Small-world models[*]

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Small-world networks are classes of networks that have both the small-world property and exhibit strong clustering. Two constructions of such networks are implemented in IONTW. Here we study, both theoretically and with simulation experiments, the structure of these networks and how it influences effectiveness of a certain vaccination strategy.

1   Small worlds

1.1   The small-world property and small-world networks

In our module *Exploring distances with IONTW* [2] you learned that a class of networks has the *small-world property* if, for some fixed positive constants P and c, the P-th percentile of the distance between two randomly chosen nodes will asymptotically almost surely[1] (a.a.s.) be less than $c \ln(N)$. Thus the small-world property can be conceptualized in terms of a small median (or other percentile) distance between two randomly chosen nodes of a network. It is defined only for classes of networks that contain instances of arbitrarily large size N.

In our module *Clustering coefficients* [1] we introduced the notion of *normalized network clustering coefficients* C_{norm} and the property of *strong clustering* for a class of networks. This property holds if, and only if, $C_{norm} \to \infty$ a.a.s. as $N \to \infty$ for networks in this class. Classes of networks that have both the small-world property and exhibit strong clustering will be called *small-world networks.*

While many empirically studied contact networks appear to be small-world networks [8], none of the classes of graphs that we have explored in our earlier modules at this website[2] qualify. Erdős-Rényi random graphs $G_{ER}(N, \lambda)$, random regular graphs $G_{Reg}(N, k)$ and many other graphs $G_D(N, \bar{q})$ that are generic for a given degree distribution have the small-world property but don't exhibit strong clustering. On the other hand, as you saw in our modules [1, 2], the classes of nearest neighbor networks $G_{NN}^1(N, d)$ and $G_{NN}^2(N, d)$ with $d > 1$ exhibit strong clustering, but do not have the small-world property.

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[1]The phrase "asymptotically almost surely" means that this probability will approach 1 as $N \to \infty$.

[2]http://www.ohio.edu/people/just/IONTW/

In the next subsection we will construct specific examples of small-world networks and call them *small-world models.* Readers should be aware that there is no consensus about the usage of the terms "small-world property," "small-world network," and "small-world model." Different sources in the literature use different mappings between these phrases and the three concepts. One always needs to carefully read the definition in each source to figure out the intended meaning.

1.2 Small-world models

Consider the monastic order of the Sisters of the Round Table that we introduced in [3]. The sisters spend most of their lives in their individual cells, where they devote themselves to prayer and meditation. The only time they have contact with each other is during meals that they take seated in a fixed order around a giant round table. Within this community, diseases can be transmitted only during mealtime.

The probability of transmission will be largest between sisters who sit next to each other, and then decrease with the distance at the table. When constructing a network model, we need to decide on a cutoff. Let us assume that there is a significant probability of transmission from sister i to sister j if at most one sister sits in between. This will be the case if $|i - j| \leq 2$. But since the table is round, this will also be the case if, for example, $i = 2$ and $j = N$. This construction gives a contact network that is modeled by the graph $G_{NN}^1(N, 2)$. More generally, if there is a significant probability of transmission between sisters that have at most $d - 1$ seats between them, the resulting contact network will be modeled by the graph $G_{NN}^1(N, d)$. Recall form our module [6] that these graphs are called *one-dimensional nearest-neighbor networks.* In IONTW, they are implemented by the option

network-type → Nearest-neighbor 1.

There is a problem though with our story: Even in a strictly monastic setting, contact networks will rarely have such a rigid structure as $G_{NN}^1(N, d)$. The Sisters will not necessarily head straight to the table from their cells. More likely, along the way they will exchange a few kind words with their next-cell neighbors who may be seated across the table. We can incorporate these more informal contacts into our network model by adding a few randomly chosen edges to a graph $G_{NN}^1(N, d)$. This will result in a class of networks that simultaneously exhibits strong clustering and has the small-world property. Quite literally, the Sisters can have the best of both worlds!

We construct our *small-world models* by adding a few randomly chosen edges to a graph $G_{NN}^1(N, d)$ or $G_{NN}^2(N, d)$. As you will see, for fixed parameters $d > 1$, these constructions result in classes of graphs that exhibit both strong clustering and have the small-world property, that is, in classes of small-world networks.

The construction that we are describing here is a modification of the networks that Watts and Strogatz introduced in their seminal paper [8]. The main difference is that the small-world models of [8] were constructed by randomly *rewiring* a small fraction of the edges in networks $G_{NN}^1(N, d)$ or $G_{NN}^2(N, d)$ rather than randomly adding new edges. The

modification we are using here was proposed in [7]. It gives networks that have similar properties to the ones studied by Watts and Strogatz in [8], but are slightly easier to define and a lot easier to analyze mathematically.

Let us give a mathematically rigorous definition of our small-world networks. We construct random graphs $G = G_{SW}^{dim}(N, d, \lambda)$ for $dim = 1, 2$, $d \geq 1$, and $\lambda > 0$ as follows:

- Let $G_{short} = G_{NN}^{dim}(N, d)$.

- Randomly choose an Erdős-Rényi graph $G_{long} = G_{ER}(N, \lambda)$.

- Define $V(G) = \{1, \ldots, N\}$ and $E(G) = E(G_{short}) \cup E(G_{long})$.

We can think of $G_{SW}^{dim}(N, d, \lambda)$ as having two types of edges: *short edges* that it inherits from $G_{NN}^{dim}(N, d)$ and *long edges* that it inherits from $G_{ER}(N\lambda)$. In the example of the Sisters of the Round Table the short edges represent the contacts between sisters that are seated close to each other at the table, and the long edges represent the more informal contacts that they make in the hall with sisters from across the table. It is possible that some edges will be simultaneously "short" and "long" in this sense; we will always treat them as long ones.

In Section 2 we will take a close look at some of the networks $G_{SW}^{dim}(N, d, \lambda)$. In particular, we will show that for $d > 1, \lambda > 0$ our small-world models really are small-world networks. In Section 3 we will explore how diseases spread in our small-world models. More precisely, we will investigate how well a vaccination strategy that we developed in Section 9.4.3 of [4] for $G_{NN}^1(N, 2)$ will work when the contact network is a small-world model $G_{SW}^1(N, 2, \lambda)$.

Let us mention that one can easily generalize the definition given above by starting with higher-dimensional versions of nearest-neighbor networks and/or using different types of random networks (such as the scale-free networks of [5]) instead of Erdős-Rényi networks $G_{ER}(N, \lambda)$ for the long edges. Exploration of the resulting small-world models is of great interest and an active area of research, but goes beyond the scope of this module.

2 Exploring small worlds with IONTW

2.1 Small world models in IONTW

Open IONTW, click **Defaults**, and change the following parameter settings:

    network-type $\rightarrow$ **Small World 2**
    num-nodes: 100
    lambda: 0
    d: 2

Click **New**. In the **World** window you will see a network $G_{NN}^2(100, 2)$, which is a 10-by-10 rectangular grid with diagonals. Since we have set **lambda** $= 0$, this isn't really a small-world model; it has no long edges whatsoever. But we want to use it as a baseline for comparison. Click **Metrics** to record its properties for future reference.

Now change

lambda: 0.2

and press **New** again. This will create an instance of $G_{SW}^2(100, 2, 0.2)$. You will see a few long edges in addition to the short ones. Click **Metrics**, and repeat for

lambda: 0.4, 0.6. 0.8

Be sure to click **Metrics** after creation of each network.

You will have seen more and more long edges appearing as you increased **lambda**. Let us see what their addition did to the network properties. Enlarge the **Command Center** by clicking on the double-arrow icon and look at the data. Most likely you will observe that as the parameter λ increases in increments of 0.2, the following changes in the characteristic properties of $G_{SW}^2(100, 2, \lambda)$ occur:

- The mean degree $\langle k \rangle$ increases; roughly in increments of 0.2.

- The edge density slightly increases.

- The (normalized) network clustering coefficients decrease.

- The mean distance decreases.

- The diameter may decrease.

Your data may not give a clear-cut picture due to the inherent randomness, but the general pattern should be discernible.

Let us look at a larger network. Set

num-nodes: 400

Repeat the previous experiment for

lambda: 0, 0.8

You may observe that although the mean degree increases by only about 10%, in $G_{SW}^2(400, 2, 0.8)$ the mean distance and the diameter are less than half in of what they were in $G_{SW}^2(400, 2, 0)$.

How could adding a few random long edges have such a large effect? Let us look at some examples of type $G_{SW}^1(N, d, \lambda)$. Change

network-type $\rightarrow$ **Small World 1**
num-nodes: 10
lambda: 0
d: 1

Click **New**. In the **World** window, you will see the network $G_{SW}^1(10, 1, 0) = G_{NN}^1(10, 1)$. Let's turn this nearest-neighbor network into a true small-world network by changing

lambda: 1

After clicking **New**, you will see an instance of $G^1_{SW}(10, 1, 1)$ with a few long edges in addition to the short ones of $G^1_{NN}(10, 1)$.

Now let us try something bigger. Change

num-nodes: 101
lambda: 0

Click **New** to create the network $G^1_{NN}(101, 1)$. Click **Labels** to see how the nodes are numbered.

Nodes 40 and 90 are opposite each other in this network. The distance $d(40, 90) = 50$; it is the diameter of this network. By looking at the picture you can see why mathematicians call the maximum distance of any two nodes in a graph "diameter." Click **Metrics** and look up the diameter and the mean distance between any nodes in the **Command Center.**

Of course, if you add the edge $\{40, 90\}$, you will get a distance $d(40, 90) = 1$. But when we add a few random edges as in the construction of $G^1_{SW}(101, 1, \lambda)$ it is unlikely that one of them will be $\{40, 90\}$. Instead of having the computer add random long edges, let's study the effect on the network properties of adding the particular edge $\{50, 67\}$. You can create this edge manually by choosing

spawn-kill $\rightarrow$ Spawn

clicking **Link**, and then first clicking on node 50 and next on node 67. Click **Metrics** and look up the diameter and the mean distance between any nodes in the **Command Center.**

Exercise 1 *(a) Find $d(40, 90)$ in the new network and explain how you calculated it.*

(b) Explain why the addition of the new edge changed the mean distance, but only slightly.

(c) Explain why the addition of the new edge did not change the diameter.

Can you see that it doesn't matter all that much where we put the new edge? Had we added the edge $\{52, 69\}$ instead of $\{50, 67\}$, the effect on $d(40, 90)$ would have been exactly the same.

Now choose

spawn-kill $\rightarrow$ Kill

and remove the new edge by first pressing **Link** and then clicking on its endpoints. Return to

spawn-kill $\rightarrow$ Spawn

and add two new edges $\{42, 70\}$ and $\{66, 93\}$ to $G^1_{NN}(101, 1)$. Click **Metrics** and look up the diameter and the mean distance between any nodes in the **Command Center.**

Exercise 2 *(a) Find $d(40, 90)$ in the new network and explain how you calculated it.*

(b) Explain why the addition of the new edges changed the mean distance rather significantly.

(c) Explain why the addition of the new edges substantially decreased the diameter.

Can you see how addition of very few random edges may shorten the distance between nodes i, j even when none of these edges has i or j as is endpoint and how this may dramatically decrease the mean distance and the diameter?

2.2 Mathematical derivations of some properties of $G_{SW}^{dim}(N, d, \lambda)$

We will focus on three network properties here: The mean degree $\langle k \rangle$, the normalized network-clustering coefficient C_{norm}, and the median distance between any two nodes. Moreover, in order to sidestep some distracting technicalities, for the two-dimensional case we will mostly focus only on networks $G_{SW}^2(N^2, d, \lambda)$, although the results generalize to small-world models based on rectangular rather than square grids, many of which are implemented in IONTW as $G_{SW}^2(N, d, \lambda)$.

Exercise 3 *(a) Find an approximation μ to the mean degree in $G_{SW}^1(N, d, \lambda)$ that a.a.s. approaches the true mean degree $\langle k \rangle$ when $N \to \infty$.*

(b) Repeat part (a) for $G_{SW}^2(N^2, 1, \lambda)$.

(c) Find an exact formula for the expected mean degree $\langle k \rangle$ in $G_{SW}^1(10, 2, 0.2)$.

(d) Find an exact formula for the expected mean degree $\langle k \rangle$ in $G_{SW}^1(N, d, \lambda)$.

(e) Repeat parts (c) and (d) for $G_{SW}^2(16, 1, 0.1)$ and $G_{SW}^2(N^2, 1, \lambda)$.

We have optimistically labeled the graphs $S_{SW}^{dim}(N, d, \lambda)$ "small-world models." But do they live up to this name? We will need to show that our two classes of networks are small-world networks, that is, exhibit strong clustering and have the small-world property.

Strong clustering requires that normalized network clustering coefficients C_{norm} will grow without bounds when the network size increases without bound while the other two parameters, d and λ, remain fixed. By Exercise 12(b) of Module [1], the graphs $G_{NN}^1(N, d)$ and $G_{NN}^2(N^2, d)$ with $d > 1$ have the property that $tr(i) \geq 1$ for each node i, and the same will continue to hold in the corresponding small-world networks $G_{SW}^1(N, d, \lambda)$ and $G_{SW}^2(N^2, d, \lambda)$. Moreover, by Exercise 3(a)(b) above, for each of these classes the expected degree $\langle k \rangle$ will approach some finite number that depends on the particular choice of d, λ. Thus each of the classes $G_{NN}^1(N, d, \lambda)$ and $G_{NN}^2(N^2, d, \lambda)$ with fixed $d > 1$ and λ satisfies the assumptions of Theorem 1 of Module [1], and we can conclude that each of these classes exhibits strong clustering.

Exercise 4 *(a) Consider $G_{SW}^{dim}(N, d, \lambda)$, where the parameters dim, N, d are fixed, $d > 1$, and the network size N is very large. Explain why you* usually *will see a decrease in the normalized network clustering coefficient C_{norm} when you increase the value of λ.*

(b) What behavior of C_{norm} would you expect in the setup for part (a) when you set $d = 1$? How would you explain the predicted differences from part (a)?

Finally, let us consider the small-world property. As you have seen in our module [2], a class of networks will have this property if for some constants P and c, the P-th percentile of

the distance between two randomly chosen nodes will asymptotically almost surely (a.a.s.) be less than $c \ln(N)$.

If $\lambda > 1$, then the graph $G_{long} = G_{ER}(N, \lambda)$ is predicted to a.a.s. contain a giant component of diameter $< c_{diam} \ln(N)$ for some constant c_{diam}. Thus a.a.s., if i, j are nodes in the giant component of $G_{long} = G_{ER}(N, \lambda)$, there will be a path of length $< c_{diam} \ln(N)$ from i to j. Since each such path is also a path in $G_{SW}^{dim}(N, d, \lambda)$, the small-world property of the latter class of networks follows. If λ is sufficiently large, then the percentile P in the definition of this property can be taken to be the median Q_2 (see Exercise 10 of [2]).

However, if $0 < \lambda < 1$, then the class of graphs $G_{ER}(N, \lambda)$ does not have the small-world property, but our small-world models do. Even if λ is very small but positive, we will still be able to travel from any node i to any node j along a path of very small length. This path may need to contain both short and long edges though.

The following exercise gives important insights into the structure of typical small-world networks. Part (a) of the above exercise should be accessible to all of our students; part (b) is intended for students with an advanced mathematical background. While we phrased the exercise only in terms of networks $G_{SW}^1(N, d, \lambda)$, the analogous result is also true for networks $G_{SW}^2(N, d, \lambda)$ and can be proved similarly.

Exercise 5 *(a) Give an intuitive explanation why for any choice of d and of $\lambda > 0$ the class of networks $G_{SW}^1(N, d, \lambda)$ should have the small-world property. A hint will be given in Section 4 after the references, but first try doing the problem without a hint.*
(b) Give a formal prove that for any choice of d and of $\lambda > 0$ the networks $G_{SW}^1(N, d, \lambda)$ have the small-world property.

3 Vaccination strategies in small-world models

Consider a next-generation SIR-model on a network $G_{NN}^1(120, 2)$ with infection probability $b = 0.9$. In such a model, $R_0 = 3.6$ and the *herd immunity threshold* is

$$HIT = 1 - \frac{1}{R_0} = 1 - \frac{1}{3.6} = 0.7222. \tag{1}$$

The uniform mixing assumption implies in this case that if we want to prevent all major outbreaks by vaccinating a certain fraction of *randomly chosen* hosts, we would need to administer vaccine to at least 72.22% or 87 of our 120 hosts. Vaccinating a substantially smaller fraction of randomly chosen nodes would hardly make a difference (see Project 9.1 of [4]). However, here we are looking at a network model with a very particular, highly nonuniform mixing pattern. For the given contact network very few doses of vaccine may go a long way if we administer them to an *optimal subset of hosts.*

The problem finding such an optimal set of hosts was explored in Section 9.4.3 of [4]. In a series of exercises we discovered a vaccination strategy that appears to be optimal when only 12 doses of vaccine are available. Relative to a strategy of randomly vaccinating 12 hosts, it reduces the mean number of unvaccinated hosts that experienced infection from near 67 to slightly less than 18. It boiled down to vaccinating the following vector of hosts:

```
[ 1 2 21 22 41 42 61 62 81 82 101 102 ]
```
Note that this vaccination strategy works by creating barriers that the pathogens cannot cross. Of course, the contact network in these explorations of [4] was the limiting case $G_{NN}^1(120, 2)$ of $G_{SW}^1(120, 2, \lambda)$ for $\lambda = 0$. For $\lambda > 0$, some or all of these barriers might no longer give reliable protection against the spread of the pathogens. Here we want to explore whether and how the effectiveness of the above vaccination strategy will deteriorate when we increase λ.

Open IONTW, click **Defaults**, and change the following parameter settings:

infection-prob: 0.9
end-infection-prob: 1
network-type $\rightarrow$ **Small World 1**
num-nodes: 120
lambda: 0
d: 2

Click **New**. This will set up a next-generation SIR-model on a network $G_{NN}^1(120, 2)$. Click **Metrics,** look up the value of R0 for this model in the **Command Center** and convince yourself that it is $R_0 = 3.6$.

Create a plain text file that contains the above vector of hosts to be vaccinated as a single line and save it under the name `vaccinate.txt` in the same directory in which you are running IONTW.

To get a set of baseline data, run a batch of 100 simulations for this vaccination strategy. Set up a **New** experiment with 100 repetitions. Use

Reporters:
`count turtles with [removed?]`

Setup commands:
`new-network`
`ask turtles-from-file "vaccinate.txt" [become-removed]`
`ask n-of 1 turtles with [susceptible?]  [become-infectious]`

Exercise 6 *Run the experiment and analyze your output file. Record the minimum, maximum, and mean number of removed nodes at the end. Also record the numbers of runs where you found exactly 30, 48, 66, 84, 102, and 120 hosts who experienced infection.*

Now **Edit** your setup for batch processing by setting

`["lambda" 0.05]`

and specifying a new suggestive output file name. Run the experiment.
Then **Edit** your setup for batch processing again by setting

`["lambda" 0.1]`

and specifying another suggestive output file name. Run the experiment.

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Exercise 7 *(a) Analyze your output files as you did for Exercise 6.*

(b) Verbally summarize your findings. How does the parameter λ appear to influence disease transmission? Does the strategy of vaccinating evenly spaced groups of two adjacent hosts provide as good protection when the contact network is $G^1_{SW}(120, 2, \lambda)$ as it does for contact networks $G^1_{NN}(120, 2)$? Does it provide better protection than randomly vaccinating 12 hosts?

(c) Find an explanation for the high frequencies with which you observed outcomes of 30, 48, 66, 84, 102, or 120 hosts who experienced infection. How is this pattern related to the structure of the network and the disease transmission parameters?

References

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4   Hints

Hint for Exercise 5: Partition the vertex set V of the small-world model into pairwise disjoint sets of consecutive nodes V_i of roughly equal size. The left panel of Figure 1 shows such a partition for an instance of $G_{SW}^1(24, 1, 0.5)$. Here the sets V_i are color-coded. They comprise the following nodes (recall that NETLOGO always will start the numbering of nodes with 1):

$$V_1 = \{0, 1, 2, 3\}, \quad V_2 = \{4, 5, 6, 7\}, \quad V_3 = \{8, 9, 10, 11\},$$
$$V_4 = \{12, 13, 14, 15\}, \quad V_5 = \{16, 17, 18, 19\}, \quad V_6 = \{20, 21, 22, 23\}. \tag{2}$$

Then form a new graph G_I by making each of the *sets* V_i a vertex and drawing an edge $\{V_i, V_j\}$ if, and only if, there is at least one $v^* \in V_i$ and at least one $v^{**} \in V_j$ such that $\{v^*, v^{**}\}$ forms an edge in the original small-world network. The right panel of Figure 1 shows the resulting graph G_I for the small-world model on the left.

It can be shown that (a slight modification of) this construction will give Erdős-Rényi random graphs G_I with sufficiently large mean degrees.
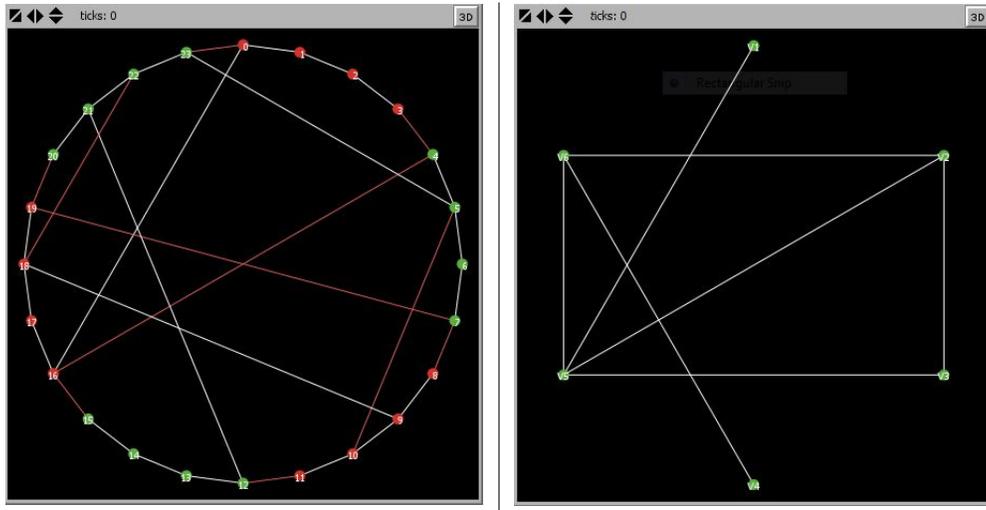

Figure 1: The left panel shows an instance of $G_{SW}^1(24, 1, 0.5)$. The right panel shows a corresponding graph G_I.