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ON MATRIX WREATH PRODUCTS OF ALGEBRAS

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Abstract. We introduce a new construction of matrix wreath products of algebras that is similar to the construction of wreath products of groups introduced by L. Kaloujnine and M. Krasner [17]. We then illustrate its usefulness by proving embedding theorems into finitely generated algebras and constructing nil algebras with prescribed Gelfand-Kirillov dimension.

1. Matrix Wreath Products

Let F be a field and let A, B be two associative F-algebras. Let $\mathrm{Lin}(A, B)$ denote the vector space of all F-linear transformations $A \to B$.

We will define multiplication on $\mathrm{Lin}(B, B \otimes_F A)$. Let $f, g \in \mathrm{Lin}(B, B \otimes_F A)$. For an arbitrary element $b \in B$, let $g(b) = \sum_i b_i \otimes a_i$, where $a_i \in A$ and $b_i \in B$. Let $f(b_i) = \sum_j b_{ij} \otimes a_{ij}$, where $a_{ij} \in A$ and $b_{ij} \in B$. Define

$$(fg)(b) = \sum_{i,j} b_{ij} \otimes a_{ij} a_i.$$

We define a structure of a B-bimodule on $\mathrm{Lin}(B, B \otimes_F A)$. For an arbitrary element $b \in B$ and a linear transformation $f : B \to B \otimes_F A$, we define linear transformations fb and bf via

$$(fb)(b') = f(bb') \quad \text{and}$$
$$(bf)(b') = (b \otimes 1) f(b'), \ b' \in B.$$

In other words, if $f(b') = \sum_i b_i \otimes a_i$ then $(bf)(b') = \sum_i bb_i \otimes a_i$. Now consider the semidirect sum

$$A \wr B = B + \mathrm{Lin}(B, B \otimes_F A)$$

that extends multiplication on B and on $\mathrm{Lin}(B, B \otimes_F A)$.

Theorem 1. *$A \wr B$ is an associative algebra.*

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Choose a basis $\{b_i\}_{i\in I}$ of the algebra B. For a linear transformation $f : B \to B \otimes_F A$, let

$$f(b_j) = \sum_i b_i \otimes a_{ij}.$$

Consider the $I \times I$ matrix $A_f = (a_{ij})_{I\times I}$. Each column of this matrix contains only finitely many nonzero entries.

Let $M_{I\times I}(A)$ denote the algebra of $I \times I$ matrices over A having finitely many nonzero entries in each column. Then $f \to A_f$, $f \in \mathrm{Lin}(B, B \otimes_F A)$, is an isomorphism $\mathrm{Lin}(B, B \otimes_F A) \to M_{I\times I}(A)$.

The wreath product $G_1 \wr G_2$ of two groups G_1 and G_2 embeds in the multiplicative group of the matrix wreath product $FG_1 \wr FG_2$ of group algebras.

Indeed, let $\mathrm{Fun}(G_2, G_1)$ be the group of mappings from G_2 to G_1 with pointwise multiplication: $(fg)(a) = f(a)g(a)$ for all $f, g \in \mathrm{Fun}(G_2, G_1)$ with $a \in G_2$. Then $G_1 \wr G_2$ is the semidirect product of $G_2\,\mathrm{Fun}(G_2, G_1)$ with $(b^{-1}fb)(a) = f(ba)$ for arbitrary elements $a, b \in G_2$.

For a mapping $f : G_2 \to G_1$, consider the "diagonal" linear transformation $\mathrm{diag}(f) : g \to g \otimes f(g)$ for $g \in G_2$. The mappings $b \to b$ and $f \to \mathrm{diag}(f)$ for $b \in G_2$ and $f \in \mathrm{Fun}(G_2, G_1)$ extend to an embedding of $G_1 \wr G_2$ into the multiplicative group $FG_1 \wr FG_2$.

If ${}_BM$ is a left module over the algebra B, then we can define

$$A \wr_M B = B + \mathrm{Lin}(M, M \otimes_F A).$$

Different constructions of wreath products of Lie algebras were introduced by A. L. Smel'kin [27] and V. Petrogradsky, Y. Razmyslov, E. Shishkin [24] and L. Bartholdi [3].

In what follows, we will always assume that the algebra B is finitely generated, infinite dimensional, and, moreover, $\{b \in B \mid \dim bB < \infty\} = (0)$.

Along with the algebra of matrices $M_{I\times I}(A)$, we will consider two important subalgebras:

(1) $M_\infty(A)$ that consists of $I \times I$ matrices having finitely many nonzero entries, and
(2) the subalgebra $S(A, B)$ that consists of matrices having finitely many nonzero rows. In the language of linear transformations $\varphi : B \to B\otimes_F A$, the subalgebra $S(A, B)$ consists of such φ for which there exists a finite dimensional subspace $V \subset B$ with $\varphi(B) \subseteq V \otimes_F A$.

Clearly $M_\infty(A) \subset S(A, B)$.

Theorem 2. *Let $M_\infty(A) \subseteq S \subseteq S(A, B)$ be a subalgebra such that $BS + SB \subseteq S$. Then*

(1) the algebra $B + S$ is prime if and only if the algebra A is prime, and
(2) the algebra $B + S$ is (left) primitive if and only if the algebra A is primitive.

We say that a linear transformation $\gamma : B \to A$ is a generating linear transformation if $\gamma(B)$ generates the algebra A. Suppose that $1 \in B$. Let $\gamma : B \to A$ be a generating linear transformation. Consider the element

$$c_\gamma : b \to 1 \otimes \gamma(b) \in B \otimes_F A.$$

Consider the subalgebra $\langle B, c_\gamma\rangle$ generated in $A \wr B$ by B and the element c_γ.

For an element $a \in A$ and two indices $i, j \in I$, let $e_{ij}(a)$ denote the matrix whose (i, j)-entry is a and all other entries are equal to zero. For a fixed element

$u \in A$, we consider also the subalgebra $\langle B, c_\gamma, e_{11}(u)\rangle$. Clearly, $\langle B, c_\gamma, e_{11}(u)\rangle$ lies in $B + S(A,B)$. If $u = 1$, then $M_\infty(A) \subseteq \langle B, c_\gamma, e_{11}(1)\rangle$.

Since we always assume that the algebra B is finitely generated, the algebras $\langle B, c_\gamma\rangle$, $\langle B, c_\gamma, e_{11}(u)\rangle$ are finitely generated as well. Our immediate goal now is to estimate growth of these algebras.

We start with some general definitions. Consider an F-algebra R generated by a finite dimensional subspace V. Let

$$V^n = \operatorname{span}_F \big\{ v_1 \cdots v_k \,|\, k \leq n, v_i \in V, 1 \leq i \leq k \big\}.$$

Then $\dim_F V^n < \infty$ and R is the union of the ascending chain $V^1 \subseteq V^2 \subseteq \cdots$. The function $g(V,n) = \dim_F V^n$ is called the growth function of the algebra R that corresponds to the generating subspace V.

Let $\mathbb{N}$ denote the set of positive integers. Given two functions $f, g : \mathbb{N} \to [1,\infty)$, we say that $f \preceq g$ (f is asymptotically less than or equal to g) if there exists a constant $c \in \mathbb{N}$ such that $f(n) \leq cg(cn)$ for all $n \in \mathbb{N}$. If $f \preceq g$ and $g \preceq f$, then f and g are said to be asymptotically equivalent, i.e., $f \sim g$. If V and W are finite dimensional generating subspaces of A, then $g(V,n) \sim g(W,n)$. We will denote the class of equivalence of $g(V,n)$ as g_A.

A finitely generated algebra R has polynomially bounded growth if there exists $\alpha > 0$ such that $g_R(n) \preceq n^\alpha$. Then

$$\operatorname{GKdim}(R) = \inf \big\{ \alpha > 0 \,|\, g_R(n) \preceq n^\alpha \big\}$$

is called the Gelfand-Kirillov dimension of R. If the growth of R is not polynomially bounded, then we let $\operatorname{GKdim}(R) = \infty$. If the algebra R is not finitely generated, then the Gelfand-Kirillov dimension of R is defined as the supremum of Gelfand-Kirillov dimensions of all finitely generated subalgebras of R.

For $n \geq 1$, consider the vector space

$$W_n = \sum_{i_1 + \cdots + i_r \leq n} \gamma(V^{i_1}) \cdots \gamma(V^{i_r}),$$

and let $A = \bigcup_{n \geq 1} W_n$. Clearly, $\dim_F W_n < \infty$ and $W_1 \subseteq W_2 \subseteq \cdots \subseteq A$. Denote $w_\gamma(n) = \dim_F W_n$. We call $w_\gamma(n)$ the growth function of the linear transformation γ.

A linear transformation $\gamma : B \to A$ is said to be *dense* if for arbitrary linearly independent elements $b_1, \ldots, b_n \in B$ and an arbitrary nonzero element $a \in A$, there exists an element $b \in B$ such that $\gamma(b_i b) = 0$, $1 \leq i \leq n-1$, and $a\gamma(b_n b) \neq 0$.

Theorem 3.

(1) $g_{\langle B, c_\gamma, e_{11}(u)\rangle} \preceq g_B^2(n) w_\gamma(n)$.

(2) If the generating linear transformation γ is dense, then $g_{\langle B, c_\gamma\rangle}(n) \sim g_B^2(n) w_\gamma(n)$.

2. Embedding Theorems

G. Higman, H. Neumann, and B. H. Neumann [15] proved that every countable group embeds in a finitely generated group. The papers [4], [23], [25], and [30] show that some important properties can be inherited by these embeddings. Much of this work relies on wreath products of groups.

Following [15], A. I. Malcev [21] showed that every countable dimensional associative algebra over a field is embeddable in a finitely generated algebra, and

A. I. Shirshov [26] showed that every countable dimensional Lie algebra is embeddable in a finitely generated Lie algebra.

Let A be an associative algebra, and let I be a countable set. As above, we consider the algebra $M_\infty(A)$ of $I\times I$ matrices having finitely many nonzero entries. Clearly, the algebra A is embeddable in $M_\infty(A)$ in many ways. We say that an algebra A is M_∞-embeddable in an algebra B if there exists an embedding $\varphi : M_\infty(A)\to B$. We say that A is M_∞-embeddable in B as a (left, right) ideal if the image of φ is a (left, right) ideal in B.

Observe that [1] extended the theorem of Malcev in the following way: every countable dimensional associative algebra over a field is M_∞-embeddable in a finitely generated algebra as an ideal.

3. Radical Algebras

S. Amitsur [2] asked if a finitely generated algebra can have a non-nil Jacobson radical. The first examples of such algebras were constructed by K. Beidar [9]. J. Bell [6] constructed examples having finite Gelfand-Kirillov dimension. Finally, L. Bartholdi and A. Smoktunowicz [29] constructed a finitely generated Jacobson radical non-nil algebra of Gelfand-Kirillov dimension two.

Theorem 4. *An arbitrary countable dimensional Jacobson radical algebra is embeddable in a finitely generated Jacobson radical algebra.*

Theorem 5. *An arbitrary countable dimensional Jacobson radical algebra of Gelfand-Kirillov dimension d over a countable field is embeddable in a finitely generated Jacobson radical algebra of Gelfand-Kirillov dimension $\leq d+6$.*

We will start with the following lemma.

Lemma 1. *For an arbitrary Jacobson radical algebra A, there exists a Jacobson radical algebra $\widetilde{A}$ and an element $u\in\widetilde{A}$, with $u^3=0$, such that A is embeddable in the right ideal $u\widetilde{A}$ (resp. left ideal $\widetilde{A}u$.)*

Proof. Consider the two dimensional nilpotent algebra $B=Fb+Fb^2$, $b^3=0$. Then $\widetilde{A}=A\wr B=B+M_2(A)$ is a Jacobson radical algebra and $e_{22}(A)\subseteq b\widetilde{A}$. $\square$

Sketch of the proof of Theorem 4. Let B be a finitely generated infinite dimensional nil algebra of E. S. Golod [11]. Let $\widehat{B}=B+F\cdot 1$ be its unital hull. Let $\widetilde{A}$ be the Jacobson radical algebra of Lemma 1, $u\in\widetilde{A}$, $u^3=0$, $A\leq\widetilde{A}u$. Consider a generating linear transformation $\gamma:\widehat{B}\to\widetilde{A}$ and the element $c_\gamma\in\mathrm{Lin}(\widehat{B},\widehat{B}\otimes_F\widetilde{A})$. Then the algebra $\langle B,c_\gamma,e_{11}(u)\rangle$ is finitely generated and Jacobson radical. Hence, the algebra A is embeddable in a finitely generated Jacobson radical algebra $\langle B,c_\gamma,e_{11}(u)\rangle$. $\square$

To prove Theorem 5, we will need the following lemma.

Lemma 2. *Let A be a countable dimensional algebra of Gelfand-Kirillov dimension $\leq d$. Let B be an arbitrary finitely generated algebra. Then there exists a generating linear transformation $\gamma:B\to A$ such that $w_\gamma(n)\leq n^{d+\epsilon_n}$ where $\epsilon_n>0$, $\epsilon_n\to 0$ as $n\to\infty$.*

Instead of the Golod nil algebra B, we will consider a finitely generated nil algebra B of polynomially bounded growth. Existence of such algebras was established by T. Lenagan and A. Smoktunowicz in [19] under the assumption that the ground field is countable. In [20], T. Lenagan, A. Smoktunowicz, and A. Young refined the argument of [19] and obtained a finitely generated nil algebra of Gelfand-Kirillov dimension ≤ 3.

Let $A \hookrightarrow \widetilde{A}u$, $u^3 = 0$, be the embedding of Lemma 1, and let B be the nil algebra of [20]. Arguing as above, we embed the algebra A in the finitely generated subalgebra $\langle B, c_\gamma, e_{11}(u)\rangle$ of $\widetilde{A} \wr \widehat{B}$, where γ is a generating linear transformation of Lemma 2. By Theorem 3(1), we have

$$g_{\langle B, c_\gamma, e_{11}(u)\rangle} \preceq g_B(u)^2 w_\gamma(u),$$

which implies $\mathrm{GKdim}\langle B, c_\gamma, e_{11}(u)\rangle \leq d+6$.

4. Nil Algebras

We say that a nil algebra A is *stable nil* (resp. *stable algebraic*) if all matrix algebras $M_n(A)$ are nil (resp. algebraic).

Theorem 6. *An arbitrary stable nil algebra A is embeddable in a finitely generated stable nil algebra. If $\mathrm{GKdim}\, A = d < \infty$ and the ground field is countable, then A is embeddable in a finitely generated nil algebra of Gelfand-Kirillov dimension $\leq d+6$.*

To use the wreath product constructions as above, we will need a finitely generated infinite dimensional stable nil algebra. Existence of such algebras can be established using methods from E. S. Golod [11] based on Golod-Shafarevich inequalities [12].

More precisely, let $F\langle x_1, \cdots, x_m\rangle$ be the associative algebra on m free generators, $m \geq 2$. We consider the free algebra without 1, i.e., it consists of formal linear combinations of nonempty words in $x_1, \ldots, x_m$. Assigning degree 1 to all variables $x_1, \ldots, x_m$, we make $F\langle x_1, \ldots, x_m\rangle$ a graded algebra. The degree $\deg(a)$ of an arbitrary element $a \in F\langle x_1, \ldots, x_m\rangle$ is defined as the minimal degree of a nonzero homogeneous component of a.

Let $R \subset F\langle x_1, \ldots, x_m\rangle$ be a subset containing finitely many elements of each degree.

Golod-Shafarevich Condition: *If there exists a number $\frac{1}{m} < t_0 < 1$ such that $\sum_{a\in R} t_0^{\deg(a)} < \infty$ and $1 - mt_0 + \sum_{a\in R} t_0^{\deg(a)} < 0$, then the algebra $\langle x_1, \ldots, x_m \,|\, R = (0)\rangle$ presented by the set of generators $x_1, \ldots, x_m$ and the set of relations R is infinite dimensional.*

Lemma 3. *For $m \geq 2$, there exists a subset $R \subset F\langle x_1, \ldots, x_m\rangle$ satisfying the Golod-Shafarevich Condition and such that the algebra $\langle x_1, \ldots, x_m \,|\, R = (0)\rangle$ is stable nil.*

For a stable nil algebra A and its extension $A \subset \widetilde{A}u$, $u^3 = 0$, of Lemma 1 and an algebra B of Lemma 3, the finitely generated algebra $\langle B, c_\gamma, e_{11}(u)\rangle$ is stable nil. It implies the first part of Theorem 6.

Now let F be a countable field, let B be the Lenagan-Smoktunowicz-Young algebra [20], and let A be a countable dimensional stable nil algebra of $\mathrm{GKdim}\, A \leq d$. Then the algebra $\langle B, c_\gamma, e_{11}(u)\rangle$ is nil and has Gelfand-Kirillov dimension $\leq d+6$. We do not know if this finitely generated algebra is stable nil.

5. Primitive Algebras

I. Kaplansky [18] asked if there exists an infinite dimensional finitely generated algebraic primitive algebra, a particular case of the celebrated Kurosh Problem. Such examples were constructed by J. Bell and L. Small in [7]. Then J. Bell, L. Small, and A. Smoktunowicz [8] constructed finitely generated algebraic primitive algebras of finite Gelfand-Kirillov dimension provided that the ground field is countable.

Theorem 7. *An arbitrary countable dimensional stable algebraic primitive algebra is M_∞-embeddable as a left ideal in a 2-generated algebraic primitive algebra.*

This theorem answers the first part of question 7 from [8].

Theorem 8. *Let F be a countable field. An arbitrary countable dimensional stable algebraic primitive algebra of Gelfand-Kirillov dimension $\leq d$ is M_∞-embeddable as a left ideal in a finitely generated algebraic primitive algebra of Gelfand-Kirillov dimension $\leq d+6$.*

Without loss of generality, we assume that a countable dimensional stable algebraic algebra A is unital. As above, we start with Golod's finitely generated infinite dimensional nil algebra B and a generating linear transformation $\gamma : \widehat{B} \to A$. Then the algebra $\langle \widehat{B}, c_\gamma, e_{11}(1)\rangle$ is primitive by Theorem 2(2) and contains $M_\infty(A)$ as a left ideal.

The same argument with the Lenagan-Smoktunowicz-Young algebra B and a linear transformation of Lemma 2 implies Theorem 8.

6. Algebras of Locally Subexponential Growth

Recently, L. Bartholdi and A. Erschler [4] proved that a countable group of locally subexponential growth embeds in a finitely generated group of subexponential growth. We prove the analog of Bartholdi-Erschler theorem for algebras and semigroups and establish some related results.

Given two functions $f, g : \mathbb{N} \to [1, \infty)$, we say that f is *weakly asymptotically less than or equal to* g if for arbitrary $\alpha > 0$, we have $f \preceq gn^\alpha$ (denoted $f \preceq_w g$).

A function f is *subexponential* if $\lim\limits_{n\to\infty} \dfrac{f(n)}{e^{\alpha n}} = 0$ for any $\alpha > 0$. In the seminal paper [14], R. I. Grigorchuk constructed the first example of a group with an intermediate growth function: subexponential but growing faster than any polynomial. Finitely generated associative algebras with intermediate growth functions come as universal enveloping algebras of certain Lie algebras (see [28]).

A not necessarily finitely generated algebra A is of locally subexponential growth if every finitely generated subalgebra of A has a subexponential growth function.

The growth of A is locally (resp. weakly) bounded by a function $f(n)$ if for every finitely generated subalgebra of A its growth function is $\preceq f(n)$ (resp. $\preceq_w f(n)$).

A function $h(n)$ is *superlinear* if $\dfrac{h(n)}{n} \to \infty$ as $n \to \infty$.

Theorem 9. *Let $f(n)$ be an increasing function. Let A be a countable dimensional associative algebra whose growth is locally weakly bounded by $f(n)$. Let $h(n)$ be a superlinear function. Then the algebra A is M_∞-embeddable as a left ideal in a 2-generated algebra whose growth is weakly bounded by $f(h(n))n^2$.*

We then use Theorem 9 to derive an analog of the Bartholdi-Erschler Theorem ([4]).

Theorem 10. *A countable dimensional associative algebra of locally subexponential growth is M_∞-embeddable in a 2-generated algebra of subexponential growth as a left ideal.*

The idea of the proofs of Theorems 9 and 10 is the same as in previous sections. We consider the matrix wreath product $A \wr F[t^{-1}, t]$ with the algebra $F[t^{-1}, t]$ of Laurent polynomials and choose a generating linear transformation $\gamma : F[t^{-1}, t] \to A$ with appropriate subexponential growth function $w_\gamma(n)$. The algebra A is then M_∞-embeddable as a left ideal in the finitely generated algebra $C = \langle F[t^{-1}, t], c_\gamma, e_{11}(1)\rangle$. By V. T. Markov's theorem [22], the matrix algebra $M_n(C)$ is 2-generated for a sufficiently large n, which yields the result.

Using [28] and Theorem 10, we can prove an embedding theorem for countable dimensional Lie algebra of locally subexponential growth.

Theorem 11. *Let F be a field of characteristic $\neq 2$. Every countable dimensional Lie F-algebra of locally subexponential growth is embeddable in a finitely generated Lie algebra of subexponential growth.*

7. Gelfand-Kirillov Dimension of Nil Algebras

In this section, we assume that the ground field F is countable. Question 1 from [8] asks if an arbitrary sufficiently big $\alpha \geq 2$ is the Gelfand-Kirillov dimension of some finitely generated nil algebra.

Theorem 12. *Let F be a countable field. For an arbitrary $d \geq 8$, there exists a finitely generated nil F-algebra of Gelfand-Kirillov dimension d.*

Let B be the finitely generated infinite dimensional algebra of Lenagan-Smoktunowicz-Young [20] with $\operatorname{GKdim} B \leq 3$.

For an arbitrary $\alpha \geq 2$, W. Bohro and H. P. Kraft [10] constructed a graded F-algebra $R = \sum_{i=1}^{\infty} R_i$, generated by two elements $x, y \in R_1$, such that for any $\epsilon > 0$ we have

$$n^{\alpha-\epsilon} \leq \dim \sum_{i=1}^{n} R_i \leq n^{\alpha+\epsilon}$$

for all sufficiently large n.

Using the Bohro-Kraft algebra, we construct a countable dimensional locally nilpotent algebra A and a dense generating linear transformation $\gamma : B \to A$ of growth $w_\gamma(n)$ such that for an arbitrary $0 < \epsilon < \alpha$, we have

$$\left(\frac{n}{\ln n}\right)^{n-\epsilon} \preceq w_\gamma(n) \preceq n^{\alpha+\epsilon}(\ln(\ln n))^2.$$

By Theorem 3, for the finitely generated algebra $C = \langle B, c_\gamma\rangle$, we have $g_c(n) \sim g_B(n)^2 w_\gamma(n)$, and therefore $\operatorname{GKdim}(C) = 2\operatorname{GKdim}(B) + \alpha$, which implies Theorem 12.

Question. *Let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $n^2 \preceq g(n)$ and $g(m + n) \leq g(m)g(n)$ for all $m, n \in \mathbb{N}$. Is $g(n)$ asymptotically equivalent to the growth function of some finitely generated associative algebra?*

Conjecture. *For all sufficiently large functions $g : \mathbb{N} \to \mathbb{N}$, the following assertions are equivalent:*

(1) g is asymptotically equivalent to the growth function of some finitely generated associative algebra,
(2) g is asymptotically equivalent to the growth function of some finitely generated primitive algebra,
(3) g is asymptotically equivalent to the growth function of some finitely generated nil algebra.

L. Bartholdi and A. Smoktunowicz [5] proved that if g is an increasing submultiplicative function such that $g(Cn) \geq ng(n)$ for some $C \in \mathbb{N}$ and all $n \in \mathbb{N}$ then g is asymptotically equivalent to the growth function of a finitely generated associative algebra. Moreover, B. Greenfeld [13] showed that in this case there exists a finitely generated primitive monomial algebra with the growth function equivalent to g. This partially answers the questions above.

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