GROUP ALGEBRAS IN WHICH COMPLEMENTS ARE DIRECT SUMMANDS

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Abstract. It is shown that (i) every almost selfinjective group algebra is selfinjective, and (ii) if the group algebra KG is continuous then G is a locally finite group. Furthermore, it follows that a CS group algebra KG is continuous if and only if KG is principally selfinjective if and only if G is locally finite.

1. Introduction

It is well-known that the group algebra KG of a group G over a field K is selfinjective if and only if G is a finite group, and is principally selfinjective if and only if G is a locally finite group. Every right self-injective ring is right continuous in the sense of von Neumann, that is, every complement right ideal is a summand and every right ideal isomorphic to a summand is itself a summand. Rings in which complement right ideals are summands are called right CS rings. A ring R is said to be a right almost selfinjective ring if for each right ideal K of R and for each R-homomorphism $f : K \longrightarrow R$ either f can be extended to R or there exists an R-homomorphism $g : R \longrightarrow R$ such that $g \circ f = I_K$, the identity on K. Prime and semiprime CS group algebras of polycyclic-by-finite groups have been studied in [1], [2], [11], and [12]. Since a prime regular right and left continuous (equivalently, CS) ring is simple, it follows that there does not exist a nontrivial prime regular continuous group algebra. The purpose of this paper is to study: (1) when is a group algebra almost selfinjective? and (2) when is a group algebra continuous? We show that every almost selfinjective group algebra is selfinjective (Theorem 3.9); and if the group algebra KG is continuous then G is a locally finite group (Theorem 4.3). We conclude the paper with a number of examples and a question.

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2. Definitions and Notation

Throughout this paper, unless otherwise stated, all rings have identity $1 \neq 0$ and all modules are right unital. A submodule K of a right R-module M is said to be essential in M, denoted by $K \subset_e M$, if for any nonzero submodule L of M, $K \cap L \neq 0$. M is called a CS (or extending) module if every submodule of M is essential in a direct summand of M, equivalently, if every closed submodule of M is a direct summand of M. A CS module M is called continuous if a submodule N of M isomorphic to a direct summand of M is itself a direct summand of M. M is called principally injective if for any $a \in R$, any R-homomorphism $f : aR \longrightarrow M$ can be extended to an R-homomorphism from R_R to M.

A ring R is said to be right CS (right continuous, right principally selfinjective) if the right R-module R is CS (resp. continuous, principally injective). R is called a right almost selfinjective ring if for each right ideal K of R and for each R-homomorphism $f : K \longrightarrow R$ either f can be extended to R or there exists a R-homomorphism $g : R \longrightarrow R$ such that $g \circ f = I_K$, the identity on K. R is called right quasi-continuous (equivalently, π-injective) if for any two right ideals A_1 and A_2 of R with $A_1 \cap A_2 = 0$, each projection $\pi_i : A_1 \oplus A_2 \longrightarrow A_i$, $i = 1$, 2, can be extended to an endomorphism of R_R. Obviously, a right almost self-injective ring is right π-injective. R is called directly finite if for $a, b \in R$, $ab = 1$ implies $ba = 1$. Clearly, if R has no nontrivial idempotents then R is directly finite.

For a ring R, $Z(R_R)$ will denote the right singular ideal of R and $U(R)$ will denote the multiplicative group of units of R. For a nonempty subset X of a ring R, $r.ann_R(X)$ ($l.ann_R(X)$) will denote the right (left) annihilator of X in R. If X is the singleton $\{a\}$ then we write $r.ann_R(X) = r.ann_R(a)$ ($l.ann_R(X) = l.ann_R(a)$). For any $a \in R$, l_a will denote the left multiplication by a.

A group G is called locally finite if every finitely generated subgroup of G is finite. For a group G, $\triangle(G)$ will denote the FC subgroup of G and $\omega(KG)$ will denote the augmentation ideal of the group ring KG. If H is a subgroup of G, for the sake of simplicity we will write $\omega(H)$ to denote $\omega(KH)KG$ (or $KG\omega(KH)$). However, if H is a normal subgroup then $\omega(KH)KG = KG\omega(KH)$. It is known that a subgroup H of a group G is infinite if and only if $r.ann_{KG}(\omega(H)) = 0$ (equivalently, $l.ann_{KG}(\omega(H)) = 0$).

The involution $* : KG \longrightarrow KG$ given by $(\sum_{i=1}^{n}\alpha_i g_i)^* = \sum_{i=1}^{n}\alpha_i g_i^{-1}$ defines an anti-automorphism of KG of order 2. From this we deduce that KG enjoys similar right and left properties. In particular, we observe that KG is right CS (right continuous, right quasi-continuous, right almost selfinjective) if and only if KG is left CS (resp. left continuous, left quasi-continuous, left almost selfinjective). In view of this we will omit right or left prefix while working with CS (continuous, quasi-continuous, almost selfinjective) group algebras.

3. Almost Selfinjective Group Algebras

Lemma 3.1. *Let R be a right almost selfinjective ring. Then R is right quasi-continuous (π-injective).*

Proof. It is straightforward. ∎

Lemma 3.2. *Let R be a right almost selfinjective ring. Then for every a, $b \in R$ with $r.ann_R(a) = 0$ and $r.ann_R(b) \neq 0$, $Rb \subset Ra$ and if $r.ann_R(a) = 0$ and $r.ann_R(b) = 0$, then either $Ra \subset Rb$ or $Rb \subset Ra$.*

Proof. Define $f : aR \longrightarrow bR$ by $f(ar) = br$. Clearly, f is a well defined R-homomorphism. First assume that $r.ann_R(a) = 0$ and $r.ann_R(b) \neq 0$. Then f is not one-one. Since R is right almost selfinjective, f can be extended to R. Hence there exists $s \in R$ such that $f = l_s$ on aR. Thus $b = f(a) = sa$. Consequently $Rb \subset Ra$. Now let $r.ann_R(a) = 0$ and $r.ann_R(b) = 0$. In this case f is one-one. So either $f = l_s$ on aR for some $s \in R$ or there exists $s \in R$ such that $l_s \circ f = I_{aR}$. If $f = l_s$ on aR then as above $Rb \subset Ra$. If $l_s \circ f = I_{aR}$ then $a = (l_s \circ f)(a) = sf(a) = sb \in Rb$. Thus $Ra \subset Rb$. ∎

Lemma 3.3. *([13], Proposition 1.5) A module N is $(\bigoplus_{i\in I} A_i)$-injective if and only if N is A_i-injective for every $i \in I$.*

Lemma 3.4. *Let R be a right almost selfinjective ring. If R has a nontrivial idempotent element, then R is right self-injective.*

Proof. Let e be a nontrivial idempotent element of R. By Lemma 3.3, it is enough to prove R_R is both eR-injective and $(1-e)R$-injective. Let X be a nonzero submodule of eR and $f : X \longrightarrow R$ be an R-homomorphism. Define $g : X \oplus (1-e)R \longrightarrow R$ by $g(x+(1-e)r) = f(x)$. Then g is an R-homomorphism which is not one-one. Since R is almost selfinjective of R, there exists a homomorphism $h : R \longrightarrow R$ such that $h\mid_{X\oplus(1-e)R} = g$. But then $h\mid_X = f$. Thus R is eR-injective. Similarly, R is $(1-e)R$-injective. ∎

Lemma 3.5. *Let R be a right almost selfinjective ring with no nontrivial idempotent element, and let $T = \sum Ra$, where $r.ann_R(a) = 0$ and a is not invertible. Then T is a two-sided ideal of R.*

Remark 3.1. *If there is no element a such that $r.ann_R(a) = 0$ and a is not invertible, then by convention $T = 0$.*

Proof. of Lemma 3.5 Let $T \neq 0$. It is enough to show that $ar \in T$ for each $r \in R$ and for each noninvertible element $a \in R$ with $r.ann_R(a) = 0$. So let $r \in R$ and a be a noninvertible element of R such that $r.ann_R(a) = 0$. If $r.ann_R(ar) \neq 0$, then by Lemma 3.2 $ar \in T$. We show that if $r.ann_R(ar) = 0$, then ar is not invertible. Let, if possible, ar is invertible. Then there exists $x \in R$ such that $xar = arx = 1$. Since R has no nontrivial idempotents, R is directly finite. Thus $rxa = arx = 1$, a contradiction because a is not invertible. Hence $ar \in T$. ∎

Theorem 3.6. *Let R be a right almost selfinjective ring. Then either R is right selfinjective or local.*

Proof. By Lemma 3.4 if R has a nontrivial idempotent then R is right selfinjective. So assume R has no nontrivial idempotents. Since R is almost selfinjective, by Lemma 3.1, R is quasi-continuous and hence by ([13], Proposition 1.6) R_R is uniform. Let $F = \{a \in R \mid r.ann_R(a) = 0$ and a is not invertible$\}$. If F is empty, then $a \in R$ is invertible if and only if $r.ann_R(a) = 0$. Since R_R is uniform, $Z(R_R) = R \setminus U(R)$. It follows that $R \setminus U(R)$ is a two sided ideal. Hence R is local. If F is not empty, let $T = \sum_{a \in F} Ra$. By Lemma 3.2, $R \setminus U(R) \subset T$. Now let $t \in T$. We show that t is not invertible. By Lemma 3.2, $t = xc$ for some $c \in F$. Now if t is invertible, then c is left invertible. Since R has no nontrivial idempotents, c is invertible, a contradiction because $c \in F$. Thus $T = R \setminus U(R)$. Since T is a two-sided ideal of R, it follows that R is local. ∎

Remark 3.2. *Since a right almost selfinjective ring is right quasi-continuous, a local right almost selfinjective ring is right uniform.*

Lemma 3.7. *(*[14]*, Lemma 1.13, p415) Let H be a nonidentity subgroup of a group G. If $\omega(KH) \subset J(KG)$, then $\omega(KH) = J(KH)$, K is a field of characteristic p for some prime p, and H is a p-group.*

Theorem 3.8. *Let KG be almost selfinjective. Then $J(KG) = Z(KG_{KG})$.*

Proof. By Theorem 3.6, the group algebra KG is either selfinjective or local. If KG is selfinjective then $J(KG) = Z(KG_{KG})$.

If KG is local, then $J(KG) = \omega(KG)$. By Lemma 3.7, G is a p-group. Thus for every $g \in G$, $r.ann_{KG}(1 - g) \neq 0$, because g is of finite order. Since KG is almost selfinjective, KG_{KG} is uniform. Consequently, $1 - g \in Z(KG_{KG})$ for every $g \in G$. Thus $\omega(KG) \subset Z(KG_{KG})$ which implies $J(KG) = \omega(KG) = Z(KG_{KG})$. ∎

Theorem 3.9. *Every almost selfinjective group algebra KG is selfinjective and hence G is finite.*

Proof. Let KG be almost selfinjective. Then by Theorem 3.6, KG is either selfinjective or local. Assume KG is local. Since KG is almost selfinjective, KG is quasi-continuous and hence CS. Thus, by ([3], Theorem 4.1), char$K = p$ and G is a locally finite p-group. Hence $J(KG)$ is nil and consequently, by ([3], Theorem 3.5 and Corollary 3.7), KG is selfinjective. For the sake of completeness, we give here a direct proof. Let L be a right ideal of KG and $\varphi : L \longrightarrow KG$ be a KG-homomorphism. Assume that there exists $\psi : KG \longrightarrow KG$ with $\psi \circ \varphi = I_L$. If ker$\psi \neq 0$ then, because KG is uniform, kerψ $\cap \varphi(L) \neq 0$. Let $0 \neq \varphi(x) \in$kerψ $\cap\varphi(L)$. Then $x = (\psi \circ \varphi)(x) = \psi(\varphi(x)) = 0$. Consequently, $\varphi(x) = 0$, a contradiction. Thus kerψ $= 0$, that is, $r.ann_{KG}(\psi(1)) = 0$. Consequently $\psi(1) \notin Z(KG_{KG})$. Since $J(KG) = Z(KG_{KG})$ by Theorem 3.8 and KG is local, $\psi(1)$ is invertible. Define $\eta : KG \longrightarrow KG$ by $\eta(\alpha) = \psi(1)^{-1}\alpha$ for every $\alpha \in KG$. Then η is a KG-homomorphism and for every $x \in L$, $\eta(x) = \psi(1)^{-1}x = \psi(1)^{-1}(\psi \circ \varphi(x)) = \psi(1)^{-1}(\psi(1)\varphi(x)) = \varphi(x)$. Thus φ can be extended to an endomorphism of KG. ∎

4. Continuous Group Algebras

In this section we study continuous group algebras. We begin with the following Lemma.

Lemma 4.1. *If KG is continuous then G is a torsion group.*

Proof. Let $g \in G$ and let, if possible, $o(g)$ be infinite. Then $r.ann_{KG}(1-g) = 0 = l.ann_{KG}(1-g)$. Since KG is continuous, $1-g$ is invertible in KG, a contradiction because $1-g \in \omega(KG)$. ∎

Theorem 4.2. *If G is a torsion group and KG is quasi-continuous then G is a locally finite group.*

Proof. Let $R = KG$. To prove G is locally finite, let H be a finitely generated subgroup of G. We apply induction on the number of generators of H. Let $H = \langle g_1 \rangle$ where $g_1 \in G$. Then H is finite because G is torsion.

Assume that $H_0 = \langle g_1, g_2, \ldots , g_n\rangle$ is finite and let $H = \langle g_1, g_2, \ldots , g_n, g_{n+1}\rangle$. Then

$$\begin{aligned} \omega(H) &= KG(H-1) \\ &= \omega(H_0) + KG(1-g_{n+1}). \end{aligned}$$

Note that $\omega(H_0) = KG(H_0 - 1) = KG\omega(KH_0)$. Since H_0 is finite, $r.ann_R(\omega(H_0)) \neq 0$. We show that $r.ann_R(\omega(H)) \neq 0$. Let, if possible,

$r.ann_R(\omega(H)) = 0$. Then $r.ann_R(\omega(H_0) + KG(1 - g_{n+1})) = 0$, that is, $\widehat{H}_0 KG \cap r.ann_R(1 - g_{n+1}) = 0$. Since KG is quasi-continuous, there exist idempotents e_1 and e_2 in R such that $\widehat{H}_0 KG \subset_e e_1 R$ and $r.ann_R(1 - g_{n+1}) \subset_e e_2 R$. Then $e_1 R \cap e_2 R = 0$. But then there exists an idempotent $e \in R$ such that $e_1 R = eR$, $e_2 R \subset (1 - e)R$. Thus $l.ann_R(e_1 R) = l.ann_R(eR)$ and $l.ann_R((1 - e)R) \subset l.ann_R(e_2 R)$, that is, $R(1 - e_1) = R(1 - e)$ and $Re \subset R(1 - e_2)$. Now

$$R(1 - e_1) = l.ann_R(e_1 R) \subset l.ann_R(\widehat{H}_0 KG) = \omega(H_0)$$

and

$$R(1 - e_2) = l.ann_R(e_2 R) \subset l.ann_R(r.ann_R(1 - g_{n+1})) = R(1 - g_{n+1}).$$

Hence

$$\begin{aligned} R &= Re + R(1 - e) \subset R(1 - e_2) + R(1 - e_1) \\ &\subset \omega(H_0) + R(1 - g_{n+1}) = \omega(H), \end{aligned}$$

a contradiction, because $1 \notin \omega(H)$. Hence $r.ann_R(\omega(H)) \neq 0$ and consequently H is finite. ∎

Observe that for a quasi-continuous group algebra KG, the group G need not be torsion. For example, the group algebra of an infinite cyclic group is quasi-continuous.

As a consequence of Lemma 4.1 and Theorem 4.2, we have the following theorem.

Theorem 4.3. *If KG is continuous then G is a locally finite group.*

Since KG is principally selfinjective if and only if G is locally finite ([7], Theorem, p26), we have the following corollary.

Corollary 4.4. *If KG is continuous then KG is principally selfinjective.*

We now give an example to show that the above result is not true for arbitrary rings.

Example 4.1. *Let $R = \mathbb{Q}(x_1, x_2, \dots, x_n \dots)$, $S = \mathbb{Q}(x_1^2, x_2^2, \dots, x_n^2 \dots)$, and $A = \begin{pmatrix} R & 0 \\ R & S \end{pmatrix}$. Let f be the ring homomorphism $f(a) = a$ for all $a \in \mathbb{Q}$ and $f(x_i) = x_i^2$. Let $T = \left\{ \begin{pmatrix} r & 0 \\ r' & f(r) \end{pmatrix} \mid r, r' \in R \right\}$. Then T is a subring of A. The only nontrivial right ideal of T is $\begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}$ which is principal. Thus T is right continuous. If T is right selfinjective then*

T is quasi-Frobenius and hence left artinian which is not true. Therefore, T is not right selfinjective. If T is right principally injective then T is right selfinjective, a contradiction.

For a CS group algebra, we have the following theorem.

Theorem 4.5. *If G is a group such that KG is CS then the following are equivalent.*
(i) KG is continuous.
(ii) KG is principally selfinjective.
(iii) G is locally finite.

We now give examples of countable locally finite groups such that KG is not continuous.

Example 4.2. *Let $G = S_\infty = \bigcup_{n=1}^{\infty} S_n$ and K be any field. Then G is a countable locally finite group and $\triangle(G) = 1$. Thus the group algebra KG is prime. By ([14], Theorem 4.8, p295) KG is semisimple. By ([14], Theorem 2.5, p368) KG is primitive. Since a primitive continuous ring is simple, it follows that KG is not continuous.*

Example 4.3. *Let p be a prime, $P = \mathbb{Z}_{p^\infty}$, the Prüfer p-group, and H be any finite group. Let $G = P \times H$ and $R = \mathbb{Q}G$. Then R is not continuous. For if R is continuous then by ([6], Corollary 10.11) R is semiperfect. Since R is regular, R is semisimple artinian, a contradiction because G is an infinite group. In particular, $\mathbb{Q}\mathbb{Z}_{p^\infty}$ is not continuous.*

Remark 4.1. *It can be similarly shown that if p and q are distinct primes, $P = \mathbb{Z}_{p^\infty}$, H a finite group of order $p^n m$ where $m > 1$, q does not divide m, and $G = P \times H$ then $\mathbb{Z}_q G$ is not continuous.*

We now give an example of a prime local continuous group algebra.

Example 4.4. *Let p be a prime and $G = P_\infty = \bigcup_{n=1}^{\infty} P_n$ where for each n, P_n is a Sylow p-subgroup of S_{p^n} and $P_n \subset P_{n+1}$. Then G is a locally finite p-group and $\triangle(G) = 1$. Let K be a field of characteristic p. Then KG is a prime local continuous group algebra.*

Since a prime regular continuous ring is simple, we note that there does not exist any nontrivial prime regular continuous (equivalently, CS) group algebra KG. We conclude this section with a question.

Question: Is it true that a regular continuous group ring (equivalently, CS group ring) selfinjective?

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