Conversation 12: Solving Systems of Linear Equations with Back-Substitution

Winfried Just
Department of Mathematics, Ohio University

MATH 3200: Applied Linear Algebra

Solving systems of linear equations

Cindy: Will we talk today about how old the three girls are? I really want to know.

Denny: And I want to know how much Dan and Cody spent on those beverages.

Alice: Let's start talking today about *solving* linear systems.

Theo: Which means *finding* the solution set.

Bob: Should we start by reviewing the two systems that we discussed in Conversation 11?

Shall we try something simpler first?

Frank: Give me a break! These systems are too complicated. Let's try solving something simpler first.

Alice: Excellent idea! In fact, let's start with the simplest possible systems, like this one:

$$\begin{array}{rcr} x_1 \qquad & = & 3 \\ x_2 & = & -5 \end{array}$$

Denny: That one is not really a system though. It's already the solution!

Cindy: Yes, it already gives away the solution. But the way it is written, with these equality signs, it also looks like a system.

I'm confused. Is it a system or the solution?

Systems *vs.* solutions

Theo: It is formally a system. We can write it as

$$\begin{array}{rcrcr} 1\,x_1 & + & 0\,x_2 & = & 3 \\ 0\,x_1 & + & 1\,x_2 & = & -5 \end{array}$$

The solution would be the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

What Denny observed was that for this system we can immediately read off the solution set.

Alice: Notice that for this system the coefficient matrix is $\mathbf{I}_2$.

Bob: Are you suggesting that whenever we have a system whose coefficient matrix is an identity matrix, we can immediately read off the solution?

Systems with identity coefficient matrices

Alice: Yes! When we can write the system in matrix form as $\mathbf{I}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, or

$$\begin{array}{ccccccc} x_1 & & & & = & b_1 \\ & x_2 & & & = & b_2 \\ & & \dots & & & \\ & & & x_n & = & b_n \end{array}$$

then the only solution is $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Denny: Cool!

Frank: But boring. Are there other examples of systems for which we can immediately read off the solutions?

Systems where we can immediately read off the solution

Alice: There are. Consider, for example, systems with coefficient matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ They can be written as:

$$\begin{array}{ccccccc} x_1 & + & 0\,x_2 & + & 0\,x_3 & = & b_1 \\ 0\,x_1 & + & x_2 & + & 0\,x_3 & = & b_2 \\ 0\,x_1 & + & 0\,x_2 & + & 0\,x_3 & = & b_3 \end{array}$$

or simply

$$\begin{array}{cccccc} x_1 & & & = & b_1 \\ & x_2 & & = & b_2 \\ & & 0 & = & b_3 \end{array}$$

Frank: But wait!! When, for example, $b_3 = 1$, the last line becomes $0 = 1$. This doesn't make sense!

Systems where we can immediately read off the solution

Alice: What, exactly, do you mean by "doesn't make sense"?

Question C12.1: What can we say about the following system and its solutions?

$$\begin{array}{rcl} x_1 \quad\quad & = & 3 \\ x_2 \quad & = & 2 \\ 0 & = & 1 \end{array}$$

Alice: This system has no solutions; it is *inconsistent*. We can see this right from its third equation.

Cindy: But what if we have $b_3 = 0$? I mean, what could we say about the solution set of a system like this one:

$$\begin{array}{rcl} x_1 \quad\quad & = & 3 \\ x_2 \quad & = & 2 \\ 0 & = & 0 \end{array}$$

Question C12.2: How would you respond to Cindy?

This is an underdetermined system

Bob: The system

$$\begin{aligned} x_1 \quad\quad &= 3 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

is consistent.

The first two lines tell us what x_1 and x_2 are, but there is no constraint on x_3.

Denny: So the solution must be $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Question C12.3: Did Denny get this right?

Theo: No, because we are treating this as a system of three variables. The solution set comprises all vectors of the form $\begin{bmatrix} 3 \\ 2 \\ x_3 \end{bmatrix}$

As Bob said, there is no constraint on x_3, so the system is *underconstrained* or *underdetermined.*

Why consider such weird systems?

Denny: OK, OK. But why consider systems with weird equations like $0 = 0$ or $0 = 1$?

Alice: Because they naturally pop up in the process of solving more complicated systems. We will see examples soon.

Bob: I think we are now ready to look at some more complicated examples.

Cindy: But not too complicated ones, please!

Frank: I'm with you, Cindy!

Alice: The next simplest systems would be the ones with an upper-triangular coefficient matrix like $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

A system with an upper-triangular coefficient matrix

Alice: Here is an example of such a system:

$$\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -3 \\ & & x_2 & = & -2 \end{array}$$

Cindy: Again we can read off the value for x_2 in the solution, but it is not immediately clear what x_1 would be.

Bob: You can do the following: Substitute the value $x_2 = -2$ in the first equation. This gives:

$$\begin{array}{rcrcr} x_1 & + & 4(-2) & = & -3 \\ & & x_2 & = & -2 \end{array}$$

Then move the $4(-2) = -8$ to the right of the first equation:

$$\begin{array}{rcrcr} x_1 & & & = & 5 \\ & & x_2 & = & -2 \end{array}$$

The method of back-substitution

Denny: Cool! So now we can read off the solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

Theo: What Bob and Denny have shown us here is the *method of back-substitution.* It works by substituting numerical values for some variables that you get from one equation into another equation.

Cindy: But wait! The system

$$\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -3 \\ & & x_2 & = & -2 \end{array}$$

doesn't seem to be the same as the system

$$\begin{array}{rcrcr} x_1 & & & = & 5 \\ & & x_2 & = & -2 \end{array}$$

Denny gave us the solution of the second system. Is this also the solution of the first system? I'm not sure this is correct.

Question C12.4: How would you respond to Cindy?

Theo: The systems

$$\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -3 \\ & & x_2 & = & -2 \end{array}$$

and

$$\begin{array}{rcr} x_1 & = & 5 \\ x_2 & = & -2 \end{array}$$

are *different systems* but they *have the same solution sets.* Two systems with the same solution sets are called *equivalent systems.*

Cindy: But how can I be sure that the two systems are equivalent?

Denny: Come on, Cindy! This is obvious.

Theo: No, Denny. Cindy asked a very good question. In mathematics we always need to carefully examine whether every step of an argument is correct. When solving systems of linear equations, we can only use methods that lead to equivalent systems. The method of back-substitution is one of them.

Another example of back-substitution

Bob: Let's practice back-substitution with another example:

$$\begin{array}{ccccccc} x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 1 \\ & & & & x_3 & = & 2 \end{array}$$

Cindy: I can substitute $x_3 = 2$ in the second equation and get $x_2 - 2 = 1$. So we get an equivalent system

$$\begin{array}{ccccccc} x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ & & x_2 & & & = & 3 \\ & & & & x_3 & = & 2 \end{array}$$

But now what do I do?

Question C12.5: What should Cindy do next?

Another example of back-substitution, completed

Recall that after substituting $x_3 = 2$ in the second equation, Cindy got an equivalent system

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ & & x_2 & & & = & 3 \\ & & & & x_3 & = & 2 \end{array}$$

Bob: You can substitute *both* $x_3 = 2$ and $x_2 = 3$ into the first equation. This gives $x_1 + 9 - 4 = 0$ and leads to the equivalent system

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -5 \\ & & x_2 & & & = & 3 \\ & & & & x_3 & = & 2 \end{array}$$

Denny: For which we can immediately read off the solution. Cool!

A third example of back-substitution

Cindy: But what if our system were:

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 3x_3 & - & 2x_4 & = & 0 \\ & & & & x_3 & - & x_4 & = & 1 \\ & & & & & & x_4 & = & 2 \end{array}$$

Bob: You can start as before.

Cindy: I know. But then I obtain the equivalent system

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & & & & & = & -5 \\ & & & & x_3 & & & = & 3 \\ & & & & & & x_4 & = & 2 \end{array}$$

How do I get the solution for this system?

Question C12.6: What should Cindy do now?

A third example of back-substitution, continued

Alice: The system

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & & & & & = & -5 \\ & & & & x_3 & & & = & 3 \\ & & & & & & x_4 & = & 2 \end{array}$$

is *underdetermined* and has infinitely many solutions.

You need to choose one of the variables as your *free variable* aka *free parameter.* Let's make x_2 your free variable.

You can pick any number for x_2. Then $x_1 = x_2 - 5$ by the first equation, and your solution set will consist of all vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - 5 \\ x_2 \\ 3 \\ 2 \end{bmatrix}$$

A third example of back-substitution, completed

The system

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & & & & & = & -5 \\ & & & & x_3 & & & = & 3 \\ & & & & & & x_4 & = & 2 \end{array}$$

is underdetermined and has infinitely many solutions.

Cindy: Could I use x_1 as my free variable instead of x_2?

Question C12.7: Could Cindy do this?

Alice: Yes you could. Your solution set would then consist of all vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + 5 \\ 3 \\ 2 \end{bmatrix}$$

When does back-substitution work?

Frank: So when, exactly, can we use back-substitution? Are there some clear guidelines?

Alice: Let's look at the extended matrices for the systems in our three examples:

$$\begin{array}{rcrcl} x_1 & + & 4x_2 & = & -3 \\ & & x_2 & = & -2 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 1 \\ & & & & x_3 & = & 2 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & 3x_3 & - & 2x_4 & = & 0 \\ & & & & x_3 & - & x_4 & = & 1 \\ & & & & & & x_4 & = & 2 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

All of these matrices are in so-called *row echelon form* or simply *echelon form*. We can use back-substitution whenever the extended matrix is in this form.

Frank: So what, exactly, does "row echelon form" mean?

Bob: I think the exact definition will be given in Lecture 13.

Take-home message

Let us consider what we have been doing here. Essentially, *solving* a linear system that has exactly one solution boils down to *transforming a system* of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

into an equivalent system of the form:

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ &\vdots \\ x_n &= c_n \end{aligned}$$

where $c_1, c_2, \ldots, c_n$ are numbers.

Take-home message, completed.

When the extended matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & & \dots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

of the system has a sufficiently nice form, we can find the solution set with the method of *back-substitution.*

A particularly nice form of the extended matrix, called the *(row) echelon form*, that allows us to use back-substitution will be covered in Lecture 13.

When the system is underdetermined, it's solution set can be described by leaving at least one of the variables in symbolic form, as a *free variable* or *free parameter* that can take any values.