When is a Semilocal Group Algebra Continuous?

S.K.Jain, Pramod Kanwar, and J.B.Srivastava

Department of Mathematics, Ohio University, Athens, OH 45701, USA
Email: jain@ohiou.edu
Department of Mathematics Ohio University-Zanesville, Zanesville, OH 43701, USA
Email: pkanwar@math.ohiou.edu
Department of Mathematics Indian Institute of Technology, Delhi 110016, India
Email: jbsrivas@maths.iitd.ernet.in

Abstract

It is shown that (i) a semilocal group algebra KG of an infinite nilpotent group G over a field K of characteristic $p > 0$ is CS (equivalently continuous) if and only if $G = P \times H$, where P is a locally finite, infinite p-group and H is a finite abelian group whose order is not divisible by p, (ii) if K is a field of characteristic $p > 0$ and $G = P \times H$ where P is an infinite locally finite p-group (not necessarily nilpotent) and H is a finite group whose order is not divisible by p then KG is CS if and only if H is abelian. Furthermore, commutative semilocal group algebra is always continuous and for PI group algebras this holds for local group algebras; however this result is not true, in general.

1 Introduction

Semilocal finitely $\sum -CS$ group algebras of a solvable or linear group were characterized earlier as precisely self-injective group algebras (See Theorem 0.1 in [3] and related results in [2]). The purpose of this paper is to continue our investigation as to when a semilocal group algebra KG is continuous? We consider the cases when (i) G is nilpotent, and (ii) $G = P \times H$ where P is an infinite locally finite $p-$group and H is a finite group whose order is not divisible by p ($=$ char K). It is known in general that if KG is continuous then G is locally finite [1]. Theorem 4.3 shows that a semilocal group algebra KG of an infinite nilpotent group G over a field K of characteristic $p > 0$ is CS (equivalently continuous) if and only if $G = P \times H$

where P is an infinite locally finite $p-$group and H is finite abelian group whose order is not divisible by p. Theorem 4.1 shows that every commutative semilocal group algebra is continuous. This raises a natural question as to whether every $PI-$semilocal group algebra is also continuous? Example 5.1 shows that this is not true, in general. However, the result holds for any local PI group algebra, that is, such a group algebra is continuous.

2 Definitions and Notation

Unless otherwise stated throughout K will denote a field of characteristic $p > 0$ and G, a group. KG or $K[G]$ will denote the group algebra of a group G over a field K. R will denote a ring with identity and $J(R)$ its Jacobson radical. R is called local if it has a unique maximal right ideal. If $R = KG$ is local over a field K then char $K = p > 0$ and G is a $p-$group. A ring R is called semilocal if $R/J(R)$ is semisimple artinian. If $R = KG$ is semilocal then G is a torsion group. A ring R is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted. For a locally finite group G, KG is semiperfect if and only if $G/O_p(G)$ is finite, where $O_p(G)$ denotes the maximal normal $p-$subgroup of G.

A ring R is called a right $CS-$ring if it satisfies any one of the following equivalent conditions referred to as $(C_1)-$condition : (i) Each right ideal is essential in eR, $e = e^2$; (ii) Each complement right ideal is of the form eR, $e = e^2$. A ring R is called right finitely $\sum -CS$ ring if for each positive integer n, the $n \times n$ matrix ring is also a right $CS-$ring. A ring R is called right quasi-continuous (also known as right $\pi-$injective) if it satisfies any one of the following equivalent statements:

(i) For all right ideals A_1, A_2 with $A_1 \cap A_2 = (0)$, each projection π_i : $A_1 \oplus A_2 \longrightarrow A_i$, $i = 1,\ 2$ can be lifted to an endomorphism of R_R.

(ii) R satisfies the condition (C_1) given above and the condition (C_3): If $eR \cap fR = (0)$, $e = e^2$, $f = f^2$ then $eR \oplus fR = gR$ where $g = g^2$.

A ring R is known as right continuous (as defined by von Neumann) if it satisfies the condition (C_1) and the condition (C_2): If $aR \simeq eR$, $e = e^2$ then $aR = fR$, $f = f^2$. It is known $(C_2) \Rightarrow (C_3)$ and so every right continuous

ring is right quasi-continuous ring (also known as right $\pi-$injective ring). A ring R is called principally right self-injective if each $R-$homomorphism from $aR \longrightarrow R$, $a \in R$, can be lifted to an $R-$endomorphism of R_R. The concepts of CS, quasi-continuous, continuous, principally injective and injective are right-left symmetric for group algebras. Thus we will omit the prefix right or left when dealing with these concepts for group algebras.
A group G is called locally finite if each finite subset generates a finite subgroup.

3 Preliminaries

In this section, we give the results that are used often in the proofs of main results. We begin with a lemma that if KG is $\pi-$injective ($=$ quasi-continuous) then the torsion elements form a subgroup of G.

Lemma 3.1. *If KG is quasi-continuous then the torsion elements of G form a locally finite normal subgroup of G. In particular, if KG is continuous then G is locally finite, ([1], Theorem 4.3).*

Proof. Let T be the set of torsion elements of G. Let $a,\ b \in T$. Let $H =< a, b >$. Then $\omega(H) = KG(a-1)+KG(b-1)$ and $r.ann\,\omega(H) = r.ann\ KG(a-1) \bigcap r.ann\ KG(b-1)$. Suppose $r.ann\ KG(a-1) \bigcap r.ann\ KG(b-1) = (0)$. Then the projection $\pi_1 : r.ann\ KG(a-1) \oplus r.ann\ KG(b-1) \longrightarrow r.ann\ KG(a-1)$ can be extended to $\pi_1^* : KG_{KG} \longrightarrow KG_{KG}$. Let $\pi_1^*(1) = x \in KG$. Now $\pi_1^*(y) = \pi_1(y) = y$, for all $y \in r.ann\ KG(a-1)$. Also, $\pi_1^*(y) = xy$. This implies $(x-1) \in l.ann(r.ann\ KG(a-1)) = KG(a-1)$. Furthermore, $\pi_1^*(r.ann\ KG(b-1)) = 0$ and so $x(r.ann\ KG(b-1)) = 0$. This implies $x \in l.ann(r.ann\ KG(b-1)) = KG(b-1)$. Therefore, $1 = x + (1-x) \in KG(b-1) + KG(a-1) \subseteq \omega(KG)$, a contradiction. Thus $r.ann\,\omega(H) \neq 0$, which yields that $H =< a, b >$ is finite. This gives $H \subset T$ and so T is a subgroup of G. The above argument, using ([7], 3.1.2, p.68), can be extended to show by induction that any finite subset of T generates a finite subgroup.

Let KG be continuous. If $g \in G$ has infinite order then $1-g$ is regular and hence invertible, a contradiction. Hence G is torsion. That G is locally finite follows from the first part. This completes the proof. $\square$

The following lemma is stated in ([5], Theorem 4.1) for a group algebra over a field. However, the proof carries over to group algebra over a division ring.

Lemma 3.2. *Let D be a division algebra over a field K with characteristic $p > 0$. Let P be a locally finite $p-$group. Then DP is a continuous local ring.*

Proof. We sketch its proof briefly just for the convenience of the reader. Firstly, $\omega(DP)$ is nil and hence $J(DP) = \omega(DP)$ ([4], Corollary, p.682). Thus DP is a local algebra. Next $\alpha,\ \beta \in DP \Rightarrow \alpha,\ \beta \in DH$, where $H =< Supp(\alpha) \cup Supp(\beta) >$ is a finite $p-$group. Thus DH is local selfinjective and therefore uniform. This implies DP is local uniform with nil radical. Hence DP is continuous. $\square$

Next, we record below a wellknown fact.

Lemma 3.3. *If G is a locally finite group and K is a field with char $K = p > 0$ then KG is semilocal if and only if KG is semiperfect.*

The lemmas which follow are stated for easy reference.

Lemma 3.4 ([7], Lemma 10.1.6)**.** *If KG is semilocal then G is torsion.*

Lemma 3.5 ([8], Theorem 7.4.10, p.230)**.** *Let K be a field of characteristic $p > 0$, and let G be a finite group. Then KG has no nonzero nilpotent elements if and only if G is an abelian $p'-$group.*

The following theorem is due to Farkas.

Theorem 3.1 ([7], Exercise 10(ii), p.107)**.** *KG is principally selfinjective if and only if G is locally finite.*

Next we state a fact which is a consequence of the above theorem and the result that if KG is principally self-injective then KG satisfies the condition (C_2) (Lemma [6], page 119, Ex.46).

Lemma 3.6. *Let G be a locally finite group. Then KG is continuous if and only if KG is CS.*

4 Main Results

The following theorem provides plenty of examples of continuous rings.

Theorem 4.1. *Let KG be a semilocal group algebra of an abelian group G over a field K of characteristic $p > 0$. Then KG is continuous.*

Proof. Since KG is semilocal, G is torsion by Lemma 3.4. But then G is locally finite because G is abelian. Thus $J(KG) = N^*(KG)$ and KG is semiperfect. Therefore, $G/O_p(G)$ is finite ([7], 10.1.3, p.409). This yields, $G \simeq O_p(G) \times A$, where A is a finite abelian group such that $p \nmid |A|$. Write $P = O_p(G)$. Now, $KG \simeq KP \otimes_K KA \simeq \oplus \sum_{i=1}^{n} K_i P$, where $KA \cong \oplus \sum_{i=1}^{n} K_i$ and $K_i's$ are field extensions over K. Since P is a locally finite $p-$group, $K_i P$ is a continuous ring (Lemma 3.2). This yields KG is a continuous ring. $\square$

The question arises whether the above theorem holds for PI group algebras. The answer is, in general, negative (See Example 5.1). However, the result is true for local group algebras.

Theorem 4.2. *Let KG be a local PI group algebra of a group G over a field K of char $K = p > 0$. Then KG is continuous.*

Proof. Since KG is local, G is a $p-$group. Furthermore, since KG has PI, G contains a $p-$abelian subgroup of finite index and hence G is solvable-by-finite. Because torsion solvable-by-finite groups are locally finite, G is locally finite $p-$group. Therefore, by Lemma 3.2, KG is continuous. $\square$

Next, we give a complete characterization of a semilocal continuous group algebra of a nilpotent group.

Theorem 4.3. *Let G be an infinite nilpotent group and let K be a field of characteristic $p > 0$. Then the following statements are equivalent:*

1. *KG is semiperfect continuous.*

2. *KG is semilocal continuous.*

3. *KG is semiperfect CS.*

4. *KG is semilocal CS.*

5. *$G = P \times A$, where P is infinite locally finite $p-$group and A is a finite abelian group such that $p \nmid |A|$.*

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3) : Since KG is continuous, G is locally finite (Lemma 3.1). Thus $J(KG)$ is nil and so KG is semiperfect.
(3) $\Rightarrow$ (4) is obvious.
(4) $\Rightarrow$ (5) : Since KG is semilocal, G is torsion (Lemma 3.4). Since G is nilpotent, G is locally finite. Thus KG is semiperfect and so $G/O_p(G)$ is finite ([7], Theorem 10.1.5, p. 409). Let $P = O_p(G)$. Since G is infinite, P must be infinite. Furthermore, G being nilpotent and locally finite, $P = O_p(G)$ is the unique $p-$Sylow subgroup of G, where p does not divide the order of G/P. Let $|G/P| = n = q_1^{e_1} \cdots q_s^{e_s}$ where $q_i's$ are primes different from p. Then $G = P \times H$ where $H = Q_1 Q_2 \cdots Q_s$ and $Q_i = \{x \in G \mid o(x) = \textit{some power of } q_i\}$. It may be noted that each Q_i is a normal subgroup of G, because G is a locally finite nilpotent group.

By hypothesis KG is CS and since G is shown to be locally finite, by Lemma 3.6 KG is continuous. Now

$$KG = K[P \times H] \simeq KP \bigotimes_K KH \simeq KP \bigotimes \oplus \sum_{i=1}^{s} \mathbb{M}_{n_i}(D_i)$$

$$\simeq \oplus \sum_{i=1}^{s} KP \bigotimes_K \mathbb{M}_{n_i}(D_i),$$

where D_i is a finite dimensional division algebra over K. Then

$$KG \simeq \oplus \sum_{i=1}^{s} \mathbb{M}_{n_i}(D_i P)$$

By Utumi [9], KG is continuous if and only if either $D_i P$ is selfinjective or $n_i = 1$. Since P is infinite, $D_i P$ is not selfinjective. Hence each $n_i = 1$. So, $KH \simeq \oplus \sum_{i=1}^{s} D_i$. Because KH has no nonzero nilpotent elements, it follows by ([8], Theorem 7.4.10, p. 230) that H must be abelian.

(5) $\Rightarrow$ (1) : Under the given hypothesis KG is semiperfect. The rest of the argument is exactly similar to the argument in Theorem 4.1. $\square$

We close this section with the following result that provides a possible direction for further investigation of continuous group algebras.

Proposition 4.1. *Let KG be a group algebra of a group $G = P \times H$ over a field of characteristic $p > 0$, where P is an infinite locally finite $p-$group and H is a finite group with $p \nmid |H|$. Then KG is continuous if and only if H is abelian.*

Proof. Under the given hypothesis KG is semiperfect. If KG is continuous then the fact that H is abelian follows as in the proof of (4) $\Rightarrow$ (5) of Theorem 4.3. For the converse, we argue as in Theorem 4.1. $\square$

5 Examples

Our first example is of a semiperfect PI group algebra which is not continuous.

Example 5.1 Let F be a finite field of characteristic $p > 0$. Let $U_n(F)$ be the group of all $n \times n$ upper triangular matrices whose entries are in F with diagonal entries all equal to 1. $U_n(F)$ is a finite $p-$group which is nilpotent of class $n-1$([10], Exercise 1.3 (iv), p.16). Let $P = U_n(F) \times P_0$, $n \geq 3$, where $P_0 = \prod_{i=1}^{\infty} C_p^{(i)} = \{(x_i) \mid x_i \in C_p^{(i)}$ *and* $x_i = e_i$ *for all but finitely many* $i\}$ is the restricted direct product of $C_p^{(i)}$, $1 \leq i \leq \infty$ and for all i, $C_p^{(i)} \simeq C_p$, the cyclic group of order p. Let $H = U_n(K)$, where K is a finite field of characteristic $q \neq p$. Let $G = P \times U_n(K)$, $n \geq 3$ which is a nilpotent group. KG satisfies PI because P is a $p-$abelian subgroup of finite index. Since $U_n(K)$ is not abelian, by Theorem 4.3, KG is not continuous. Here KG is semiperfect ([7], Theorem 10.1.5, p.409) with PI.

The example which follows shows that KG can be continuous without G being nilpotent.

Example 5.2 Under the notation in Example 5.1, let $G = P \times A$, where

$P = \prod_{n=1}^{\infty} U_n(F)$ is the restricted direct product and A is any finite abelian group whose order is not divisible by p. Then KG is semiperfect and by Proposition 4.1, KG is continuous. Here G is not nilpotent.

References

[1] A. N. Alahmadi, S. K. Jain, P. Kanwar, and J.B. Shrivastava. Group algebras in which the complements are direct summands. *Fundamental and Applied Math.*, 11(3):3–11, 2005.

[2] K. I. Beidar, S. K. Jain, Pramod Kanwar, and J. B. Srivastava. CS matrix rings over local rings. *J. Algebra*, 264(1):251–261, 2003.

[3] K. I. Beidar, S. K. Jain, Pramod Kanwar, and J. B. Srivastava. Semilocal CS matrix rings of order > 1 over group algebras of solvable groups are selfinjective. *J. Algebra*, 275(2):856–858, 2004.

[4] I. G. Connell. On the group ring. *Can. J. Math.*, 15:650–685, 1963.

[5] S. K. Jain, Pramod Kanwar, and J. B. Srivastava. *Survey of some recent results on CS-group algebras and open questions.* Advances in Algebra, World Scientific Publishers, 401-408, 2003.

[6] T.Y. Lam. *Lectures on Modules and Rings.* Springer-Verlag, New York, 1999.

[7] Donald S. Passman. *The algebraic structure of group rings.* Wiley-Interscience [John Wiley & Sons], New York, 1977.

[8] César Polcino Milies and Sudarshan K. Sehgal. *An introduction to group rings.* Algebras and Applications. Kluwer Academic Publishers, 2002.

[9] Yuzo Utumi. On continuous rings and self injective rings. *Trans.Amer.Math.Soc.*, 118:158–173, 1965.

[10] B. A. F. Wehrfritz. *Infinite Linear Groups.* Springer-Verlag, Berlin, 1973.