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RIGHT-LEFT SYMMETRY OF RIGHT NONSINGULAR RIGHT MAX-MIN CS PRIME RINGS

S. K. Jain, Husain S. Al-Hazmi, and Adel N. Alahmadi
Department of Mathematics, Ohio University, Athens, Ohio, USA

In this article we show, among others, that if R is a prime ring which is not a domain, then R is right nonsingular, right max-min CS with uniform right ideal if and only if R is left nonsingular, left max-min CS with uniform left ideal. The above result gives, in particular, Huynh et al. (2000) Theorem for prime rings of finite uniform dimension.

Key Words: CS ring; Goldie ring; Max CS ring; Min CS ring; Nonsingular; Uniform dimension; Uniform right ideal.

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1. INTRODUCTION

A ring in which each closed right ideal is a direct summand is called a *right CS ring*. For example, right self-injective rings and right continuous rings are right CS rings. A ring R is called *right min CS* if every minimal closed (equivalently, uniform closed) right ideal is a direct summand of R. A left min CS ring is defined similarly. We refer to Dung et al. (1994) for background on CS rings and modules. We call a ring R to be a *right max CS* if every maximal closed right ideal with nonzero left annihilator is a direct summand of R. We define left max CS ring similarly. It was shown by Huynh et al. (2000, Theorem 1) that a prime right Goldie, right CS ring R with right uniform dimension at least 2 is left Goldie, and left CS. In this article, we generalize this result to infinite uniform dimension. We consider a prime right nonsingular right CS ring of possibly infinite uniform dimension with a uniform right ideal. We show that for a prime ring R with right uniform dimension at least 2, R is a right nonsingular right max and right min CS ring with uniform right ideal if and only if R is a left nonsingular, left max and left min CS ring with uniform left ideal.

2. DEFINITIONS AND PRELIMINARIES

Throughout this article, unless otherwise stated, all rings have unity and all modules are unital. A submodule K of an R-module M is said to be a complement

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Address correspondence to S. K. Jain, Department of Mathematics, Ohio University, Athens, OH 45701, USA; E-mail: jain@ohiou.edu

to a submodule N of M if K is maximal with respect to the property that $K \cap N = 0$. A submodule N of an R-module M is called *essential in* M, denoted by $N \subseteq_e M$, if for any nonzero submodule L of M, $L \cap M \neq 0$. A submodule N of M is called *closed in* M if it has no proper essential *extension in* M. Closed submodules are precisely complement submodules (Lam, 1999, Proposition 6.32). A nonzero module M is called *uniform* if any two nonzero submodules of M intersect nontrivially. Clearly, uniform closed submodules of M are precisely minimal closed submodules of M. A ring R is called *right max-min CS* if R is both right max CS and right min CS. Similarly, we define left max-min CS.

A right R-module M has finite uniform dimension if it does not contain any infinite direct sum of nonzero submodules. It is known that for a module M with finite uniform dimension, there exists a positive integer n such that every direct sum of nonzero submodules of M does not have more than n summands, and the least positive integer n with this property is called the *uniform dimension of* M. We denote the uniform dimension of M by $u.dim(M)$. A ring R has finite right (left) uniform dimension if R has finite uniform dimension as a right (left) R-module.

A right quotient ring of a ring R is a ring Q which contains R as a subring such that Q_R is a rational extension of R_R (see Goodearl, 1976, p. 57). For a right nonsingular ring R, Q is a right quotient ring of R if R is a subring of Q and Q_R is an essential extension of R_R. A left quotient ring is defined similarly. For a ring R, the maximal right (left) quotient ring of R is denoted by $Q^r_{max}(R)$ ($Q^l_{max}(R)$), see Goodearl (1976). For a right nonsingular ring R, the injective hull $E(R_R)$ is a ring that coincides with the right maximal quotient ring $Q^r_{max}(R)$ (Goodearl, 1976, Corollary 2.31). Furthermore, $Q^r_{max}(R)$ is a von Neumann regular ring and it is injective as a right module over itself as well as over R. A ring Q is called a *two-sided quotient ring of* R if Q is both a left and right quotient ring of R. *Nonsingular ring* will mean both right and left nonsingular. For a nonsingular ring R, the maximal two-sided quotient ring of R, denoted by $Q^t_{max}(R)$, is the maximal essential extension of ${}_RR$ in $Q^r_{max}(R)$ (equivalently, $Q^t_{max}(R)$ is the maximal essential extention of R_R in $Q^l_{max}(R)$), see Utumi (1963). It is known that for a right quotient ring Q of a right nonsingular ring R, the lattices of closed right ideals of R and Q are isomorphic under the correspondence $A \to A \cap R$, where A is a closed right ideal of Q (Johnson, 1961, Corollary 2.6). For simplicity, we will denote $Q^r_{max}(R)$, $Q^l_{max}(R)$ and $Q^t_{max}(R)$ by Q^r, Q^l, and Q^t respectively. Observe that, from the above lattice isomorphism, each minimal closed right ideal of a right nonsingular ring R is of the form $eQ^r \cap R$, where eQ^r is a minimal right ideal of Q^r. A ring R is called *right (left) Goldie ring* if the right (left) uniform dimension of R is finite and R has ascending chain condition on right (left) annihilators. A ring R is called *directly finite* (or "*von Neumann finite*", or "Dedekind finite") if $ab = 1$ implies $ba = 1$, for all $a, b \in R$. More generally, a module M over a ring R is called *directly finite* (or "von Neumann finite", or "Dedekind finite") if its endomorphism ring $End(M_R)$ is directly finite (or "von Neumann finite", or "Dedekind finite"), see Goodearl (1979). For a subset S of a ring R, $r.ann_R(S)$ and $l.ann_R(S)$ will denote the right annihilator and left annihilator of S in R, respectively. For a right R-module M, $Z(M)$ will denote the singular submodule of M.

3. THE RESULTS

Lemma 3.1. *Let R be a prime ring with a uniform right ideal. If R is right min CS and right nonsingular, then R is left nonsingular.*

Proof. By our assumption R has a minimal closed right ideal, say U. Then $U = eQ^r \cap R$, where eQ^r is a minimal right ideal of Q^r and $e = e^2 \in Q^r$. Since R is a right min CS ring, $U = fR$, for some idempotent f in R. So we have $fR = eQ^r \cap R$ and hence $fQ^r = eQ^r$. Since $fRf \subseteq fQ^r f$ and $fQ^r f$ is a division ring, fRf is a domain. We claim that $Z(Rf) = 0$. Assume on the contrary, that $Z(Rf) \neq 0$. Let $0 \neq xf \in Z(Rf)$. Then there exists an essential left ideal E of R such that $Exf = 0$. So, for each $0 \neq rf \in E \cap Rf$, we have $rfxf = 0$. This implies $(frf)(fxf) = 0$. Since fRf is a domain, $frf = 0$ or $fxf = 0$. If $fxf \neq 0$, then $frf = 0$, for all $rf \in E \cap Rf$. This implies $(Rf)(E \cap Rf) = 0$. Since R is prime and $E \cap Rf \neq 0$, $Rf = 0$, which is a contradiction. Thus $fxf = 0$, for all $xf \in Z(Rf)$. But then we have $(Rf)(Z(Rf)) = 0$, and so, since R is prime, $Rf = 0$, a contradiction. Thus, $Z(Rf) = 0$. Now if $Z(_RR) \neq 0$, then $Z(_RR)$ is essential as a left ideal of R. So $Z(_RR) \cap (Rf) \neq 0$. This means $Z(Rf) \neq 0$, a contradiction. Thus R is left nonsingular. □

Lemma 3.2 (Utumi, 1963, Lemma 1.4). *Let R be a nonsingular ring. Then R has the maximal two-sided quotient ring Q^t. Q^t may be regarded as the subring of Q^l consisting of elements x such that the set of $y \in R$ with $xy \in R$ forms a large (essential) right ideal of R.*

Lemma 3.3 (Utumi, 1963, Lemma 2.5). *Let R be a prime nonsingular ring with a uniform left ideal and a uniform right ideal. Then the two-sided quotient ring Q^t is a primitive ring with nonzero socle.*

Theorem 3.1. *Let R be a prime ring with a uniform right ideal. If R is nonsingular and right max CS, then R is left min CS.*

Proof. By Lemma 3.2, R has the maximal two-sided quotient ring Q^t. If R has no uniform left ideal, then R is trivially a left min CS ring. So, we assume that R has uniform left ideals and so R has minimal closed left ideals. Let U be a minimal closed left ideal of R. Then by Lemma 3.3, Q^t is primitive and has nonzero socle. Thus, by the lattice isomorphism (Johnson, 1961, Corollary 2.6) $U = Q^t e \cap R$, where $Q^t e$ is a minimal left ideal of Q^t and $e = e^2 \in Q^t$, because $Soc(_{Q^t}Q^t)$ is essential as a left ideal of Q^t. Define $F = \{a \in R \mid (1-e)a \in R\}$. Since $R_R \subseteq_e Q^t_R$, $F \subseteq_e R_R$. Clearly, $(1-e)Q^t$ is a maximal closed right ideal of Q^t, and hence $(1-e)Q^t \cap R$ is a maximal closed right ideal of R. We have $l.ann_R((1-e)Q^t \cap R) = l.ann_R((1-e)F) = \{x \in R \mid x(1-e)F = 0\} = \{x \in R \mid x(1-e) \in Z(Q^t_R)\} = \{x \in R \mid x(1-e) = 0\} = l.ann_{Q^t}(1-e) \cap R = Q^t e \cap R = U \neq 0$. Then by our hypothesis, $(1-e)Q^t \cap R = fR$, for some idempotent $f \in R$. Therefore, $U = R(1-f)$ is a direct summand of R. Thus R is a left min CS ring. □

The next theorem is probably the best right-left symmetry of the CS property that one can obtain for prime right nonsingular right CS rings with uniform right ideals.

Theorem 3.2. *Let R be a prime ring with uniform right ideal. If R is right nonsingular and right CS, then R is left min CS.*

Proof. The proof follows from Lemma 3.1 and Theorem 3.1. □

Theorem 3.3 (Lam, 1999, Theorem 6.48). *Let R be a ring such that both R_R and ${}_RR$ are CS modules. Then R is Dedekind finite.*

The example that follows shows that one cannot obtain the full strength of left CS property, in general, for right CS rings.

Example 3.1. *Let V_D be a right vector space over a division ring D with $dim V_D = \infty$. Let $R = Hom(V_D, V_D)$. It is well known that R is a regular primitive right self-injective ring which is not a Dedekind finite ring. If R is a left CS ring, then by Theorem* 3.3, *R is Dedekind finite, a contradiction. Thus, R is not a left CS ring. This example shows that R in Theorem* 3.2 *need not be left CS.*

Lemma 3.4. *Let R be a right nonsingular prime ring which is not a domain. If R is right min CS with a uniform right ideal, then R has a uniform left ideal.*

Proof. Since R has a uniform right ideal, R has a minimal closed right ideal, say U. Then by the lattice isomorphism and right min CS, $U = e_1Q^r \cap R = e_1R$ where e_1Q^r is a minimal right ideal of Q^r and $e_1 = e_1^2 \in R$. So, $R = e_1R \oplus (1 - e_1)R$ and $Q^r = e_1Q^r \oplus (1 - e_1)Q^r$. Since $Soc(Q^r_{Q^r}) \subseteq_e Q^r_{Q^r}$, $(1 - e_1)Q^r$ contains a minimal right ideal, say e_2Q^r where e_2 is idempotent. We may choose $e_2 \in R$ because R is a right min *CS* ring. Hence, $(1 - e_1)Q^r = e_2Q^r \oplus e_3Q^r$, $e_3 = e_3^2 \in Q^r$. Thus, $Q^r = e_1Q^r \oplus e_2Q^r \oplus e_3Q^r$ and $R = e_1R \oplus e_2R \oplus fR$ where $f = f^2 \in R$.

We will show that e_1Re_1 is a left Ore domain. If $e_1Re_1 = e_1Q^re_1$ we are done because $e_1Q^re_1$ is a division ring. So, assume that $e_1Re_1 \subsetneq e_1Q^re_1$.

We have:

$$R = e_1R \oplus e_2R \oplus fR \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 & e_1Rf \\ e_2Re_1 & e_2Re_2 & e_2Rf \\ fRe_1 & fRe_2 & fRf \end{pmatrix} \text{ and}$$

$$Q^r = e_1Q^r \oplus e_2Q^r \oplus e_3Q^r \cong \begin{pmatrix} e_1Q^re_1 & e_1Q^re_2 & e_1Q^re_3 \\ e_2Q^re_1 & e_2Q^re_2 & e_2Q^re_3 \\ e_3Q^re_1 & e_3Q^re_2 & e_3Q^re_3 \end{pmatrix}.$$

Let $0 \neq a_1 \in e_1Q^re_1 \backslash e_1Re_1$ and $0 \neq a_2 \in e_2Re_1$. Set $\alpha = \left(\begin{smallmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Let $I = \left(\begin{smallmatrix} e_1Q^re_1 & e_1Q^re_2 & e_1Q^re_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ be a nonzero right ideal of Q^r and note that αI is a nonzero right ideal of Q^r. Since Q^r is prime and has a nonzero socle, αI contains a minimal right ideal of Q^r, say N. So, $R \cap N$ is a minimal closed right ideal of R. Consequently, $R \cap N$ is generated by an idempotent $e^* \in R$. Therefore, there exists an element $\beta = \left(\begin{smallmatrix} x_1 & x_2 & x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in M$ such that $\alpha\beta = \left(\begin{smallmatrix} a_1x_1 & a_1x_2 & a_1x_3 \\ a_2x_1 & a_2x_2 & a_2x_3 \\ 0 & 0 & 0 \end{smallmatrix}\right) = e^* \in R$, where $x_i \in e_1Q^re_i$. Hence, $a_ix_j \in e_iRe_j$, and $a_ix_3 \in e_iRf$, for $i, j = 1, 2$. Note that at least one x_i is nonzero.

After squaring this matrix and comparing the corresponding entries of the first row of this matrix and the first row of its square we get the following equations:

$$a_1x_1a_1x_1 + a_1x_2a_2x_1 = a_1x_1 \tag{1}$$

$$a_1x_1a_1x_2 + a_1x_2a_2x_2 = a_1x_2 \tag{2}$$

$$a_1x_1a_1x_3 + a_1x_2a_2x_3 = a_1x_3. \tag{3}$$

We claim that $x_1 \neq 0$. For if $x_1 = 0$, then we have two cases. First case, $x_2 \neq 0$. Then from equation (2) we get $a_1x_2a_2x_2 = a_1x_2$. Since $a_1 \neq 0$, $x_2a_2x_2 = x_2$. As $x_2 \in e_1Q^re_2$, there exists $x_2^* \in e_2Q^re_1$ such that $x_2x_2^* = e_1$ because $e_1Q^r \cong e_2Q^r$. Consequently, $x_2a_2 = e_1$. Multiplying this with a_1 on the left we get $(a_1x_2)a_2 = a_1e_1 = a_1$. From this and $a_1x_2 \in e_1Re_2$, it follows that $a_1 \in e_1Re_1$, a contradiction. Second case, $x_2 = 0$. Then we must have $x_3 \neq 0$. Thus, equation (3) becomes $a_1x_3 = 0$, and so, $x_3 = 0$, a contradiction. Hence, $x_1 \neq 0$. Thus, equation (1) becomes $x_1a_1 + x_2a_2 = e_1$, and hence

$$x_1a_1 = e_1 - x_2a_2 \tag{4}$$

Let $0 \neq y \in e_1Re_2 \subset e_1Q^re_2$. Since $e_1Q^r \cong e_2Q^r$, there exists $y' \in e_2Q^re_1$ such that $y'y = e_2$. Then $ya_2 \neq 0$. For, if $ya_2 = 0$, then $y'(ya_2) = (y'y)a_2 = 0$, and so, $a_2 = 0$, a contradiction. So, $(ya_2)x_1 \neq 0$. Now $0 \neq y(a_2x_1) \in (e_1Re_2)(e_2Re_1) \subseteq e_1Re_1$. Also $ya_2x_2 \in e_1Re_2$, and $a_2 \in e_2Re_1$ which implies $ya_2x_2a_2 \in e_1Re_1$. Next, multiplying (4) on the left by ya_2 we get $(ya_2x_1)a_1 = ya_2 - ya_2x_2a_2$. Consequently, $a_1 = (ya_2x_1)^{-1}(ya_2 - ya_2x_2a_2)$. Thus, e_1Re_1 is a left Ore domain.

We claim Re_1 is uniform. Assume to the contrary that Re_1 is not uniform. This means there exists nonzero submodules A and B of Re_1 such that $A \cap B = 0$. This implies $e_1A \cap e_1B = 0$. Since e_1A and e_1B are left ideals of the left Ore domain e_1Re_1, either $e_1A = 0$ or $e_1B = 0$. Consequently, either $BA = 0$ or $AB = 0$. This is a contradiction because R is a prime ring. This completes the proof. □

Remark 3.1. *We note that Lemma* 3.4 *is not true, in general. For let R be a right Ore domain which is not a left Ore domain, see* (*Goodearl,* 1976). *If R contains a uniform left ideal, then by Lemma* 3.3, *the maximal two-sided quotient ring Q^t is a primitive ring with nonzero socle. This implies that Q^t is a division ring, and hence R is a left uniform ring, a contradiction.*

As a consequence of Lemmas 3.1, 3.3, and 3.4, we have the following corollary.

Corollary 3.1. *Let R be a right nonsingular prime ring with a uniform right ideal. If R is right min CS, then R has the maximal two-sided quotient ring Q^t. Moreover, if R is not a domain, then Q^t is a primitive ring with nonzero socle.*

We are now ready to prove our stated goal of the right-left CS ring property of certain classes of prime rings.

Theorem 3.4. *For a nondomain prime ring R, the following conditions are equivalent*:

(1) *R is right nonsingular, right max-min CS with a uniform right ideal*;
(2) *R is left nonsingular, left max-min CS with a uniform left ideal.*

Proof. (1) $\Rightarrow$ (2) R is left nonsingular by Lemma 3.1 and by Theorem 3.2 R is left min CS ring. By Lemma 3.4, R has a left uniform ideal. Therefore, we need only to prove that R is a left max CS ring. R has Q^t by Lemma 3.2, and $Soc(Q^t_{Q^t}) \neq 0$ by Lemma 3.3. Let M be a maximal closed left ideal of R with $r.ann_R(M) \neq 0$. Then $M = M^* \cap R$, for some maximal closed left ideal M^* of Q^t with $r.ann_{Q^t}(M^*) \neq 0$. Since $0 \neq Soc(Q^t_{Q^t}) \subseteq_e Q^t$, $r.ann_{Q^t}(M^*)$ contains a minimal right ideal, say eQ^t, $e = e^2 \in Q^t$. Then $Q^t(1-e) \supseteq l.ann_{Q^t}(r.ann_{Q^t}(M^*)) \supseteq M^*$. This implies $M^* = Q^t(1-e)$ by maximality of M^*. We have $eQ^t \cap R = fR$, for some idempotent $f \in R$ by (1). Then $eQ^t = fQ^t$ by minimality of eQ^t. By taking the left annihilator of both sides we get $Q^t(1-e) = Q^t(1-f)$. Thus, $M = M^* \cap R = Q^t(1-f) \cap R = R(1-f)$. Therefore, R is right max CS ring.

(2) $\Rightarrow$ (1) is dual to (1) $\Rightarrow$ (2). □

From the proof of (1) $\Rightarrow$ (2) in Theorem 3.4, one may obtain the following Corollary.

Corollary 3.2. *Let R be a prime ring which is not a domain. If R is right nonsingular right min CS with a uniform right ideal, then R is left nonsingular left max CS with a uniform left ideal.*

Lemma 3.5 (Goodearl, 1976, Theorem 2.38). *Let R be a nonsingular ring. Then the maximal right and left quotient rings of R coincide if and only if every closed one-sided ideal of R is an annihilator.*

Lemma 3.6 (Faith, 1967, Theorem 8, p. 73). *Let R be a prime ring satisfying $Z(R_R) = 0$, and containing a minimal closed right ideal. Then the maximal right quotient ring of R is isomorphic to the full ring of linear transformations in a right vector space over a division ring.*

Lemma 3.7 (Dung et al., 1994, Corollary 7.8, p. 59). *A module with finite uniform dimension is extending if and only if it is uniform extending.*

For a prime ring with finite uniform dimension, we have the following proposition.

Proposition 3.1. *Let R be a prime ring with a uniform right ideal. If R is right CS and right nonsingular, then the following are equivalent:*

(1) *R is left CS;*
(2) *$u.dim(_RR) < \infty$.*

Proof. (1) $\Longrightarrow$ (2) By Lemmas 3.5 and 3.6, $Q^l = Q^r \cong Hom(V_D, V_D)$ for some right vector space V over a division ring D. Then Q^r(and also Q^l) is a semisimple artinian ring. Thus $u.dim(_RR) < \infty$ (also $u.dim(R_R) < \infty$).

(2) $\Longrightarrow$ (1) follows by Theorem 3.2 and Lemma 3.7. □

We may now derive the Theorem of Huynh et al. (2000) in the following proposition.

Proposition 3.2. *Let R be a prime right nonsingular right CS ring with a uniform right ideal. If $u.dim(R_R) = n$, where $n \geq 2$, then $u.dim({}_R R) = n$. Thus, a prime right Goldie, right CS ring R with right uniform dimension at least 2, is left Goldie, and left CS.*

Proof. Since R is right CS, $R = e_1 R \oplus \cdots \oplus e_n R$ where each $e_i R$ is uniform and $\{e_i\}_{i=1}^n$ is a system of orthogonal idempotents of R. Since R is nonsingular (Lemma 3.1), R has the maximal two-sided quotient ring Q^t. By Lemma 3.4, R has a uniform left ideal, and so Q^t has a nonzero socle (Lemma 3.3). Thus, $Q^t = e_1 Q^t \oplus \cdots \oplus e_n Q^t$ where each $e_i Q^t$ is a minimal right ideal of Q^t. Therefore, Q^t is a semisimple artinian ring. Since ${}_R R \subseteq_{eR} Q^t$, $u.dim({}_R R) = n$. □

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