Conversation 11: Translating Word Problems into Linear Systems, Part I

Winfried Just
Department of Mathematics, Ohio University

MATH 3200: Applied Linear Algebra

A question from Module 23

Bob: It says here in Module 23:

"Translate the following description into a system of linear equations. Be sure to choose appropriate variables."

"Anne is twice as old as Beth and is one year younger than Clara. The ages of the three girls add up to 11 years."

Cindy: I'm terrible with word problems!!
I never know what they want us to do with them.

Theo: They want us to find a system like

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

with specific numbers for the symbols a_{ij} and the b_i.

Choosing our variables

Cindy: But I never know how to get started with these word problems!

Bob: It says here "Be sure to choose appropriate variables." That's probably a hint that we should start with choosing our variables.

Denny: How many of these variables do we need?

Frank: There are three girls, so we need three variables, x_1, x_2, x_3.

Alice: Very good, Frank! But what, exactly, should these variables stand for?

Bob: Let's read the text again:

"Anne is twice as old as Beth and is one year younger than Clara. The ages of the three girls add up to 11 years."

I think the variables should signify their ages.

Theo: Ages—in what units?

The meaning of our variables

Denny: What do you mean, Theo?

Bob: He means that I should have said "the variables signify their ages in years." Then we don't need to write later on "x_2 yrs" but can simple write "x_2" for the age.

Denny: I like that! The less writing, the better!

Cindy: Just to make sure, x_2 is the age of Beth?

Question C11.1: Did Cindy get this right?

Bob: Yes, that's what I had in mind. x_1 is the age of Anne, x_2 is the age of Beth, and x_3 is the age of Clara.

Frank: But you didn't tell us. We cannot read your mind.

Theo: Exactly. It could also have been the other way around.

Bob: But it's only natural that we would assign the variables in this order!

A better labeling for our variables

Cindy: Now you got me all confused ...

Alice: To avoid any confusion, we need to explicitly specify which variable signifies whose age. I agree that Bob's order is the most natural one, but we still need to write down that this is the one that we will use.

Denny: Like with that unit thing? Of course Bob meant "in years."

Theo: Exactly! In mathematics, all these details need to be stated explicitly, even if the interpretation is suggested by common sense.

Cindy: Could we perhaps use x_a, x_b, x_c instead of x_1, x_2, x_3?
I mean: x_a for the age of Anne in years, x_b for the age of Beth in years, x_c for the age of Clara in years. So as not to get mixed up later on.
Are we allowed to do this?

Theo: Absolutely! You could use any symbols for your variables.

Alice: Excellent idea, Cindy! This is a great strategy for translating word problems: Use suggestive names for your variables.

Bob: Let's go with Cindy's variable names then.

Translating words into equations

Cindy: Now what do we do?

Theo: Now we need to translate the information given in the text into linear equations.

Denny: After all this talk about the variables, I have long forgotten what the text said.

Bob: Let's read it again:

"Anne is twice as old as Beth and is one year younger than Clara. The ages of the three girls add up to 11 years."

Cindy: Oh, I see! The translation should be $x_a + x_b + x_c = 11$, right?

Theo: That is a correct translation of the second sentence, but it is not the whole story.

Cindy: Should I have first translated the first sentence, then?

Question C11.2: Should she?

Translating the first sentence

Alice: Not necessarily. You can write down your equations in any order, but you need to translate each piece of information into an equation. You made a good start, and now you can continue by translating the first sentence:

"Anne is twice as old as Beth and is one year younger than Clara."

Cindy: This one is a lot more complicated though ...

Alice: It gives two pieces of information. Break it up into two parts and translate each of them individually.

Cindy: I will start with the second part. You told me that this is ok, right? *"Anne is one year younger than Clara."*

Would that be $x_a - 1 = x_c$?
Wait! No, I think it should be $x_a + 1 = x_c$.
Or should it be $x_a = x_c - 1$?

Question C11.3: Which translation is correct?

Translating the first sentence, continued

Bob: Don't rush it, Cindy! Both $x_a + 1 = x_c$ and $x_a = x_c - 1$ are correct translations of *"Anne is one year younger than Clara."*

Theo: But neither of them is in the form

$a_{21}x_a + a_{22}x_b + a_{23}x_c = b_2$

that we want for the second equation of our system.

Cindy: So I messed up? I told you I'm terrible with word problems!

Alice: Not at all! The trick is to split up your work into little steps. You completed a correct step, and now you take one more step and transform your version into the one that Theo would like to see.

Cindy: $x_a - x_c = -1$. Like this?

Theo: Right! Then we would have $a_{21} = 1$, $a_{23} = -1$, and $b_2 = -1$ in my notation.

Question C11.4: What is a_{22} here?

Obtaining the last equation

Denny: Nothing-burger. $a_{22} = 0$.

Cindy: Ha-ha! Now we need to translate: *"Anne is twice as old as Beth."* What would that be, Denny?

Denny: How about $2x_a = x_b$?

Cindy: This looks right, but I'm not sure.

Frank: Doesn't make sense.

Denny: Why not?

Frank: If Anne is $x_a = 10$ years old, for example—

Denny: —then we would get $x_b = 2x_a = 20$. So it should be $x_a = 2x_b$. But this wouldn't satisfy Theo's mathematical notation.

Cindy: Can we rewrite it as $x_a - 2x_b = 0$?

Denny: Fine with me. And this would make the system homogenous!

The linear system

Theo: Cindy's version of the last equation is correct, and gives us the following linear system:

$$\begin{array}{rcrcrcr} x_a & + & x_b & + & x_c & = & 11 \\ x_a & & & - & x_c & = & -1 \\ x_a & - & 2x_b & & & = & 0 \end{array}$$

I agree with you, Denny, on the point that this is a fine translation of the word problem. But the system is not homogenous. Only the right-hand side of the last equation is equal to zero; for a homogenous system we would need zeros on the right in every equation.

Cindy: Now I'm wondering: How old are these girls?

Alice: For this we would need to *solve* the system.

Denny: But solving wasn't covered in the lectures yet.

Bob: For now, let's focus on what was assigned from Module 23.

Cindy: But I really would like to know about the girls!!

Alice: We will talk about solving next time.

Cindy: (Sigh) Ok, I will be patient!

Take-home message

This conversation illustrates a number of useful strategies for translating word problems into systems of linear equations.

- Read the word problem closely. Keep a copy of the text handy and read it closely, several times, while working on the translation.
- Always start by setting up your variables. This is the key step.
  - Write down what each variable represents, including the units of the respective quantity.
  - It is helpful to use suggestive names for your variables.
  - Specify variables for all quantities that you might need.

Take-home message, completed

- Break up the text into small pieces of info and translate one such piece at a time.
- Don't rush! After each step, double-check whether your translation is correct. If in doubt, you can test on a numerical example whether it makes sense.
- You want to eventually write each of your equations in the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$.

  But you don't need to do so right away. You can first express the info in a more convenient way, and then transform your equation into the standard format.
- You can translate the individual pieces of info in any order, but be sure to translate all of them.
- Be sure to also translate info that is implicit in common usage of English words or phrases and that relates your variables to each other.
- At the end, double-check whether you have correctly translated all of the given info.