.1.** *Let $S$ be a cyclic multiplicative semigroup of nonnegative matrices generated by $A$. Then, $S$ is finite if and only if there exists a permutation matrix $P$ such that $PAP^T$ is a direct sum of $B$ and $N$, where $B$ is a direct sum of nonnegative matrices: $(I)$ rank 1, idempotent matrices $xy^T$, $x, y$ are positive vectors with $y^T x = 1$, and $(II)$ rank $d, d \geqq 2$, which are $d \times d$ cyclic block matrices,*

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & .... & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & .... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{d1}x_dy_1^T & 0 & 0 & & 0 \end{bmatrix}$$

*such that $\beta_{12}, ..., \beta_{d1}$ are positive numbers, the product $\beta_{12}\beta_{23}...\beta_{d1} = 1$, and $N$ is nilpotent.*

*Proof.* We assume that $S$ is finite. Since $S$ is a finite multiplicative cyclic semigroup generated by $A$, $A^m = A^l$ for distinct positive integers $m$ and $l$. We assume that $m$ is greater than $l$. We write $m = l + r$. Then, $A^l(I - A^r) = 0$. Thus, $A$ satisfies $p(x) = x^l - x^{l+r}$. Since the index of $A$ is less than or equal to the multiplicity of the root 0 of $p(x)$, we obtain the index of $A \leqq l$. Rewriting, gives $p(x) = x^l(1 - x(x^{r-1})) = x^l(1 - x(q(x)))$. We now invoke the Greville formula for computing the Drazin inverse $A^D$ of $A$ ([1], Ex. 39, p. 148), namely $A^D = A^l q(A)^{l+1} = A^l(A^{r-1})^{l+1} = A^u$, where $u = l + (r-1)(l+1)$. Therefore, $A^D$ is nonnegative.

We decompose $A = A^2A^D \oplus (A - A^2A^D) = B \oplus N$.

We know that $B$ is of index 1 and $B^\# = A^D$ and thus $B^\#$ is nonnegative. Next, $A^u = (B \oplus N)^u = B^u \oplus N^u = B^u$, because the index of $A$ is less than $l$, and hence less than $u$. Thus, $B^\# = B^u$. We note that by Theorem 1.1, there exists a permutation matrix $P$ such that $PBP^T$ is a direct sum of matrices of types $(I)$ and $(II)$. Type $(I)$ is a direct sum of matrices $\beta x y^T$, where $\beta$ is a positive root of $x^{u+1} = 1$ and $x$ and $y$ are positive vectors with $y^T x = 1$. Thus, $\beta = 1$ and so type $(I)$ consists of summands $xy^T$. Type $(II)$ consists of $d \times d$ cyclic block matrices

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & .... & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & .... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{d1}x_dy_1^T & 0 & 0 & & 0 \end{bmatrix},$$

where $\beta_{12}, \beta_{23}, ..., \beta_{d1}$ are positive integers such that their product is a common root of the equation $x^{\frac{u+1}{d}} = 1$, where $d$ divides $u+1$, and equations $x^{\frac{u+1-k}{d}} = 0$, $k = 1, 2, ...d-1$ admitting only those equations for which $d$ divides $u+1-k$. Since such equations all have roots 0, it is clear that the only possibility for obtaining the desired common root is to choose $d = u+1$, in which case none of the latter equations exist and the positive root of the first equation is 1. Thus, $\beta_{12}\beta_{23}...\beta_{d1} = 1$ and we have only the summands of order $(u+1) \times (u+1)$. Conversely, since each summand generates a finite semigroup and $N$ is nilpotent, the semigroup generated by $A$ is finite. This completes the proof. □

Next, we consider a semiring of nonnegative matrices.

We assume that $A$ is a noninvertible nonnegative matrix. The description for nonnegative invertible matrices with a nonnegative inverse is straightforward (c.f., [3]).

**Theorem 2.2.** *Let $S$ be a semiring of nonnegative $n\times$ $n$ matrices with additive identity $0$ such that for $A \in S-\{0\}$, each entry on the diag$A \geqq I$. Suppose that the multiplicative semigroup $S-\{0\}$ is a cyclic semigroup generated by $A$. Then, (i) index of $A=1$ and (ii) if $A+A^2=A^n$, for some positive integer $n$, then there exists a permutation matrix $P$ such that $PAP^T$ is a direct sum of matrices of type $(I)$ : $\beta xy^T$, $x$, $y$ are positive vectors with $y^Tx=1$ and $\beta \in ]1,2[$ is the positive root of $(x^{n-1}-x)=1$, and type*

$$(II):\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \dots & 0\\ 0 & 0 & \beta_{23}x_2y_3^T & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta_{d1}x_dy_1^T & 0 & 0 & \dots & 0 \end{bmatrix},$$

*where the product $\beta_{12}\beta_{23}...\beta_{d1}$ is a positive common root of at most $d$ equations: $x^{\frac{2}{d}}-2x^{\frac{n}{d}}+x^{\frac{2n-2}{d}}=1$, and $x^{\frac{2-k}{d}}-2x^{\frac{n-k}{d}}+x^{\frac{2n-2-k}{d}}=0$, $k=1,2,...,d-1$ ($d$ is chosen in such a way that at least one of the exponents of terms in the first equation is an integer; we apply the convention that if any other exponent is not an integer, then that term is absent).*

*Proof.* Since $A$ is a generator of the multiplicative semigroup of $S$, we can write $A+A^2=A^n$ for some positive integer $n$. Thus, $A$ satisfies $p(x)=x+x^2-x^n$. It is known that the index of a matrix $A$ is less than or equal to the multiplicity of the root 0 of $p(x)$. Thus, $A$ is of index 1, since $A$ is not invertible. Rewriting $p(x)=x(1-x(-1+x^{n-2}))=x(1-x(q(x)))$ and invoking the Greville formula $A^{\#}=Aq(A)^2$ ([1], p. 148, Ex. 39), we obtain $A^{\#}=A(-I+A^{n-2})^2=t(A)$. Note that $A^{n-2}-I\geqq 0$ because $n\geqq 3$ and diag$A\geqq I$. Thus, $A^{\#}\geqq 0$.

Therefore, by Theorem 1.1, there exists a permutation matrix $P$ such that $PAP^T$ is a direct sum of matrices of types $(I)$ and $(II)$.

Now, type $(I)$ consists of summands $\beta xy^T$, where $x, y$ are positive vectors with $y^Tx=1$ and $\beta$ is a positive root of $xt(x)=1$; that is, $x^2(x^{n-2}-1)^2=1$ (i.e., $(x^{n-1}-x)^2=1$). Equivalently, $\beta$ is a positive root of $(x^{n-1}-x)=1$ or $(x^{n-1}-x)=-1$. The former equation has exactly one positive root between 1 and 2, whereas the latter has no positive root by the intermediate value theorem of calculus.

Type $(II)$ matrices consist of $d\times d$ block matrices, as indicated by the statement that where the product $\beta_{12}\beta_{23}...\beta_{d1}$ is a positive common root of at most $d$ equations: $x^{\frac{2}{d}}-2x^{\frac{n}{d}}+x^{\frac{2n-2}{d}}=1$, and $x^{\frac{2-k}{d}}-2x^{\frac{n-k}{d}}+x^{\frac{2n-2-k}{d}}=0$, $k=1,2,...,d-1$ ($d$ is chosen in such a way that at least one of the exponents in the first equation is an integer, with the convention that if any other exponent is not an integer then that term is absent). □

As an illustration, we determine all possible summands of matrices $A$ for which $A^2 + A^3 = A^6$. Here, $A$ satisfies $p(x) = x^2 + x^5 - x^6 = x^2(1 - x(x^3 - x^2)) = x^2(1 - x(q(x))$. Then by the Greville formula, $A^{\#} = A^2(q(A))^3 = A^2(A^3 - A^2)^3 = t(A) = \sum \alpha_i A^{m_i}$, in the notation of Theorem 1.1. Then, the summands of type $(I)$ are $\beta xy^T, x, y$ are positive vectors with $y^T x = 1$, and $\beta$ is a positive root of $xt(x) = 1$; that is, $x^3(x^3 - x^2)^3 = 1$, or $(x^4 - x^3)^3 = 1$. The positive roots, if any, are the roots of $x^4 - x^3 = 1$. This equation has only one positive root and it lies between 1 and 2. So the summand of type $(I)$ is $\beta xy^T$, where $\beta$ is the positive root of $x^4 = 1 + x^3$. Thus, $\beta = 1.3803$.

Next, we determine summands of rank $d$ greater than 1. The possible choices for $d$ are the divisors of exponents of terms appearing in the expression, $xt(x) = (x^4 - x^3)^3 = x^{12} - 3x^{11} + 3x^{10} - x^9$, which are $d = 2, 3, 4, 5, 6, 9, 10, 11$, and 12.

We begin with $d = 2$. Then, $\beta_{12}\beta_{21}$ must be a positive common root of $x^{\frac{12}{2}} - 3x^{\frac{10}{2}} = 1$ and $-3x^{\frac{10}{2}} - x^{\frac{8}{2}} = 0$, which does not exist. Thus, there are no summands of rank 2.

For $d = 3$, $\beta_{12}\beta_{23}\beta_{31}$ must be a positive common root of $x^{\frac{12}{3}} - x^{\frac{9}{3}} = 1$, $3x^{\frac{9}{3}} = 0$ and $-3x^{\frac{9}{3}} = 0$, which again does not exist. Thus, there are no summands of rank 3.

For $d = 4$, we need to find a positive common root of the equations $x^{\frac{12}{4}} = 1, -x^{\frac{8}{4}} = 0, \ldots.$ Since even the first two equations do not have a common solution, it follows that there are no summands of rank 4

Proceeding similarly, we can show that there are no summands of ranks $5, 6, 9$, 10, and 11.

For $d = 12$, $\beta_{12}\beta_{23}...\beta_{12,1}$ must be a positive common root of the only possible equation $x^{\frac{12}{12}} = 1$. This implies that there exists a summand of rank 12 and $\beta_{12}\beta_{23}...\beta_{12,1} = 1$. In particular, this means that this summand is of order 12, and it looks like

$$(II): \begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & .... & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & .... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{12,1}x_dy_1^T & 0 & 0 & & 0 \end{bmatrix},$$

where $\beta_{12}\beta_{23}...\beta_{12,1} = 1$.

We conclude that a summand of rank 1 will generate an infinite cyclic semigroup and that a summand of rank 12 has a finite order of 12. This completes the characterization of the matrix $A$, where $A^2 + A^3 = A^6$.

**Remark 2.1.** *Unlike for fields for which if the multiplicative group is cyclic then the field must be finite, a semiring with a multiplicative cyclic semigroup need not be finite.*

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