DIRECT SUMS OF INJECTIVE AND PROJECTIVE MODULES

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Abstract. It is well-known that a countably injective module is Σ-injective. In Proc. Amer. Math. Soc. 316, 10 (2008), 3461-3466, Beidar, Jain and Srivastava extended it and showed that an injective module M is Σ-injective if and only if each essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules. This paper extends and simplifies this result further and shows that an injective module M is Σ-injective if and onIy if each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective. Some consequences and generalizations are also obtained.

1. Introduction

All rings considered in this paper are associative rings with identity and all modules are right unital. A module M is said to be Σ-injective if $M^{(\alpha)}$ is injective for any cardinal α, where we denote by $M^{(\alpha)}$, the direct sum of α copies of M. It is well-known that a module M is Σ-injective if and only if $M^{(\aleph_0)}$ is injective ([5], [11]). Several other characterizations for an injective module to be Σ-injective are given by Cailleau [3], Faith [5], and Goursaud - Valette [6].

Recently, Beidar, Jain, and Srivastava [2] gave the following characterization for an injective module to be Σ-injective.

Theorem 1. [2] *An injective module M is Σ-injective if and only if each essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules.*

In this paper we extend the above theorem and provide the following new characterization for an injective module to be Σ-injective in terms of the direct sums of injective modules and projective modules.

Theorem 2. *Let M be any module. Then the injective hull $E(M)$ is Σ-injective if each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective.*

As a consequence, we obtain that an injective module M is Σ-injective if and only if each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective.

It also follows from the above theorem that an arbitrary module M is Σ-injective if and only if each essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules.

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In this result we have not only strengthened the Theorem 1, but we have also provided a much more succinct proof for it.

In the later part of the paper, we characterize Σ-injective modules in terms of the direct sums of quasi-injective and projective modules.

Theorem 3. *Let M be an injective R-module. Then M is Σ-injective if and only if R is right q.f.d. relative to M and each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either quasi-injective or projective.*

Note that a ring R is called right *q.f.d.* relative to M if no cyclic right R-module contains an infinite direct sum of modules isomorphic to submodules of M. We shall write $N \subseteq_e M$ whenever N is an essential submodule of M. The reader is referred to [8] for the details on quasi-injective modules; and [4] and [10] for the general references on module theory.

These characterizations of Σ-injective modules lead to new characterizations of right noetherian rings which extend the results of Bass [1] and Beidar-Jain-Srivastava [2].

We begin with a proof of Theorem 2, which is an adaptation of the techniques used by Guil Asensio and Simson in [7].

2. Proof of Theorem 2

Suppose each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective. Assume to the contrary that $E(M)$ is not Σ-injective. Set $E = E(M)$. Then $\oplus_{i\in\mathcal{I}}E_i$, $(E_i \cong E)$ is not injective for some infinite index set $\mathcal{I}$. Thus, by Baer's injectivity criterion, there exists a right ideal A of R and a right R-homomorphism $g : A \to \oplus_{i\in\mathcal{I}}E_i$ such that the set $I^{'} = \{j \in \mathcal{I} : \pi_j \circ g \neq 0\}$ is infinite, where $\pi_j : \oplus_{i\in\mathcal{I}}E_i \to E_j$ is the canonical projection. Because otherwise $Im(g)$ would be contained in a finite direct subsum of $\oplus_{i\in\mathcal{I}}E_i$; and since any finite direct sum of injective modules is injective, the map g would extend to R and this will contradict our assumption that $\oplus_{i\in\mathcal{I}}E_i$ is not injective. Let $\mathcal{J}$ be a countably infinite subset of $I^{'}$. Now, choose an element $a_j \in A$. Let $b_j = g(a_j)$ and $N_j = b_jR$. Then N_j is a cyclic submodule of E_j. Since $\mathcal{J}$ is countable and each N_j is cyclic, $\oplus_{j\in\mathcal{J}}N_j$ is countably generated. Denote by Q_j, an injective hull of N_j in E_j. Let $Q = E(\oplus_{j\in\mathcal{J}}Q_j)$ be an injective hull of $\oplus_{j\in\mathcal{J}}Q_j$. Let $\pi : \oplus_{i\in\mathcal{I}}E_i \to \oplus_{j\in\mathcal{J}}Q_j$ be the epimorphism that carries E_i to zero if $i \in \mathcal{I} \setminus \mathcal{J}$; whereas for all $i \in \mathcal{J}$, the restriction of π to E_i, $\pi|_{E_i} = \beta \circ \alpha$ where $\alpha : E_i \to Q_i$ is the natural direct summand projection and $\beta : Q_i \to \oplus_{j\in\mathcal{J}}Q_j$ is the canonical monomorphism. We claim that the homomorphism $f = \pi \circ g : A \to \oplus_{j\in\mathcal{J}}Q_j$ cannot be extended to a homomorphism $h : R \to \oplus_{j\in\mathcal{J}}Q_j$ along the monomorphism $\mu : A \to R$. In particular, we claim that $\oplus_{j\in\mathcal{J}}Q_j$ is not injective. Suppose to the contrary that f admits such an extension h. Since $h(1)$ is contained only in a finite direct subsum of $\oplus_{j\in\mathcal{J}}Q_j$, $Im(f)$ is contained in $\oplus_{j\in\mathcal{F}}Q_j$ for some finite subset $\mathcal{F}$ of $\mathcal{J}$. Thus, $\pi_j \circ f = 0$ for each $j \in \mathcal{J} \setminus \mathcal{F}$. But this is not possible as $\pi_j \circ f : A \to Q_j$ and each Q_j is an injective envelope of N_j in E_j.

Consider the set Ω of submodules P of Q satisfying the following three conditions:

(1) $\oplus_{j\in\mathcal{J}}Q_j \subseteq P \subseteq Q$,

(2) P is a direct sum of injective submodules of Q,
(3) $f = \pi \circ g : A \to \oplus_{j\in\mathcal{J}} Q_j \subseteq P$ cannot be extended to a homomorphism $h : R \to P$ along the monomorphism $\mu : A \to R$.

Clearly, Ω is non-empty as $\oplus_{j\in\mathcal{J}} Q_j \in \Omega$. Define partial order $\le$ on Ω as $P_1 \le P_2$ if and only if $P_1 \subseteq P_2$. We claim that Ω is an inductive set under this partial order. Let $\{P_k\}_{k\in\mathcal{K}}$ be a chain in Ω. Let $P = \cup_{k\in\mathcal{K}} P_k$. As $\oplus_{j\in\mathcal{J}} Q_j \subseteq_e Q = E(\oplus_{j\in\mathcal{J}} Q_j)$, we have $\oplus_{j\in\mathcal{J}} Q_j \subseteq_e P$. Hence, $\oplus_{j\in\mathcal{J}} E_j \subseteq_e P$. But, we have $\oplus_{j\in\mathcal{J}} M_j \subseteq_e \oplus_{j\in\mathcal{J}} E_j$, ($M_j \cong M$). Therefore, $\oplus_{j\in\mathcal{J}} M_j \subseteq_e P$. By assumption, $P = (\oplus_{u\in\mathcal{U}} C_u) \oplus (\oplus_{v\in\mathcal{U}'} C_v^{'})$, where the C_u are injective modules and $C_v^{'}$ are projective modules. By Kaplansky [9], we know that each projective module is a direct sum of countably generated modules. Hence, we have $P = (\oplus_{u\in\mathcal{U}} C_u) \oplus (\oplus_{v\in\mathcal{V}} D_v)$, where each C_u is an injective module and each D_v, a countably generated module. Moreover, $\mathcal{U}$ and $\mathcal{V}$ are countable sets, because P contains a countably generated submodule $\oplus_{j\in\mathcal{J}} N_j$ such that $\oplus_{j\in\mathcal{J}} N_j \subseteq_e P$. Thus, $D = \oplus_{v\in\mathcal{V}} D_v$ is countably generated. We may write $D = \Sigma_{n\in\mathbb{N}} D_n^{'}$ as a countable sum of finitely generated submodules. Since $D_1^{'}$ is finitely generated, $D_1^{'} \subset \cup_{k\in\mathcal{F}} P_k$ for some finite subset $\mathcal{F} \subset \mathcal{K}$. Furthermore, since each P_k is a direct sum of injective submodules, P contains an injective hull $E(D_1^{'})$ of $D_1^{'}$. Moreover, $E(D_1^{'}) \cap (\oplus_{u\in\mathcal{U}} C_u) = 0$, because $D_1^{'} \cap (\oplus_{u\in\mathcal{U}} C_u) = 0$. Thus

$$E(D_1^{'}) \cong \frac{(\oplus_{u\in\mathcal{U}} C_u) \oplus E(D_1^{'})}{\oplus_{u\in\mathcal{U}} C_u} \subseteq \frac{(\oplus_{u\in\mathcal{U}} C_u) \oplus (\oplus_{v\in\mathcal{V}} D_v)}{\oplus_{u\in\mathcal{U}} C_u} \cong \oplus_{v\in\mathcal{V}} D_v = D.$$

Clearly the above isomorphism fixes $D_1^{'}$. Thus, D contains the injective hull $E(D_1^{'})$ of $D_1^{'}$, and therefore we have a decomposition $D = E(D_1^{'}) \oplus D_1^{''}$. We denote by $D_{1,n}^{'}$ the image of $D_n^{'}$ under the natural projection on $D_1^{''}$ for $n \ge 2$. Set $D_{1,1}^{'} = D_1^{'}$ for simplicity. It is easy to check that $D = E(D_{1,1}^{'}) \oplus \Sigma_{n\ge 2} D_{1,n}^{'}$. This yields us a decomposition $P = (\oplus_{u\in\mathcal{U}} C_u) \oplus E(D_{1,1}^{'}) \oplus \Sigma_{n\ge 2} D_{1,n}^{'}$. By applying the same construction to P and $D_{1,2}^{'}$ we get $P = (\oplus_{u\in\mathcal{U}} C_u) \oplus E(D_{1,1}^{'}) \oplus E(D_{2,2}^{'}) \oplus \Sigma_{n\ge 3} D_{2,n}^{'}$. Repeating this process, we construct an infinite set $\{E(D_{n,n}^{'})\}_{n\in\mathbb{N}}$ of injective submodules of P such that for each $m \in \mathbb{N}$, we have that $(\oplus_{u\in\mathcal{U}} C_u) \oplus (\oplus_{n=1}^{m} E(D_{n,n}^{'})) \subseteq P$. Moreover, by construction, $D_m^{'} \subseteq \oplus_{n=1}^{m} D_{n,n}^{'}$, for each $m \in \mathbb{N}$. As a consequence, $D \subseteq \oplus_{n\in\mathbb{N}} E(D_{n,n}^{'})$, so $P = (\oplus_{u\in\mathcal{U}} C_u) \oplus (\oplus_{n\in\mathbb{N}} E(D_{n,n}^{'}))$. Thus P satisfies (2). Finally, we proceed to show that the homomorphism $f = \pi \circ g : A \to \oplus_{j\in\mathcal{J}} Q_j \subseteq P$ cannot be extended to a homomorphism $h : R \to P$ along the monomorphism $\mu : A \to R$. Suppose, if possible, that g admits such an extension h. Since $Im(h)$ is finitely generated and $\{P_k\}_{k\in\mathcal{K}}$ is a chain, there exists a $k \in \mathcal{K}$ such that $Im(h) \subseteq P_k$. This yields a contradiction because $P_k \in \Omega$ and therefore, by assumption, f cannot be extended to a homomorphism $R \to P_k$. Hence, $P \in \Omega$. This establishes our claim that Ω is an inductive set and hence by Zorn's Lemma, Ω has a maximal element, say P_0. By hypothesis, $P_0 = \oplus_{t\in\mathcal{T}} W_t$, where each W_t is injective. Let $\varphi_t : P_0 \to W_t$ be the canonical projections. Since, by hypothesis, f cannot be extended to a homomorphism $h : R \to P_0$, there exists an infinite subset $\mathcal{T}^{'} \subseteq \mathcal{T}$ such that $\varphi_t \circ f \ne 0$, for each $t \in \mathcal{T}^{'}$. Because otherwise $Im(f)$ would be contained in $\oplus_{\mathcal{F}} W_t$ where F is a finite set. Since $\oplus_{\mathcal{F}} W_t$ is injective, f would extend to a homomorphism $R \to \oplus_{\mathcal{F}} W_t \subseteq P_0$, yielding a contradiction. Let us write $\mathcal{T}$ as a disjoint union of infinite sets $\mathcal{T}_1$ and $\mathcal{T}_2$. Denote $\varphi_{\mathcal{T}_1} : \oplus_{t\in\mathcal{T}} W_t \to \oplus_{t\in\mathcal{T}_1} W_t$ and $\varphi_{\mathcal{T}_2} : \oplus_{t\in\mathcal{T}} W_t \to \oplus_{t\in\mathcal{T}_2} W_t$. Note that $\varphi_{\mathcal{T}_i} \circ f$:

$A \to \oplus_{t\in\mathcal{T}_i} W_t$ cannot be extended to a homomorphism $h : R \to \oplus_{t\in\mathcal{T}_i} W_t$ for each $i \in \{1, 2\}$. Because otherwise $Im(h) \subset \oplus_{t\in\mathcal{F}} W_t$, where $\mathcal{F}$ is a finite set and hence $\varphi_t \circ f = \varphi_t \circ \varphi_{\mathcal{T}_i} \circ f = 0$, for each $t \in \mathcal{T}_i \setminus \mathcal{F}$, a contradiction. This implies that $\oplus_{t\in\mathcal{T}_1} W_t$ is not injective and hence $\oplus_{t\in\mathcal{T}_1} W_t \neq E(\oplus_{t\in\mathcal{T}_1} W_t)$. Thus, $P_0 = \oplus_{t\in\mathcal{T}} W_t \subsetneq E(\oplus_{t\in\mathcal{T}_1} W_t) \oplus (\oplus_{t\in\mathcal{T}_2} W_t)$. Now, it may be observed that f cannot be extended to a homomorphism $R \to E(\oplus_{t\in\mathcal{T}_1} W_t) \oplus (\oplus_{t\in\mathcal{T}_2} W_t)$, because otherwise $\varphi_{\mathcal{T}_2} \circ f$ would extend to a homomorphism $R \to \oplus_{t\in\mathcal{T}_2} W_t$, a contradiction. Therefore, $E(\oplus_{t\in\mathcal{T}_1} W_t) \oplus (\oplus_{t\in\mathcal{T}_2} W_t) \in \Omega$. But this yields a contradiction to the maximality of P_0. Hence, $E(M)$ must be Σ-injective.

Corollary 4. *An injective right R-module M is Σ-injective if and only if each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective.*

Before proceeding further, we would like to introduce some terminology.

An internal direct sum $\oplus_{i\in\mathcal{I}} A_i$ of submodules of a module M is called a *local summand* of M, if given any finite subset $\mathcal{F}$ of $\mathcal{I}$, the direct sum $\oplus_{i\in\mathcal{F}} A_i$ is a direct summand of M.

Let $M = \oplus_{i\in\mathcal{I}} M_i$ be a decomposition of the module M into nonzero summands M_i. This decomposition is said to *complement direct summands* if, whenever A is a direct summand of M, there exists a subset $\mathcal{J}$ of $\mathcal{I}$ for which $M = (\oplus_{j\in\mathcal{J}} M_j) \oplus A$.

Now we are ready to prove the following.

Corollary 5. *An arbitrary right R-module M is Σ-injective if and only if each essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules.*

Proof. Suppose each essential extension of $M^{(\aleph_0)}$ is a direct sum of injective modules. Let $E = E(M)$. We have $M^{(\aleph_0)} \subset_e E^{(\aleph_0)}$. By assumption, $M^{(\aleph_0)}$ itself is a direct sum of injective modules. Therefore, $M^{(\aleph_0)}$ is a local summand of $E^{(\aleph_0)}$. Since by Theorem 2, E is Σ-injective, so is $E^{(\aleph_0)}$. Hence $E^{(\aleph_0)}$ has an indecomposable decomposition that complements direct summands. Therefore, any local summand of $E^{(\aleph_0)}$ is a direct summand (see [4], 13.6). Hence, $M^{(\aleph_0)}$ is a direct summand $E^{(\aleph_0)}$. Therefore, $M^{(\aleph_0)}$ is injective and thus M is Σ-injective. The converse is obvious. $\square$

It is well-known that a ring R is right noetherian if and only if every direct sum of injective right R-modules is injective [1]. From this it follows that a ring R is right noetherian if and only if each injective right R-module is Σ-injective. As a consequence, we have the following characterization for a right noetherian ring.

Theorem 6. *A ring R is right noetherian if and only if for each injective right R-module M, every essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective.*

This extends the result of Beidar, Jain and Srivastava (Theorem 4, [2]).

Before giving the proof of Theorem 3, we recall that a module M is said to be *locally finite dimensional* if any finitely generated submodule of M has finite Goldie dimension. We say that the Goldie dimension $G\dim_U(N)$ of N with respect to U is finite, written as $G\dim_U(N) < \infty$, if N does not contain an infinite independent family of nonzero submodules which are isomorphic to submodules of U. A module N is said to be *q.f.d.* relative to U if for any factor module $\bar{N}$ of N, $G\dim_U(\bar{N}) < \infty$.

3. Proof of Theorem 3

Proof. Let M be a Σ-injective module. Then $M^{(\aleph_0)}$ is injective. Since an injective module has no proper essential extension, we only need to show that R is right *q.f.d.* relative to M. A proof for the fact that when an injective module M is Σ-injective, R is right *q.f.d.* relative to M, is hidden in [2], but we will give a direct and shorter proof here.

Assume to the contrary that R is not right *q.f.d.* relative to M. Then there exists a cyclic right module C with an infinite independent family $\{V_i : i \in \mathcal{I}\}$ of nonzero submodules of C such that each V_i is isomorphic to a submodule of M and $\oplus_{i\in\mathcal{I}} V_i$ is essential in C. Set $M_i = M$, $i \in \mathcal{I}$. Since M is Σ-injective, the monomorphism $\varphi : \oplus_{i\in\mathcal{I}} V_i \longrightarrow \oplus_{i\in\mathcal{I}} M_i$ such that $\varphi(V_i) \subseteq M_i$ for all $i \in \mathcal{I}$ extends to a monomorphism $f : C \longrightarrow \oplus_{i\in\mathcal{I}} M_i$. Now, since C is cyclic, there exists a finite subset $\mathcal{J} \subseteq \mathcal{I}$ such that $f(C) \subseteq \oplus_{j\in\mathcal{J}} M_j$. Therefore, $f(V_k) \cap M_k \subseteq f(C) \cap M_k = 0$ for all $k \notin \mathcal{J}$, a contradiction to the fact that $f(V_i) = \varphi(V_i) \subseteq M_i$ for all i.

Thus, R is right *q.f.d.* relative to M.

Conversely, assume that R is right *q.f.d.* relative to M and each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either quasi-injective or projective. In view of the Theorem 2, to prove that M is Σ-injective, it suffices to show that every essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective.

Set $M_i = M$, $i \in \mathbb{N}$. Since R is right *q.f.d.* relative to M, it follows that every nonzero cyclic and hence every nonzero submodule of M contains a uniform submodule. Now, consider the set $\mathcal{S}$ of independent families $(M_k)_{k\in\mathcal{K}}$ of uniform injective modules $0 \neq M_k \subseteq M$. Suppose $\mathcal{S}$ is partially ordered by $(M_k)_{k\in\mathcal{K}} \leq (N_l)_{l\in\mathcal{L}}$ if and only if $\mathcal{K} \subseteq \mathcal{L}$ and $M_k = N_k$ for $k \in \mathcal{K}$. By Zorn's lemma we get a maximal independent family $(M_i)_{i\in\mathbb{N}}$ of uniform injective submodules. Clearly $\oplus_{i\in\mathbb{N}} M_i \subseteq_e M$, because otherwise we will get a contradiction to the maximality of this independent family of submodules. This yields that we have an independent family $\{M_{i_j} : j \in \mathcal{J}\}$ of uniform injective submodules such that $\oplus_{j\in\mathcal{J}} M_{i_j} \subseteq_e M_i$. Set $G = \oplus_{i,j} M_{i_j}$. So, $G \subseteq_e \oplus_{i\in\mathbb{N}} M_i$. Let $E = E(\oplus_{i\in\mathbb{N}} M_i)$.

Let V be any essential extension of $\oplus_{i\in\mathbb{N}} M_i$. By our assumption $V = (\oplus_{k\in\mathcal{K}_1} V_k) \oplus (\oplus_{k\in\mathcal{K}_2} U_k)$ where each V_k is quasi-injective and each U_k is projective. We will show that each V_k is injective.

Since $G \subseteq_e \oplus_{i\in\mathbb{N}} M_i \subseteq_e V = (\oplus_{k\in\mathcal{K}_1} V_k) \oplus (\oplus_{k\in\mathcal{K}_2} U_k)$, we have $G \cap V_k \subseteq_e V_k$. Let $A = \sum_{i=1}^n a_i R$ be any finitely generated submodule of E. Since R is *q.f.d.* relative to M, by induction it may be shown that $G\dim(\sum_{i=1}^n a_i R) < \infty$. Hence E is locally finite dimensional. Therefore, each V_k is locally finite dimensional.

Because $V_k \cap G \subset \oplus_{i\in\mathbb{N}} M_i$, $V_k \cap G$ contains an independent family $\{V'_{k_l} : l \in \mathcal{L}'_k\}$ of submodules, with $\oplus_{l\in\mathcal{L}'_k} V'_{k_l} \subseteq_e V_k \cap G \subseteq_e V_k$, where each V'_{k_l} is isomorphic to a submodule of some M_i. By a standard argument using Zorn's lemma and using the local finite dimensionality of V_k, each V'_{k_l} contains essentially a direct sum of cyclic uniform submodules. Thus, we get an independent family $\{V_{k_l} : l \in \mathcal{L}_k\}$ of cyclic uniform submodules, with $\oplus_{l\in\mathcal{L}_k} V_{k_l} \subseteq_e V_k \cap G \subseteq_e V_k$. Take any $k \in \mathcal{K}$

and $l \in \mathcal{L}_k$. Since V_{k_l} is a cyclic submodule of $G = \oplus_{i,j} M_{i_j}$, there exists a finite subset $\mathcal{T} \subseteq \mathbb{N} \times \mathcal{J}$ such that $V_{k_l} \subseteq \oplus_{(i,j)\in\mathcal{T}} M_{i_j}$. Let $\widehat{V}_{k_l}$ be an essential closure of V_{k_l} in $\oplus_{(i,j)\in\mathcal{T}} M_{i_j}$. As $\mathcal{T}$ is finite, $\oplus_{(i,j)\in\mathcal{T}} M_{i_j}$ is injective, and so is $\widehat{V}_{k_l}$. Let $\pi : \oplus_{s\in\mathcal{K}} V_s \longrightarrow V_k$ be the canonical projection. Then $\pi|_{V_{k_l}}$ is the identity map and so $\pi|_{\widehat{V}_{k_l}}$ is a monomorphism. Setting $W_{k_l} = \pi(\widehat{V}_{k_l})$, we see that W_{k_l} is an injective submodule of V_k. Since $\pi|_{V_{k_l}}$ is the identity map, $V_{k_l} \subseteq W_{k_l}$. Therefore, $\{W_{k_l} : l \in \mathcal{L}_k\}$ is an independent family of injective submodules of V_k such that $\oplus_{l\in\mathcal{L}_k} W_{k_l} \subseteq_e V_k$.

Since R is given to be right $q.f.d.$ relative to M, it follows that R is right $q.f.d.$ relative to $\oplus_{l\in\mathcal{L}_k} W_{k_l}$. Now, we claim that the injective hull of $\oplus_{l\in\mathcal{L}_k} W_{k_l}$ coincides with the quasi-injective hull of $\oplus_{l\in\mathcal{L}_k} W_{k_l}$.

Set $F = \oplus_{l\in\mathcal{L}_k} W_{k_l}$ and $E' = E(F)$. Let $\Lambda = End(E'_R)$ be the endomorphism ring of E'_R. Let $x \in E'$. Because $F \subseteq_e E'$ and $xR \subseteq E'$, $F \cap xR \subseteq_e xR$. Furthermore, since R is right $q.f.d.$ relative to F, $G\dim_F(xR) < \infty$. This gives $G\dim_{E'}(xR) < \infty$, as $F \subseteq_e E'$. Therefore, xR and hence $F \cap xR$ has finite Goldie dimension. So, there exists a finitely generated submodule $B \subset_e xR \cap F \subset_e xR$. As $F = \oplus_{l\in\mathcal{L}_k} W_{k_l}$, there exists a finite subset $\mathcal{J} \subseteq \mathcal{L}_k$ such that $B \subseteq \oplus_{j\in\mathcal{J}} W_{k_l}$. Since $\oplus_{j\in\mathcal{J}} W_{k_l}$ is an injective module containing an essential submodule B of xR, $E(xR) \cong E(B) \subset \oplus_{j\in\mathcal{J}} W_{k_l} \subset F$. Thus, $E(xR) \cong F'$ where F' is a submodule of F. If $\varphi : F' \longrightarrow E(xR)$ is an isomorphism, then it can be extended to $\overset{\Lambda}{\varphi} : E' \longrightarrow E'$. So, $\overset{\Lambda}{\varphi} F' = E(xR)$. This gives $xR \subset \Lambda F$. So, $x \in \Lambda F$. Thus, $E' \subseteq \Lambda F$ and hence $E' = \Lambda F$. This establishes our claim that the injective hull of $\oplus_{l\in\mathcal{L}_k} W_{k_l}$ coincides with the quasi-injective hull of $\oplus_{l\in\mathcal{L}_k} W_{k_l}$.

Therefore, $E(\oplus_{l\in\mathcal{L}_k} W_{k_l}) = V_k$ and hence each V_k is injective. So, V is a direct sum of modules that are either injective or projective.

Thus, we have shown that each essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either injective or projective and hence by Theorem 2, M is Σ-injective. □

As a consequence, we have the following characterization for a right noetherian ring.

Corollary 7. *A ring R is right noetherian if and only if for each injective right R-module M, R is right $q.f.d.$ relative to M and every essential extension of $M^{(\aleph_0)}$ is a direct sum of modules that are either quasi-injective or projective.*

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