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On Σ-q rings

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ABSTRACT

Nakayama [T. Nakayama, On Frobeniusean algebras II, Annals of Mathematics 42 (1941) 1–21] showed that over an artinian serial ring every module is a direct sum of uniserial modules. Hence artinian serial rings have the property that each right (left) ideal is a finite direct sum of quasi-injective right (left) ideals. A ring with the property that each right (left) ideal is a finite direct sum of quasi-injective right (left) ideals will be called a right (left) Σ-q ring. For example, commutative self-injective rings are Σ-q rings. In this paper, various classes of such rings that include local, simple, prime, right non-singular right artinian, and right serial, are studied. Prime right self-injective right Σ-q rings are shown to be simple artinian. Right artinian right non-singular right Σ-q rings are upper triangular block matrix rings over rings which are either zero rings or division rings. In general, a Σ-q ring is not left-right symmetric nor is it Morita invariant.

1. Introduction

Artinian serial rings[1] introduced by Asano and Köthe is an important class of rings that appear at a number of places in the study of theory of rings and modules (see [5,18] for details). For example, these are rings of a finite representation type. Nakayama showed that every module over an artinian serial ring is a direct sum of uniserial modules [17]. Fuller proved that every indecomposable module over an artinian serial ring is quasi-injective [7]. Therefore, every right ideal in an artinian serial ring is a finite direct sum of quasi-injective right ideals. In this paper, we study rings having the property that each right ideal is a finite direct sum of quasi-injective right ideals. Such rings will be called right *Σ-q rings*. In particular, rings in which each right ideal is quasi-injective were studied by Beidar et al. [2], Byrd [3], Hill [9], Ivanov [10,11], Jain et al. [12] and Mohamed [16] and are known as *right q-rings*. Jain et al. [12] proved that a ring is a right q-ring if and only if, it is right self-injective and each essential right ideal is two-sided. In Beidar et al. [2], among others, non-local indecomposable right q-rings were shown to be either semisimple artinian or of the form $\begin{bmatrix} D & V & 0 & . & . & 0 \\ 0 & D & V & 0 & . & . \\ . & . & D & V & 0 & . \\ . & . & . & D & V & 0 \\ . & . & . & . & D & V \\ V(\alpha) & 0 & . & . & . & D \end{bmatrix}$, where V is one-dimensional both as a left D-space and a right D-space, $V(\alpha)$ is also a one-dimensional left D-space as well as a right D-space with right scalar multiplication twisted by an automorphism α of D. The purpose of this paper is to study right Σ-q rings and to provide a description of indecomposable right non-singular right artinian right Σ-q rings.

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[1] Generalized uniserial rings in the terminology of Nakayama.

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A right Σ-q ring need not be a right q-ring. For example, a matrix ring over an artinian serial ring, that is not semisimple, is a right Σ-q ring but not a right q-ring. We show that a simple right Σ-q ring must be an artinian ring. Lemma 5 shows that if $\mathbb{M}_n(R)$ is a right Σ-q ring then R is also a right Σ-q ring. One of our main results is that a prime right self-injective right Σ-q ring is simple artinian (Theorem 7). As a consequence, it follows that the ring of all linear transformations on a vector space V is a right Σ-q ring if and only if V is finite dimensional. The notion of Σ-q ring is not left-right symmetric. We give an example of an incidence ring which is a left Σ-q ring but not a right Σ-q ring (Example 16). Also, we show that right (left) Σ-q ring is not Morita invariant. We conclude by giving the structure of a right artinian right non-singular right Σ-q ring as a triangular matrix ring of certain block matrices (Theorems 23 and 24).

2. Definitions and notations

All rings considered in this paper have unity and all modules are right unital unless stated otherwise. A right R-module M is called quasi-injective if $\mathrm{Hom}_R(-, M)$ is right exact on all short exact sequences of the form $0 \longrightarrow K \to M \longrightarrow L \longrightarrow 0$. Johnson and Wong [13] characterized quasi-injective modules as those that are fully invariant under endomorphisms of their injective hulls. A ring R is called von Neumann regular if every principal right (left) ideal of R is generated by an idempotent. A von Neumann regular ring is called abelian if all its idempotents are central. A ring R is called a right (left) duo ring if every right (left) ideal of R is two-sided. If a ring is both right duo and left duo then it is called a duo ring. It is known that a right self-injective right duo ring is a duo ring (see Remark 2.3, page 314, [2]). A ring R is called directly finite if $xy = 1$ implies $yx = 1$, for all $x, y \in R$. The index of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The index of a two-sided ideal J in R is the supremum of the indices of all nilpotent elements of J. If this supremum is finite, then J is said to have bounded index. A module is called uniserial if its submodules are linearly ordered with respect to inclusion. A ring R is called a right (left) serial ring if R_R ($_RR$) is a direct sum of uniserial modules. If a ring is both a left as well as a right serial ring then it is called a serial ring. A ring R is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted. A right R-module M is called linearly compact in the discrete topology if any finite solvable system $\{x \equiv x_a (\mathrm{mod}\ I_a) : a \in A\}$ of congruences is solvable for any index set A, where $x_a \in M$ and I_a is a submodule. A ring R is called right linearly compact ring if R_R is linearly compact. Any right linearly compact ring is semiperfect [19]. A right ideal I of R is called an essential right ideal if $I \cap J \neq 0$ for every non-zero right ideal J of R. We will denote by $Soc(M)$ and $E(M)$, respectively, the socle and injective hull of M. Two rings R and S are said to be Morita equivalent if there exists a category equivalence F : mod-$R \longrightarrow$ mod-S. A ring theoretic property $\mathcal{P}$ is said to be Morita invariant if, whenever R has the property $\mathcal{P}$, so does every ring Morita equivalent to R. Let X be a finite partially ordered set and R any ring. The incidence ring of X with coefficients in R, denoted by $I(X, R)$, is the ring of functions $\{f : X \times X \longrightarrow R$ such that $f(x, y) = 0$ for each $x \nleq y\}$; multiplication is given by $(fg)(x, y) = \sum_{x \le z \le y} f(x, z)g(z, y)$. The ring $I(X, R)$ may also be viewed as the ring of square matrices with entries in R, whose rows and columns are indexed by X, where the (x, y) entry is 0 whenever $x \nleq y$.

3. Preliminaries

We will start with a basic lemma.

Lemma 1. *Let R be a right self-injective right Σ-q ring then each essential right ideal E of R is of the form $E = e_1T_1 \oplus e_2T_2 \oplus \cdots \oplus e_nT_n$ where $T_1, \ldots, T_n$ are two-sided ideals of R and $e_1, \ldots, e_n$ are orthogonal idempotents with $e_1 + e_2 + \cdots + e_n = 1$.*

Proof. By hypothesis, $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$, where each E_i is a quasi-injective right ideal. This gives $R = \widehat{E}_1 \oplus \cdots \oplus \widehat{E}_n$, where $\widehat{E}_i$ denotes injective hull of E_i. Write each $\widehat{E}_i$ as e_iR with $e_1 + e_2 + \cdots + e_n = 1$. Since each E_i is quasi-injective, we have $e_iRe_iE_i = E_i$. This gives $e_iRE_i = E_i$. Denote each two-sided ideal RE_i as T_i. So, we have $E = e_1T_1 \oplus e_2T_2 \oplus \cdots \oplus e_nT_n$ where $T_1, \ldots, T_n$ are two-sided ideals of R and $e_1, \ldots, e_n$ are orthogonal idempotents with $e_1 + e_2 + \cdots + e_n = 1$. □

Lemma 2. *Let R be a right Σ-q ring with no nontrivial idempotents. Then R is a right q-ring and hence a duo ring.*

Proof. If R is a right Σ-q ring with no nontrivial idempotents then R is right self-injective and by Lemma 1, each essential right ideal of R is two-sided. Hence R is a right q-ring [12]. Since R has no nontrivial idempotents, it must be right duo ([12], Theorem 2.3). Now, since a right self-injective right duo ring must be a duo ring (see [2], page 314, Remark 2.3), R is a duo ring. □

Corollary 3. *Let R be a local ring. Then R is a right Σ-q ring if and only if R is a right q-ring if and only if R is a right self-injective duo ring.*

Proof. This follows from the above lemma. □

Proposition 4. *If R is a right Σ-q ring and e is an idempotent in R such that $ReR = R$ then eRe is also a right Σ-q ring.*

Proof. We know that if $ReR = R$, then mod-R and mod-eRe are Morita equivalent under the functors given by $\mathcal{F}$: mod-$R \longrightarrow$ mod-eRe, $\mathcal{G}$: mod-$eRe \longrightarrow$ mod-R such that for any M_R, $\mathcal{F}(M) = Me$ and for any module T over eRe, $\mathcal{G}(T) = T \otimes_{eRe} eR$.

Suppose R is a right Σ-q ring. Let A be any right ideal of eRe. Then $AeR \cong A \otimes_{eRe} eR$ and AeR is a right ideal of R. Therefore, $AeR = A_1 \oplus \cdots \oplus A_n$ where A_i's are quasi-injective right ideals in R. By Morita equivalence we get that each A_ie is quasi-injective as an eRe-module. Then $A = AeRe = A_1e \oplus \cdots \oplus A_ne$ is a direct sum of quasi-injective right ideals. Hence eRe is a right Σ-q ring. $\square$

Lemma 5. *If $\mathbb{M}_n(R)$ is a right Σ-q ring, then R is also a right Σ-q ring.*

Proof. We have $R \cong e_{11}\mathbb{M}_n(R)e_{11}$ and $\mathbb{M}_n(R)e_{11}\mathbb{M}_n(R) = \mathbb{M}_n(R)$, where e_{11} is the usual matrix unit. Therefore, the result follows from the above proposition. $\square$

Later, in Example 18, we will show that if R is a right Σ-q ring then $\mathbb{M}_n(R)$ need not be a right Σ-q ring.

4. Prime rings, right self-injective rings, and regular rings

First, we consider a prime right self-injective right Σ-q ring.

Lemma 6. *A prime right self-injective right Σ-q ring must be von Neumann regular.*

Proof. We prove $Z(R_R) = 0$. If possible, let x be a non-zero element in $Z(R_R)$. Then there exists an essential right ideal E of R such that $xE = 0$. Now, by Lemma 1, $E = e_1T_1 \oplus e_2T_2 \oplus \cdots \oplus e_nT_n$ where $T_1, \ldots, T_n$ are two-sided ideals of R and $e_1, \ldots, e_n$ are orthogonal idempotents with $e_1 + e_2 + \cdots + e_n = 1$. Thus, $xe_iT_i = 0$. But since R is prime, $xe_i = 0$, for all i. Therefore, $x = 0$, a contradiction. Hence R is right non-singular and therefore, R must be von Neumann regular (see [15], page 362, Corollary 13.2). $\square$

Theorem 7. *A prime right self-injective ring R is right Σ-q ring if and only if R is artinian.*

Proof. Let R be a prime right self-injective right Σ-q ring. By Lemma 6, R is von Neumann regular. Since the two-sided ideals of R are well-ordered (see [8], Proposition 8.5), R has a unique maximal two-sided ideal, say A. We claim R/A is simple artinian. If each maximal right ideal of R/A is a summand then each right ideal of R/A is a summand and hence R/A is simple artinian. Or else, let N/A be a maximal essential right ideal of R/A. Then, N is an essential maximal right ideal of R. By Lemma 1, $N = e_1T_1 \oplus e_2T_2 \oplus \cdots \oplus e_nT_n$, where $T_1, \ldots, T_n$ are two-sided ideals of R and $e_1, \ldots, e_n$ are orthogonal idempotents with $e_1 + e_2 + \cdots + e_n = 1$. Since R/N is a simple module, we have $\frac{R}{N} \cong \frac{e_iR}{e_iT_i}$. Now, since $\frac{e_iR}{e_iT_i}$ is a direct summand of $\frac{R}{T_i}$, it is projective as R/T_i-module. As R/N is an R/T_i-module and $T_i \subset A \subset N$, it follows that R/N is a simple, projective R/A-module. Therefore, R/A is a simple ring with non-zero socle and hence artinian. Thus, R has bounded index of nilpotence (see [8], page 79) and so R is artinian (see [8], Theorem 7.9).

The converse is obvious. $\square$

We remark that if in the above theorem R is not a prime ring then it needs not be artinian. The following is an example of a non-prime von Neumann regular right self-injective right Σ-q ring which is not artinian.

Example 8. Let S be an infinite boolean ring and suppose $R = Q^r_{\max}(S)$. Clearly R is commutative, von Neumann regular, self-injective and hence a Σ-q ring. But, R is not artinian.

The following result is a consequence of Theorem 7.

Corollary 9. *The ring of linear transformations, $R = End_D(V)$ of a vector space V over a division ring D is a right Σ-q ring if and only if, the vector space V is finite-dimensional.*

Next, we have the following

Corollary 10. *A simple ring R is a right Σ-q ring if and only if R is artinian.*

Proof. Let R be a simple right Σ-q ring. Then R contains a quasi-injective right ideal, say A. As R is a simple ring, $RA = R$. So, we have $R = \Sigma r_iA$ where $r_i \in R$. As $1 \in \Sigma_{i=1}^k r_iA$, we have $R = \Sigma_{i=1}^k r_iA$. Therefore, $R \cong A^k/L \cong S$ where S is a direct summand of A^k and L is a submodule of A^k. Because A is quasi-injective, A^k is quasi-injective. Therefore, S_R and hence R_R is quasi-injective. Thus, R is right self-injective. Now, by Theorem 7, R is artinian.

The converse is trivial. $\square$

Remark 11. The above proof shows, in particular, that a simple ring with a quasi-injective right ideal must be right self-injective. Dinh van Huynh has told us in a private communication that in his forthcoming joint paper with John Clark, they have obtained the same result.

Next, we give an example of a simple right self-injective ring which is not a right Σ-q ring.

Example 12. Let S be an integral domain which is not a right Ore domain. Consider $R = Q^r_{\max}(S)$. We know that R is a simple, right self-injective, von Neumann regular ring (see [15], page 377, Corollary (13.38)′). If R is a right Σ-q ring then by Corollary 10, R is artinian and hence S is right Ore, which is not true. Therefore, R is not a right Σ-q ring.

Lemma 13. *Let R be a von Neumann regular ring. Suppose every ring homomorphic image of R is a right self-injective right Σ-q ring. Then R is semisimple artinian.*

Proof. Let P be a prime ideal of R. Then R/P is a prime von Neumann regular right self-injective right Σ-q ring. By Theorem 7, R/P is artinian. Since R is a von Neumann regular right self-injective ring and each primitive factor ring of R is artinian, we have $R \cong \Pi_{i=1}^{k}\mathbb{M}_{n_i}(S_i)$ where each S_i is an abelian regular right self-injective ring (see [8], Theorem 7.20). Thus, each S_i is a right self-injective duo ring.

Now, let C_i be any right ideal of S_i. Since S_i is a duo ring, C_i is a two-sided ideal and we have $\frac{\mathbb{M}_{n_i}(S_i)}{\mathbb{M}_{n_i}(C_i)} \cong \mathbb{M}_{n_i}(S_i/C_i)$. Since each $\mathbb{M}_{n_i}(S_i)$ satisfies the property that homomorphic images are right self-injective, $\mathbb{M}_{n_i}(S_i/C_i)$ is right self-injective and hence S_i/C_i is right self-injective. So, each cyclic right S_i- module is quasi-injective. Hence by Koehler [14], S_i is a finite direct sum of rings each of which is semisimple artinian or a rank 0 duo right linearly compact ring. But, since S_i is a von Neumann regular ring, we conclude that it must be semisimple artinian. Hence R is semisimple artinian. □

Next, we show

Proposition 14. *Let R be a right Σ-q ring and suppose every ring homomorphic image of $R/J(R)$ is a right self-injective right Σ-q ring, then R is semiperfect.*

Proof. By Lemma 13, $R/J(R)$ is semisimple artinian. Now, let the composition length of $R/J(R)$ be n. Then R_R cannot be a direct sum of more than n submodules. Thus, R_R is a finite direct sum of indecomposable right ideals, each of which is quasi-injective. Let $A = eR$ be any indecomposable summand of R_R. As $A = eR$ is quasi-injective, its ring of endomorphisms eRe is a local ring (see [15], page 244, ex. 6.32). Hence by ([1], Theorem 27.6), R is semiperfect. □

Example 18 shows, among others, that a right self-injective semiperfect ring need not be a right Σ-q ring.
For right self-injective right Σ-q rings, we ask the following question;

Problem 15. Is every right self-injective right Σ-q ring a directly finite ring?

One can answer this in the affirmative for right q-rings, but we are unable to answer this, in general.

5. Examples

Clearly, each right q-ring is a right Σ-q ring. However, there are plenty of examples of right Σ-q rings that are not right q-rings.

Recall from the introduction, that an artinian serial ring is a right Σ-q ring but such rings need not be q-rings. Furthermore, a left Σ-q ring need not be a right Σ-q ring. We give an example of an incidence ring which is a left Σ-q ring but not a right Σ-q ring.

Example 16. Let $X = \{1, 2, 3, 4\}$ be a partially ordered set with $1 < 2 < 3$ and $1 < 2 < 4$. Let F be a field. The incidence ring is given as

$$R = I(X, F) = \begin{bmatrix} F & F & F & F \\ 0 & F & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}.$$

R is a left artinian, left serial ring. Note that $Soc({}_RR) = Fe_{11} + Fe_{12} + Fe_{13} + Fe_{14}$. It is a folklore that ${}_RR$ is non-singular. But for completeness, we prove this result. Suppose $Z({}_RR) \neq 0$. Then there exists $z(\neq 0) \in soc(Z({}_RR))$. Clearly, $z = a_{11}e_{11} + a_{12}e_{12} + a_{13}e_{13} + a_{14}e_{14}$. Then $e_{11}z = z$, which implies $e_{11} \notin l.ann(z)$ and therefore, $Fe_{11} \cap l.ann(z) = 0$. Hence, $l.ann(z)$ is not essential in ${}_RR$, a contradiction. So, ${}_RR$ is non-singular.

Re_{11} is simple, so quasi-injective. As Re_{11} is non-singular and quasi-injective, $End(Re_{11}) \cong End(E(Re_{11}))$ (see [15], page 272, ex. 7.32). Hence, any endomorphism of $E = E(Re_{11})$ is given as multiplication by some element of F. Next, we show that Re_{22}, Re_{33}, and Re_{44} are quasi-injective.

The ring R is left serial, so Re_{33} is uniserial and hence uniform. We have $Soc(Re_{33}) = Fe_{13} \cong Fe_{11}$ as left R-module under the mapping $ae_{13} \longrightarrow ae_{11}$. Hence Re_{33}, being an essential extension of Fe_{13}, embeds in $E = E(Re_{11})$.

So, $\sigma(Re_{33}) \subseteq Re_{33}$, $\forall \sigma \in End(E)$. Therefore, Re_{33} is quasi-injective. Similarly, Re_{44} is quasi-injective. For every left ideal $C \subseteq Re_{33}$ and $\forall \sigma \in End(E)$, we have $\sigma(C) \subseteq C$ as Re_{33} is uniserial. Hence C is quasi-injective. Now, $Re_{22} \cong Re_{23} \subseteq Re_{33}$. Therefore, Re_{22} is quasi-injective. So, ${}_RR$ is a direct sum of quasi-injectives.

Next, we show that any indecomposable left ideal is uniserial and quasi-injective. Suppose there exists a left ideal $A \neq 0$ which is indecomposable and not uniform. Choose a left ideal A of smallest composition length. Let $\pi_i :_R R \longrightarrow Re_{ii}$ be projection. If for some i, $\pi_i(A) = Re_{ii}$, we get $A = A' \oplus B'$ for some $B', A' \cong Re_{ii}$. This gives $B' = 0$ and $A \cong Re_{ii}$, a contradiction as A is not uniform. Therefore, $\pi_i(A) \subseteq J(R)e_{ii}$ for all i, which gives $A \subseteq \begin{bmatrix} 0 & F & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B_2 \oplus B_3 \oplus B_4$.

Let $\pi_i' : J(R) \longrightarrow B_i$. Suppose $\pi_2'(A) \neq 0$. Now, B_2 is minimal and $B_2 \cong Re_{11}$ which is projective. So, $A \cong B_2$ which is uniform, a contradiction. Hence $\pi_2'(A) = 0$. This gives that $A \subseteq \begin{bmatrix} 0 & 0 & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L$.

Clearly, $l.ann(L) = \begin{bmatrix} 0 & 0 & F & F \\ 0 & 0 & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}$.

Let us denote $l.ann(L)$ by U, then L is a left R/U-module.

$R/U \cong \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, which is artinian serial.

As $A \subseteq L$, we note that A is a module over $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. This implies that A is uniserial and hence uniform. Let B be any indecomposable left ideal in R then B is uniform. For some i, $\pi_i(soc(B)) \neq 0$ and so π_i is one-one on $soc(B)$. As B is uniform, $\pi_i|_B : B \longrightarrow Re_{ii}$ is one-one. We have $B \cong \pi_i(B) \subseteq Re_{ii}$. This gives that B is uniserial and quasi-injective. Now, let I be any left ideal of R. Then I is a finite direct sum of indecomposable left ideals. Since we have already proved above that any indecomposable left ideal is quasi-injective, I is a finite direct sum of quasi-injective left ideals. Thus, R is a left Σ-q ring.

Now, we will show that R is not a right Σ-q ring. We have $e_{11}R = Fe_{11} + Fe_{12} + Fe_{13} + Fe_{14}$. Note that Fe_{13}, Fe_{14} are minimal right ideals. Clearly, $e_{11}R$ is not uniform and hence not quasi-injective. Therefore, R is not a right Σ-q ring.

We give below an example analogous to the above example that might be of interest to the reader.

Example 17. Let $X = \{1, 2, 3, 4\}$ be a partially ordered set with $1 < 3 < 4$ and $2 < 3 < 4$. Let F be a field. Then the incidence ring is given as

$$R = I(X, F) = \begin{bmatrix} F & 0 & F & F \\ 0 & F & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & F \end{bmatrix}.$$

By similar arguments as above R is a right Σ-q ring, but R is not a left Σ-q ring.

Let R be any right Σ-q ring and $S = \mathbb{M}_n(R)$. Now $R \cong e_{11}\mathbb{M}_n(R)e_{11}$ is a right Σ-q ring under the equivalence functor $\mathcal{F}$: mod-$\mathbb{M}_n(R) \longrightarrow$ mod-$e_{11}\mathbb{M}_n(R)e_{11}$, where $\mathcal{F}(M) = Me_{11}$. In particular, $\mathcal{F}(e_{11}\mathbb{M}_n(R)) = e_{11}\mathbb{M}_n(R)e_{11}$ gives that any right ideal contained in $e_{11}\mathbb{M}_n(R)$ is a direct sum of finitely many quasi-injective modules. However, the ring $\mathbb{M}_n(R)$ need not be a right Σ-q ring.

The example which follows shows that matrix ring over a right Σ-q ring (in fact, even a q-ring) need not be a right Σ-q ring. Therefore, the notion of right Σ-q ring is not a Morita invariant property.

Example 18. Let F be any field. Consider the ring $R = F[x, y]$ where $x^2 = 0$ and $y^2 = 0$. Then $R = F + Fx + Fy + Fxy$ is a commutative, local, artinian ring.

Here, $J(R) = Fx + Fy + Fxy$.

Let $z = a + bx + cy + dxy \in ann(J)$. This implies that $(a + bx + cy + dxy)x = 0$, which gives $ax + cxy = 0$, and so, $a = 0$ and $c = 0$. Similarly, $(a + bx + cy + dxy)y = 0$, gives $a = 0$ and $b = 0$. So, we have $z = dxy$. Therefore, $soc(R) = Fxy$. Thus, R is a commutative local artinian ring with simple socle and hence a self-injective ring ([6], Page 217, Exercise 5). Since a commutative self-injective ring must be a q-ring, R is a q-ring.

Let $K = soc(R)$. Then R/K is of length 3 and $J(R/K) = A \oplus B$ where A and B are simple.

Let $T = \mathbb{M}_2(R)$. Then $T = e_{11}T \oplus e_{22}T$, where $e_{11}T \cong e_{22}T$.

Note $e_{11}M_2(R/K)/e_{11}M_2(J(R/K))$ is simple. Therefore, $e_{11}M_2(R/K)$ is local as $M_2(R/K)(\cong \frac{M_2(R)}{M_2(K)})$-module, and hence as $T = M_2(R)$-module. Observe that $e_{11}M_2(R/K)$ has two minimal right ideals $e_{11}A + e_{12}A$ and $e_{11}B + e_{12}B$. By suitably factoring, we then obtain a local T-module, say, N_T with $length(soc(N)) = 2$. If S_T is a simple module, then N is embeddable in $E(N) \cong E(S) \oplus E(S) \cong T_T$. Thus, N is a local, indecomposable, non-uniform module embeddable in T and so T contains an indecomposable right ideal which is not quasi-injective. Therefore, T is not a right Σ-q ring.

6. Right non-singular right artinian rings, and right serial rings

The lemma that follows is well-known.

Lemma 19 (*see [15], page 359, Theorem 13.1*). *Let R be right artinian and let eR be a right non-singular indecomposable quasi-injective right ideal, where e is an idempotent in R. Then eRe is a division ring.*

Lemma 20. *Let R be a right artinian right non-singular right Σ-q ring. Let e, f be any two indecomposable idempotents in R such that $eRf \neq 0$. Let $D = eRe$ and $D^{'} = fRf$.*

(i) *Then eRf is a one-dimensional left vector space over D.*
(ii) *For any $0 \neq z \in eRf$, there exists embedding $\sigma : D^{'} \longrightarrow D$ such that for $fbf \in D^{'}$, $zfbf = \sigma(fbf)z$.*
(iii) *If R is also right serial, then σ is an isomorphism.*

Proof. (i) Consider any two non-zero elements erf and esf in eRf. Define a map $\phi : fR \longrightarrow erfR$ where $\phi(x) = erx$, for any $x \in fR$. This is clearly a well-defined surjective right R-homomorphism. Thus, we have $erfR \cong fR/Ker(\phi)$. Note that fR is indecomposable quasi-injective and hence uniform. If $Ker(\phi) \neq 0$ then as $Ker(\phi)$ is essential in fR, $fR/Ker(\phi)$ is singular. But, $erfR$ is non-singular, which gives a contradiction. Therefore, $Ker(\phi) = 0$ and hence $erfR \cong fR$. Similarly $esfR \cong fR$. Thus, we get an R-isomorphism $\sigma : erfR \longrightarrow esfR$ such that $\sigma(erf) = esf$. Since eR is quasi-injective, we extend σ to $\eta : eR \longrightarrow eR$. Now $\eta(e) = eue$. Then $\sigma(erf) = euerf$. Therefore, $esf = euerf$. Hence eRf is one-dimensional left vector space over D.

(ii) Now $eRf = Dz$, for some $z \in eRf$. Therefore, given any $fbf \in fRf$, $zfbf = uz$, for some $u \in D$. This defines a monomorphism $\sigma : D^{'} \longrightarrow D$ such that $\sigma(fbf) = u$. Also, we have $\sigma(fbf)z = uz = zfbf$.

(iii) Consider any two non-zero elements erf and esf in eRf. As eR is uniserial, we may suppose $esfR \subseteq erfR$. Then $esf = erfuf$ for some $fuf \in fRf$. Since fRf is a division ring, it follows that $esfRf = erfRf$. Hence eRf is one-dimensional over $D^{'}$. Therefore, $eRf = Dz = zD^{'}$. From which it is immediate that σ is an isomorphism. □

Proposition 21. *Let R be a right artinian right non-singular right Σ-q ring.*

(i) *If e, f are two indecomposable idempotents in R such that $eRf \neq 0$, then for any $0 \neq z \in eRf$, $eRez = zfRf$.*
(ii) *If R is an indecomposable ring, then $eRe \cong fRf$ for any two indecomposable idempotents e and f.*

Proof. (i) As eR is quasi-injective and indecomposable, it is uniform. So, $Soc(eR)$ is simple. Let $Soc(eR) \cong R/I$ where I is a maximal right ideal of R. If I is essential in R, then R/I is singular, which is a contradiction. Therefore, I is a summand of R. Hence, $Soc(eR) \cong gR$ for some indecomposable idempotent g in R. Let $h : gR \longrightarrow Soc(eR)$ be an isomorphism. Since $h(g) \in eR$, we have $h(g) = ewg$ for some $w = ewg$. Hence $Soc(eR) = ewgR$. Let $\mu : gR \longrightarrow wgR$ be a non-zero R-homomorphism given by $\mu(gr) = wgr$. It is monic because gR is uniform. Then $gR \cong wgR$. Thus, $Soc(eR) = eRgR$. By Lemma 20, $eRew = eRg$. This yields, $Soc(eR) = eRewR = eRgR$. Consider any non-zero $erg, esg \in eRg$. As $Soc(eR)$ is simple, we get $eRgR = ergR = esgR$, and so $eRgRg = ergRg = esgRg$, whence it is easy to verify that, $eRg = eRew = wgRg$ for any non-zero $w \in eRg$. So the induced monomorphism $\eta : gRg \longrightarrow eRe$ given by $\eta(grg)w = wgrg$ is an isomorphism.

Consider an embedding $\lambda : fR \longrightarrow eR$, where $\lambda(fr) = zfr$. Then $\lambda(Soc(fR)) = zfRgR$. Let $v \in fRg$ be such that $zv \neq 0$. Then $w = zv \in eRg$. Now, v and w induce isomorphisms $\sigma_1 : gRg \longrightarrow fRf$, and $\sigma_2 : gRg \longrightarrow eRe$ respectively. Furthermore, $\sigma : fRf \longrightarrow eRe$ is a monomorphism induced by z. Because $\sigma_2 = \sigma\sigma_1$, σ is also an isomorphism.

(ii) Let $S = \{e_1, \dots, e_m\}$ be a basic orthogonal set of indecomposable idempotents in R. For any two distinct members $e, f \in S$, set $e \leq f$ if $eRf \neq 0$, equivalently, if fR embeds in eR. This is a partial ordering on S. As R is indecomposable, S is connected. We may take $e, f \in S$. There exists a path $e = e_1, e_2, \dots, e_k = f$. By definition, for any $i < k$, $e_iRe_{i+1} \neq 0$. By (i), $e_iRe_i \cong e_{i+1}Re_{i+1}$. Hence, $eRe \cong fRf$. □

If we assume that R is right serial in addition to being right artinian and right non-singular, we have the following equivalence. First recall that by Warfield [20], such a ring is right hereditary.

Theorem 22. *Let R be a right artinian right non-singular right serial ring. Then the following are equivalent;*

(i) *R is a right Σ-q ring.*
(ii) *For any two indecomposable idempotents $e, f \in R$, if $eRf \neq 0$ then eRf is one-dimensional left vector space over eRe and one-dimensional right vector space over fRf.*

Proof. (i) $\Longrightarrow$ (ii) follows by Proposition 21.

Conversely, suppose (ii) holds. Let e be any indecomposable idempotent in R. Let A be a non-zero right ideal contained in eR. As R is right non-singular right artinian and right serial, A is projective and indecomposable. Therefore, $A \cong fR$ for some indecomposable idempotent $f \in R$. Let $\sigma : A \longrightarrow eR$ be a non-zero R-homomorphism. Then σ is a monomorphism and hence $A \cong \sigma(A)$. Because eR is uniserial, we have $\sigma(A) \subseteq A$. Since $Soc(eR)$ is simple, there exists an indecomposable idempotent $g \in R$ such that $Soc(eR) = eugR$ for some $u \in R$ and $\sigma(eug) = eugvg$ for some $v \in R$. As $eRg \neq 0$, by (ii), eRg is one-dimensional over eRe. Therefore, $eugvg = eweug$, for some non-zero ewe. Let η be the endomorphism of eR given by left multiplication by ewe. If $\lambda = \eta|_A$, then $\sigma - \lambda$ being zero on $Soc(eR)$, is zero. Hence η extends σ. So, eR is quasi-injective. This also proves that A is quasi-injective. In a right artinian right non-singular right serial ring, any right ideal is a finite direct sum of uniserial right ideals. As shown above any uniserial right ideal is quasi-injective. Hence, R is a right Σ-q ring. □

Theorem 23. *Let R be an indecomposable right artinian right non-singular right Σ-q ring. Then*

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(e_1Re_1) & \mathbb{M}_{n_1\times n_2}(e_1Re_2) & . & . & . & \mathbb{M}_{n_1\times n_k}(e_1Re_k) \\ 0 & \mathbb{M}_{n_2}(e_2Re_2) & . & . & . & \mathbb{M}_{n_2\times n_k}(e_2Re_k) \\ 0 & 0 & \mathbb{M}_{n_3}(e_3Re_3) & . & . & \mathbb{M}_{n_3\times n_k}(e_3Re_k) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(e_kRe_k) \end{bmatrix}$$

where e_iRe_i is a division ring, $e_iRe_i \cong e_jRe_j$ for each $1 \le i, j \le k$ and $n_1, \ldots, n_k$ are any positive integers. Furthermore, if $e_iRe_j \neq 0$, then it is one-dimensional left vector space over e_iRe_i and one-dimensional right vector space over e_jRe_j.

Proof. Let R be an indecomposable right artinian right non-singular right Σ-q ring. There exists an independent family $\mathcal{F} = \{e_iR : 1 \le i \le n\}$ of indecomposable right ideals such that $R = \oplus_{i=1}^{n} e_iR$. After renumbering, we may write $R = [e_1R] \oplus [e_2R] \oplus \cdots \oplus [e_kR]$, where for $1 \le i \le k$, $[e_iR]$ denotes the direct sum of those e_jR that are isomorphic to e_iR. Let $[e_iR]$ be a direct sum of n_i copies of e_iR. Consider $1 \le i < j \le k$. We arrange in such a way that $length(e_jR) \le length(e_iR)$. Suppose $e_jRe_i \neq 0$, then we have an embedding of e_iR into e_jR, hence $length(e_iR) \le length(e_jR)$. But by assumption $length(e_jR) \le length(e_iR)$, so $length(e_iR) = length(e_jR)$, we get $e_jR \cong e_iR$, which is a contradiction. Hence $e_jRe_i = 0$ for $j > i$.

Thus, we have

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(e_1Re_1) & \mathbb{M}_{n_1\times n_2}(e_1Re_2) & . & . & . & \mathbb{M}_{n_1\times n_k}(e_1Re_k) \\ 0 & \mathbb{M}_{n_2}(e_2Re_2) & . & . & . & \mathbb{M}_{n_2\times n_k}(e_2Re_k) \\ 0 & 0 & \mathbb{M}_{n_3}(e_3Re_3) & . & . & \mathbb{M}_{n_3\times n_k}(e_3Re_k) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(e_kRe_k) \end{bmatrix}.$$

We have already seen earlier that each e_iRe_i is a division ring, $e_iRe_i \cong e_jRe_j$ for each $1 \le i, j \le k$ and if $e_iRe_j \neq 0$ then it is a one-dimensional left vector space over e_iRe_i as well as a one-dimensional right vector space over e_jRe_j (see Lemma 19 and Proposition 21). □

Let us consider the following condition:

$(*)$: *For $1 \le i, j \le k$ with $i \neq j$ and primitive orthogonal idempotents e_i, e_j, either $e_iRe_j \neq 0$ or $e_jRe_i \neq 0$. In other words, for a right non-singular ring either e_iR is embeddable in e_jR or e_jR is embeddable in e_iR.*

We remark that $(*)$ holds if R is an indecomposable right non-singular serial ring.

Under the hypothesis $(*)$ we have the following

Theorem 24. *Let R be an indecomposable right artinian right non-singular ring with the condition $(*)$. Then R is a right Σ-q ring if and only if*

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(D) & \mathbb{M}_{n_1\times n_2}(D) & . & . & . & \mathbb{M}_{n_1\times n_k}(D) \\ 0 & \mathbb{M}_{n_2}(D) & . & . & . & \mathbb{M}_{n_2\times n_k}(D) \\ 0 & 0 & \mathbb{M}_{n_3}(D) & . & . & \mathbb{M}_{n_3\times n_k}(D) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(D) \end{bmatrix}$$

where D is a division ring and $n_1, \ldots, n_k$ are any positive integers.

Proof. Let R be an indecomposable right artinian right non-singular right Σ-q ring. By the above theorem,

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(e_1Re_1) & \mathbb{M}_{n_1\times n_2}(e_1Re_2) & . & . & . & \mathbb{M}_{n_1\times n_k}(e_1Re_k) \\ 0 & \mathbb{M}_{n_2}(e_2Re_2) & . & . & . & \mathbb{M}_{n_2\times n_k}(e_2Re_k) \\ 0 & 0 & \mathbb{M}_{n_3}(e_3Re_3) & . & . & \mathbb{M}_{n_3\times n_k}(e_3Re_k) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(e_kRe_k) \end{bmatrix}$$

where each e_iRe_i is a division ring and $e_iRe_i \cong e_jRe_j$ for each $1 \le i, j \le k$. Furthermore, by condition $(*)$, we have $e_iRe_j \neq 0$ for each $1 \le i < j \le k$. Therefore, each e_iRe_j is a one-dimensional left vector space over e_iRe_i as well as a one-dimensional right vector space over e_jRe_j. Let us denote division ring e_iRe_i by D. Then we have

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(D) & \mathbb{M}_{n_1\times n_2}(D) & . & . & . & \mathbb{M}_{n_1\times n_k}(D) \\ 0 & \mathbb{M}_{n_2}(D) & . & . & . & \mathbb{M}_{n_2\times n_k}(D) \\ 0 & 0 & \mathbb{M}_{n_3}(D) & . & . & \mathbb{M}_{n_3\times n_k}(D) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(D) \end{bmatrix}.$$

Conversely, suppose that

$$R \cong \begin{bmatrix} \mathbb{M}_{n_1}(D) & \mathbb{M}_{n_1\times n_2}(D) & . & . & . & \mathbb{M}_{n_1\times n_k}(D) \\ 0 & \mathbb{M}_{n_2}(D) & . & . & . & \mathbb{M}_{n_2\times n_k}(D) \\ 0 & 0 & \mathbb{M}_{n_3}(D) & . & . & \mathbb{M}_{n_3\times n_k}(D) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mathbb{M}_{n_k}(D) \end{bmatrix}$$

where D is a division ring and $n_1, \ldots, n_k$ are positive integers.

Clearly, R is an indecomposable right non-singular ring. By Eisenbud-Griffith [4], we know that R is an artinian serial ring. Therefore, R is a right Σ-q ring. □

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