Journal of the Indian Math. Soc.
42 (1978) 197–202

RINGS WHOSE CYCLIC MODULES ARE CONTINUOUS

By S.K. JAIN AND SAAD MOHAMED

[Received July 12, 1976: revised January 17, 1977]

RINGS WHOSE CYCLIC modules are quasi-injective were studied by Ahsan [1], and fully characterized by Koehler [2] as semiperfect rings which are finite direct sum of rings each of which is simple artinian or rank zero maximal valuation duo rings. Following Utumi [7] Mohamed and Bouhy [4] introduced the notion of a continuous module as a generalization of quasi-injective module. They also characterized a ring whose finitely generated modules are continuous as semisimple artinian. In this paper, we study the class of rings whose cyclic modules are continuous. Such rings are called right *cc*-rings. Our main theorem is: A semiperfect ring R is a right *cc*-ring if any only if R is a finite product of rings which are either simple artinian or right valuation right duo with nil radical.

1. Definitions, Notations and preliminaries. All rings considered have unities and all modules are unital right modules. A module M is indecomposable if 0 and M are the only direct summands of M. A module M is uniform if every two nonzero submodules of M have non-trivial intersection. An idempotent e of a ring R is indecomposable if the right R-module eR is indecomposable. A ring R is local if it has exactly one maximal right ideal. Rad R will stand for the Jacobson radical of a ring R. A ring R is semiperfect if and only if $R/\text{Rad } R$ is semisimple artinian and idempotents modulo Rad R can be lifted. R is a right valuation ring if for every two right ideals A and B of R either $A \subset B$ or $B \subset A$. R is a right duo ring if every right ideal of R is a two-sided ideal. If X is a subset of a ring R then $X^{\perp}$ will denote the right annihilator of X in R. A module M is embedded in a module N (notation $M \hookrightarrow N$) if there is a monomorphism of M into N.

Following Utumi [7], a module M is called continuous if it satisfies the following conditions:

CONDITION 1.1. Every submodule of M is essential in some direct summand of M.

CONDITION 1.2. If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.

A ring R is called a right cc-ring if every cyclic R-module is continuous.

2. We start by characterizing local right cc-rings.

LEMMA 2.1. *A homomorphic image of a right cc-ring is a rtght cc-ring.*

PROOF. Obvious.

PROPOSITION 2.2. *A local ring R is a right cc-ring if and only if R is a right valuation right duo ring with nil radical.*

PROOF. Assume that R is a right cc-ring. Let A and B be right ideals of R such that $A \not\subset B$. Consider the cyclic module $R/A \cap B$. Since R is a local ring $R/A \cap B$ is an indecomposable R-module. As $R/A \cap B$ is also continuous, $R/A \cap B$ is uniform by ([4], Prop. 2.1). Now, since

$$(A/A \cap B) \cap (B/A \cap B) = 0,$$

we get $B/A \cap B = 0$, and hence $B \subset A$. Thus R is a right valuation ring.

To show that R is a right duo ring, it is enough to prove that every principal right ideal is a two-sided ideal. Consider a principal right ideal xR and let $a \in R$. We examine two cases:

(i) $a \notin \text{Rad } R$. Then, by ([3], p. 75), a is a unit. Hence $axR \cong xR$. Suppose that $axR \not\subset xR$. Then because R is a right valuation ring, $xR \subset axR$. Since axR is continuous, then by Condition 1.2, xR is a direct summand of axR. As axR is indecomposable, we get $xR = axR$, a contradiction. Therefore $axR \subset xR$.

(ii) $a \in \text{Rad } R$. Then $(1-a) \notin \text{Rad } R$. Thus $(1-a)\,xR \subset xR$ by (i). Now, for every $r \in R$

$$axr = xr - xr + axr = xr - (1 - a)\, xr.$$

Therefore $axR \subset xR$.

In either case, we get $axR \subset xR$. Hence xR is a two sided ideal of R.

Let a be an element of R which is not nilpotent. Let P be an ideal of R which is maximal with respect to the property of being disjoint from the infinite multiplicative set $(a, a^2, a^3, \ldots)$. Then P is prime. Since, R/P is a prime right duo ring, R/P is a domain. Also, by Lemma 2.1, R/P is a continuous right (R/P)-module. Then Condition 1.2 implies that R/P is a division ring. Hence $P = \text{Rad } R$, and therefore $a \notin \text{Rad } R$. So that every element in Rad R is nilpotent. This completes the proof of the "only if" part.

Conversely, assume that R is a right valuation right duo ring with nil radical. Let A be a (right) ideal of R. Because R is a right valuation ring, R/A is a uniform right R-module. Hence condition 1.1 is trivially satisfied. Let B/A be a submodule of R/A which is isomorphic to a direct summand of R/A. Since R/A is uniform, $B/A \cong R/A$. Let ϕ be the given isomorphism of R/A onto B/A, and let $\phi(1 + A) = y + A$, for some $y \in B$. We claim that $y \notin \text{Rad } R$. Suppose not. Then y is nilpotent and we can choose a positive integer m such that $y^m \in A$ and $y^{m-1} \notin A$. But then

$$\phi(y^{m-1} + A) = y^m + A = 0.$$

Since ϕ is an isomorphism, $y^{m-1} + A = 0$. Hence $y^{m-1} \in A$, a contradiction. Thus $y \notin \text{Rad } R$, so that y is a unit. Therefore $B/A = R/A$, and hence condition 1.2 holds. Thus R/A is a continuous right R-module. This completes the proof.

LEMMA 2.3. *If $A \oplus B$ is a continuous module, and if $\phi : A \to B$ is a monomorphism, then ϕA is a direct summand of B.*

PROOF. Since $\phi A \cong A$ and $A \oplus B$ is continuous, ϕA is a direct summand of $A \oplus B$. As $\phi A \subset B$, ϕA is a direct summand of B.

The following is an immediate consequence.

COROLLARY 2.4. *Let A be a nonzero module and B is an indecomposable module such that $A \times B$ is continuous. Then*

$R = R_1 \oplus R_2$, where R_1 is semisimple artinian and R_2 is a finite product of right valuation right duo rings with nil radical.

PROOF. The "if" part follows by Proposition 2.2 and Lemma 2.7. Conversely, assume that R is a right cc-ring. Since R is semiperfect,

$$R = e_1R \oplus e_2R \oplus \ldots \oplus e_nR,$$

where $\{e_i : 1 \leqslant i \leqslant n\}$ is a set of orthogonal indecomposable idempotents. Let $[e_iR] = \Sigma e_iR$, where the Σ runs over all i for which $e_iR \cong e_tR$. Renumbering if necessary, we may write

$$R = [e_1R] \oplus [e_2R] \oplus \ldots \oplus [e_kR],$$

where $k \leqslant n$. Using Proposition 2.6, it follows that $[e_iR]$ is an ideal, $1 \leqslant i \leqslant k$. If for some i, $[e_iR]$ contains more than one summand, then again by Proposition 2.6, $[e_iR]$ is a sum of minimal right ideals of R, and hence $[e_iR]$ is a semisimple artinian ring. And if for some j, $[e_jR]$ consists of one summand, then $[e_jR]$ is a local ring (see [3], p. 76). That $[e_jR]$ is a right valuation right duo ring with nil radical follow by Proposition 2.2 and Lemma 2.7. This completes the proof.

REMARK. We expect that a right cc-ring is semiperfect but have not been able to settle this. It can be easily shown that a right cc-ring is semiperfect iff a regular right cc-ring does not contain any infinite family of orthogonal idempotents. This follows from the facts that if R is a right cc-ring then $R/\text{Rad } R$ is a regular right cc-ring and the idempotents can be lifted modulo Rad R.

(Added Dec. 20, 1978) A module M having a property $(*)$ that for each pair of submodules M_1, M_2 with $M_1 \cap M_2 = 0$, each projection $\pi_i: M_1 \oplus M_2 \to M_i$, $i = 1, 2$, can be extended to an endomorphism of M is called *π-injective* (V.K. Goel and S.K. Jain, π-injective modules and rings whose cyclics are π-injective, *Communications in Algebra*, 6 (1978), 59–73). Since a continuous module can be shown to π-injective, it follows from the above remark and the Corollary 2.7 in the aforementioned paper of Goel-Jain that a cc-ring is semi perfect. It has been brought to our attention that the module with the property $(*)$ has been called quasi-continuous by Jermy (Louis Jermy, Modules et anneaux quasi-continus, *Cand. Math. Bull.*, 17 (1974), 217–228).

REFERENCES

1. AHSAN, JAVED, Rings all of whose cyclic modules are quasi-injective, *Proc. London Math. Soc.*, 27 (1973) 425–439.
2. KOEHLER, A., Rings with quasi-injective cyclic modules, *Quart. J. Math. Oxford*, 25 (1974), 51–55.

3. LAMBEK, JOACHIM, *Lectures on rings and modules*, Blaisdell (1966).
4. MOHAMED, SAAD and THANAA BOUHY, Continuous modules, *The Arabian Journal for Science and Engineering*, Univ. of Petroleum and Minerals, Dhahran, Saudi Arabia, (to appear).
5. UTUMI, Y., On continuous regular rings and semi-simple self-injective rings, *Can. J. Math.*, 12 (1960), 597–605.
6. UTUMI, Y., On continuous regular rings, *Can. Math. Bull.*, 4 (1961), 63–69.
7. UTUMI, Y., On continuous rings and self-injective rings, *Trans. Amer. Math. Soc.*, 118 (1965), 158–173.
8. UTUMI, Y., On the continuity and self-injectivity of a complete regular ring, *Can. J. Math.*, 18 (1966), 404–412.

Ohio University
Athens, Ohio 45701
U.S.A.

Kuwait University
Kuwait