Conversation 13: Solving Systems of Linear Equations

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MATH 3200: Applied Linear Algebra

Solving systems of linear equations

Cindy: Will we talk today about how old the three girls are? I really want to know.

Denny: And I want to know how much Dan and Cody spent on those beverages.

Alice: Yes, let's talk today about solving these systems.

Cindy: Like, by back-substitution!

Frank: This method is not going to work here.

Denny: Why not?

Frank: Because you can use back-substitution only for systems with a simple extended matrix, for example, in row echelon form.

Cindy: But why wouldn't that be true for the system about the three girls?

Bob: Let's review this system and find out.

Cindy: Good idea!

Review: The ages of Anne, Beth, and Clara

Bob: We considered the following word problem:

"Anne is twice as old as Beth and is one year younger than Clara. The ages of the three girls add up to 11 years."

We translated this problem into the following system of linear equations:

$$\begin{array}{rcrcrcr} x_a & + & x_b & + & x_c & = & 11 \\ x_a & & & - & x_c & = & -1 \\ x_a & - & 2x_b & & & = & 0 \end{array}$$

where x_a, x_b, x_c denote the ages in years of Anne, Beth, and Clara, respectively.

Question C13.1: What is the extended matrix of this system? Is it in row echelon form?

The extended matrix of this system

$$\begin{array}{ccccccc} x_a & + & x_b & + & x_c & = & 11 \\ x_a & & & - & x_c & = & -1 \\ x_a & - & 2x_b & & & = & 0 \end{array}$$

The extended matrix that represents this system is:

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & 1 & 11 \\ 1 & 0 & -1 & -1 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$

$[\mathbf{A}, \vec{\mathbf{b}}]$ is not in row echelon form.

Cindy: So we cannot solve this system?

Theo: Not with back-substitution alone, but there are other methods that we will begin to learn in this conversation.

Alice: It will be best if we first introduce these methods with examples that are simpler than the one on this slide.

Example 1 of a matrix that is not in row echelon form

Frank: Yes, let's start with the easiest such systems.

Alice: Here is a very simple example:

$$\begin{array}{rcrcr} & & x_2 & = & 5 \\ x_1 & + & x_2 & = & -12 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 1 & -12 \end{bmatrix}$$

Frank: But that's practically the same thing! You just switch the equations and get a system with $[\mathbf{A}, \vec{\mathbf{b}}]$ in row echelon form:

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & -12 \\ & & x_2 & = & 5 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & -12 \\ 0 & 1 & 5 \end{bmatrix}$$

Alice: Very good! Switching the order of two equations is an operation that does not change the solution set of the system. So you end up with an equivalent system, and here the extended matrix of this equivalent system will already be in row echelon form.

Denny: Are there other such "operations" for getting equivalent systems if I may use your highfalutin expression?

Example 2 of a matrix that is not in row echelon form

Alice: Here is another very simple example:

$$\begin{array}{rcrcr} 0.5x_1 & + & x_2 & = & 15 \\ & & x_2 & = & 7 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 0.5 & 1 & 15 \\ 0 & 1 & 7 \end{bmatrix}$$

Frank: Again, practically the same thing! You just multiply the first equation by 2 and get a system with $[\mathbf{A}, \vec{\mathbf{b}}]$ in row echelon form:

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 30 \\ & & x_2 & = & 7 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 30 \\ 0 & 1 & 7 \end{bmatrix}$$

Alice: Exactly! Multiplying both sides of an equation by the same nonzero scalar is another operation that does not change the solution set and gives an equivalent system.

Question C13.2: Why did Alice say "nonzero scalar" instead of just "scalar"?

Multiplying both sides of an equation by 0 may introduce additional solutions

Consider the system:

$$\begin{array}{rcrcl} 0.5x_1 & + & x_2 & = & 15 \\ & & x_2 & = & 7 \end{array}$$

Its unique solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 7 \end{bmatrix}$

But when we multiply both sides of the first equation by 0 we obtain the system

$$\begin{array}{rcl} 0 & = & 0 \\ x_2 & = & 7 \end{array}$$

whose solution set comprises all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 7 \end{bmatrix}$

Example 3 of a matrix that is not in row echelon form

Alice: Now let's consider a third example:

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 15 \\ x_1 & + & 2x_2 & = & 7 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & 15 \\ 1 & 2 & 7 \end{bmatrix}$$

Frank: Well, that one looks different.

Bob: But when we subtract both sides of the first equation from both sides of the second equation we obtain a system with $[\mathbf{A}, \vec{\mathbf{b}}]$ in row echelon form:

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 15 \\ & & x_2 & = & -8 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & 15 \\ 0 & 1 & -8 \end{bmatrix}$$

Alice: Right! Adding a scalar multiple of one equation to another equation is yet another operation that does not change the solution set and gives an equivalent system. Here Bob subtracted, which means he added -1 times the first equation to the second.

The ages of Anne, Beth, and Clara

Cindy: Can we now find out how old the girls are? I mean, solve

$$\begin{array}{rcrcrcr} x_a & + & x_b & + & x_c & = & 11 \\ x_a & & & - & x_c & = & -1 \\ x_a & - & 2x_b & & & = & 0 \end{array}$$

where x_a, x_b, x_c are the ages in years of Anne, Beth, and Clara?

Bob: We could perhaps start by subtracting the first equation from the second and third equations. This would give the equivalent system

$$\begin{array}{rcrcrcr} x_a & + & x_b & + & x_c & = & 11 \\ & & -x_b & - & 2x_c & = & -12 \\ & - & 3x_b & - & x_c & = & -11 \end{array}$$

The ages of Anne, Beth, and Clara, continued

$$\begin{array}{rcrcrcr} x_a & + & x_b & + & x_c & = & 11 \\ & & -x_b & - & 2x_c & = & -12 \\ & - & 3x_b & - & x_c & = & -11 \end{array}$$

Bob: Then we could subtract 3 times the second equation from the third:

$$\begin{array}{rcrcrcr} x_a & + & x_b & + & x_c & = & 11 \\ & & -x_b & - & 2x_c & = & -12 \\ & & & & 5x_c & = & 25 \end{array}$$

Next we can multiply the second equation by -1 and divide the third equation by 5, which is the same as multiplying it by $\frac{1}{5}$:

$$\begin{array}{rcrcrcr} x_a & + & x_b & + & x_c & = & 11 \\ & & x_b & + & 2x_c & = & 12 \\ & & & & x_c & = & 5 \end{array}$$

Cindy: You are so smart, Bob! Thank you for working this out.

The ages of Anne, Beth, and Clara, completed

$$\begin{array}{ccccccc} x_a & + & x_b & + & x_c & = & 11 \\ & & x_b & + & 2x_c & = & 12 \\ & & & & x_c & = & 5 \end{array}$$

Cindy: By the third equation, Clara is 5 years old. Now I can use back-substitution in the second equation to get $x_b + 10 = 12$, which tells me that Beth is 2 years old. When I put these numbers into the first equation I get $x_a + 2 + 5 = 11$, so that Anne is 4 years old. These girls must be sooo cute!

Denny: Perhaps they are. But this is going by too fast for me. How on earth did you come up with these steps, Bob?

Frank: I'm with you, Denny! In a course for engineers they should teach a step-by-step procedure for doing these calculations.

Bob: I think this method will be taught in the next lecture.

Denny: Let's talk now about those beverages that Cody and Dan bought! First let's review the question.

Review: Cody and Dan go grocery shopping

Bob: Next we translated the word problem:

"Cody and Dan go shopping for food and beverages. Cody spends twice as much on beverages as Dan does and only one third as much as Dan on food. Overall, Cody ends up spending 50% more money than Dan. "

into he following system of linear equations:

$$\begin{array}{ccccccc} & & c_b & - & 2d_b & = & 0 \\ & & c_f & - & \frac{1}{3}d_f & = & 0 \\ & & c_o & - & 1.5d_o & = & 0 \\ c_b & + & c_f & - & c_o & = & 0 \\ d_b & + & d_f & - & d_o & = & 0 \end{array}$$

where c_b, c_f, c_o denote the numbers of dollars that Cody paid for beverages, food, and overall, respectively; and similarly d_b, d_f, d_o denote the numbers of dollars that Dan paid for beverages, food, and overall respectively.

How about those beverages?

Denny: So how much did Dan and Cody spend on those beverages?

Bob: I think our instructor wants to cover this in the next module.

Denny: Unfair! I really want to know.

Frank: And I bet nobody can figure out the answer.

Denny: You mean, not even Alice can figure it out?

Frank: Wanna bet?

Denny: Sure. 10 dollars.

Frank: U.S. or Canadian?

Denny: What kind of question is that?

Alice: It's a hint.

Theo: He is telling you why you will lose your bet.

Denny: What are you all talking about??

Question C13.3: What are Frank, Alice, and Theo talking about?

Can we simplify the system?

Bob: Denny, think of the system as:

$$\begin{array}{rcrcl} & & c_b & = & 2d_b \\ & & c_f & = & \frac{1}{3}d_f \\ & & c_o & = & 1.5d_o \\ c_b & + & c_f & = & c_o \\ d_b & + & d_f & = & d_o \end{array}$$

Frank: But this system is still too complicated. If we substitute the right-hand sides of the first three equations in the fourth, we obtain the following system of 2 equations in 3 variables:

$$\begin{array}{rcrcl} 2d_b & + & \frac{1}{3}d_f & = & 1.5d_o \\ d_b & + & d_f & = & d_o \end{array}$$

Denny: I still don't see why.

Solutions of this system

Bob: You can further simplify the system

$$\begin{array}{ccccc} 2d_b & + & \frac{1}{3}d_f & = & 1.5d_o \\ d_b & + & d_f & = & d_o \end{array}$$

by substituting the left-hand-side of the second equation in the right-hand side of the first, which gives one linear equation in two variables

$$2d_b + \frac{1}{3}d_f = 1.5d_b + 1.5d_f,$$

or, equivalently:

$$0.5d_b - \frac{7}{6}d_f = 0.$$

Denny: Oh, I see! This last equation does not have a unique solution. When I have one solution $[d_b, d_f]^T$ and multiply both dollar amounts by the same number λ—

Cindy: —like in the conversion from U.S. dollars into Canadian—

Denny: —then I obtain another solution $[\lambda d_b, \lambda d_f]^T$.

Take-home message: Elementary operations on systems of linear equations

Consider a system of m linear equations in n variables.

Each of the following operations preserves the set of solutions and thus transforms the system into an equivalent one:

(i) Interchanging the positions of any two equations.

(ii) Multiplying an equation by a nonzero scalar.

(iii) Adding to one equation a scalar multiple of another equation.

In practice, instead of performing operations on the equations, it is more convenient to perform *elementary row operations* on their extended matrices. A step-by-step procedure for using these to obtain a matrix in row echelon form is called *Gaussian elimination.* It will be the subject of the next two lectures.