Conversation 22: Applications of Linear Combinations to Foraging Bees

Winfried Just
Department of Mathematics, Ohio University

MATH 3200: Applied Linear Algebra

Linear combinations and the linear span: Review of the definitions

Cindy: I heard that Chapter 3 will be the hardest part of the course. This sounds so scary.

Bob: We will be fine if we really understand the meaning of all the definitions. Let's review the ones given so far.

Definition

A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$ such that

$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.

The *linear span* of a set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ is the set $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ of all linear combinations of these vectors..

Cindy: So, "$\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$" and "$\vec{\mathbf{w}}$ is in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$" mean exactly the same thing?

Bob: Right!

What do we need these definitions for?

Denny: So what's the difference between a linear combination and a linear span then?

Theo: A linear combination is a single vector, while a linear span is a set of vectors.

Frank: What do we need all this terminology for?

Theo: To get some conceptual understanding and to express certain properties very succinctly.

Bob: What properties are you talking about, Theo?
Can you give us an example?

Theo: We already saw in Lecture 22 how consistency of a system of linear equations can be expressed in terms of linear combinations and of the linear span.

Can we see a simple real-world example?

Cindy: This was all very abstract . . .

Frank: Right. In a course for engineers they should give us some real-world applications instead of these abstract concepts.

Denny: Is there any real-world example that is really simple? One that doesn't require prior knowledge of anything?

Alice: (smiling mischievously) Do you want an example about the birds or an example about the bees?

Cindy: About the bees, please! They are so cute. All worker bees in a hive are sisters. They send out some sisters to forage for new sources of nectar, and when a forager finds something, she comes back and performs a wiggle dance that tells here sisters where to find the nectar. I saw a movie of it; this looks so awesome!

An example: A forager bee

Alice: Assume a bee colony nests in a hollow tree.

Let $[0, 0, 0]$ denote the position of the hive.

A foraging bee travels t_1 time units in the direction of vector $\vec{\mathbf{v}}_1 = [2, 3, 0.5]$ and then t_2 time units in the direction of vector $\vec{\mathbf{v}}_2 = [0, -2, -0.5]$ and discovers some tasty blossoms on another tree.

We can express the position of these blossoms as the linear combination

$\vec{\mathbf{w}} = t_1\vec{\mathbf{v}}_1 + t_2\vec{\mathbf{v}}_2.$

Question C22.1: At what position are the blossoms located when $t_1 = 3$ and $t_2 = 2$?

$\vec{\mathbf{w}} = t_1\vec{\mathbf{v}}_1 + t_2\vec{\mathbf{v}}_2 = 3[2, 3, 0.5] + 2[0, -2, -0.5] = [6, 5, 0.5].$

Two sisters

Alice: On the way back to the hive, our foraging bee discovers a better route to the blossoms, and upon return to the hive, she shares this info with her sisters by performing a wiggle dance.

After the dance, sister Beezee of the foraging bee travels first t_3 time units in the direction of vector $\vec{\mathbf{v}}_3 = [1, 1, 0.2]$ and then t_4 time units in the direction of vector $\vec{\mathbf{v}}_4 = [2, 1, -0.3]$,

while sister Buzzy travels first t_5 time units in the direction of vector $\vec{\mathbf{v}}_5 = [3, 3, 0.2]$ and then t_6 time units in the direction of vector $\vec{\mathbf{v}}_6 = [0, -1, -1]$.

One of the two sisters didn't pay attention during the wiggle dance and never found the blossoms. Which one was it?

Denny: How would we know? You tell us!

Cindy: Yes, please tell us!! I'm curious.

Bob: I think we can figure this out based on the information that Alice has given us.

Beezee

Bob: If Beezee reaches $\vec{\mathbf{w}} = [6, 5, 0.5]$ after traveling first t_3 time units in the direction of vector $\vec{\mathbf{v}}_3 = [1, 1, 0.2]$ and then t_4 time units in the direction of vector $\vec{\mathbf{v}}_4 = [2, 1, -0.3]$,

then $t_3\vec{\mathbf{v}}_3 + t_4\vec{\mathbf{v}}_4 = t_3[1, 1, 0.2] + t_4[2, 1, -0.3] = \vec{\mathbf{w}} = [6, 5, 0.5]$.

This specifies a systems of linear equations. Such times t_3, t_4 exist if, and only if, the system is consistent. Which would be the same as saying that $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4$.

Cindy: Or that $\vec{\mathbf{w}}$ is in $span(\vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4)$, right?

Bob: Right! To find out, we would need to solve . . .

Denny: Oh! I can see that $4[1, 1, 0.2] + [2, 1, -0.3] = [6, 5, 0.5]$.
So, $t_3 = 4$, $t_4 = 1$, and we don't need to solve any system!

Question C22.2: Is Denny right?

Alice: Not quite. Denny's observation shows that $[6, 5, 0.5]$ is a linear combination of the vectors $[1, 1, 0.2]$ and $[2, 1, -0.3]$. But without solving a linear system, we cannot rule out that there might be other choices for the coefficients t_3, t_4.

How about Buzzy?

Cindy: So we are not 100% sure yet **how** Beezee reached the blossoms, but we know she **could** have reached them, right?

Alice: Very well put, Cindy!

Denny: Then Buzzy was the one who didn't pay attention.

Alice: How do you know?

Frank: Come on, Alice. You told us that one of them didn't pay attention, and we know now it wasn't Beezee.

Alice: But how do you know I told you the truth?
Don't you like to verify what people are telling you?

Frank: I damn sure do, in general. But I trust you, Alice.

Alice: Thank you, Frank. But it is always good to know how to verify whether other people are telling you the truth. So how would you demonstrate that Buzzy couldn't have gotten it right?

Theo: This is where the concepts of linear span and linear combination will be useful.

So how about Buzzy?

If Buzzy were to reach the blossoms at $\vec{\mathbf{w}} = [6, 5, 0.5]$ after traveling first t_5 time units in the direction of vector $\vec{\mathbf{v}}_5 = [3, 3, 0.2]$ and next t_6 time units in the direction of vector $\vec{\mathbf{v}}_6 = [0, -1, -1]$, then $t_5\vec{\mathbf{v}}_5 + t_6\vec{\mathbf{v}}_6 = t_5[3, 3, 0.2] + t_6[0, -1, -1] = \vec{\mathbf{w}} = [6, 5, 0.5]$.

Here t_5, t_6 must satisfy the linear system

$$\begin{array}{rcrcr} 3t_5 & & & = & 6 \\ 3t_5 & - & t_6 & = & 5 \\ 0.2t_5 & - & t_6 & = & 0.5 \end{array} \quad \text{with extended matrix} \quad \begin{bmatrix} 3 & 0 & 6 \\ 3 & -1 & 5 \\ 0.2 & -1 & 0.5 \end{bmatrix}$$

We can perform a Gaussian elimination:

$$\begin{bmatrix} 3 & 0 & 6 \\ 3 & -1 & 5 \\ 0.2 & -1 & 0.5 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - R1} \begin{bmatrix} 3 & 0 & 6 \\ 0 & -1 & -1 \\ 0.2 & -1 & 0.5 \end{bmatrix}$$

So how about Buzzy? Continued.

$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & -1 & -1 \\ 0.2 & -1 & 0.5 \end{bmatrix} \xrightarrow{R1 \mapsto R1/3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0.2 & -1 & 0.5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R1/5}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & 0.1 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1.1 \end{bmatrix} \xrightarrow{R2 \mapsto -R2}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1.1 \end{bmatrix} \xrightarrow{R3 \mapsto R3/1.1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Question C22.3: What does the row echelon form tell us about the solution set?

The last row of this matrix translates into an equation $0 = 1$. The system is inconsistent.

How to think about the linear span?

Bob: We can conclude that $\vec{\mathbf{w}} = [6, 5, 0.5]$ is *not* in the linear span of $\vec{\mathbf{v}}_5 = [3, 3, 0.2]$ and $\vec{\mathbf{v}}_6 = [0, -1, -1]$.
Thus Buzzy couldn't have possibly reached the blossoms.

Cindy: So, the linear span of two vectors in $\mathbb{R}^3$ would then be all the points that we can reach by moving a certain amount of time in the direction of the first vector, and then a certain amount of time in the direction of the second vector?

Question C22.4: Is Cindy right?

Alice: Almost. First of all, you need to add that you would start at the origin $[0, 0, 0]$.
Moreover, we need to be clear here that the "certain amount of time" could be zero, or it could even be a negative number.
A negative time would mean that we might have reached the point in the past, rather than that we will reach it at some time in the future.

What if we don't start at the origin?

Cindy: OK, I see why "certain amount of time" might mean zero or negative time.

But what if the hive were not at the origin? For example, if the hive were at position $[1, 2, 3]$ instead and we want to know if the bee can reach a point with coordinates $\vec{\mathbf{w}}$ if it travels a certain amount of time in the direction of a vector $\vec{\mathbf{v}}_1$ and then a certain amount of time in the direction of a vector $\vec{\mathbf{v}}_2$. How can we figure out then if it is possible to reach $\vec{\mathbf{w}}$?

Question C22.5: How can we phrase this problem in terms of linear combinations of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$?

Alice: In this case the bee must be able to achieve a *displacement* of $\vec{\mathbf{w}} - [1, 2, 3]$ by traveling only in these two directions. This will be possible if, and only if the vector $\vec{\mathbf{w}} - [1, 2, 3]$ is a linear combination of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. This will be the case if, and only if, $\vec{\mathbf{w}} - [1, 2, 3]$ is in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$.

Take-home message

This conversation illustrates how the concepts of linear combination and linear span come up in some real-world situations.

The example illustrates that a point $\vec{\mathbf{w}}$ in three-dimensional space can be reached by traveling (forward or backward in time) from the origin first in the direction of one vector $\vec{\mathbf{v}}_1$ and then in the direction of another vector $\vec{\mathbf{v}}_2$ if, and only if, $\vec{\mathbf{w}}$ is in the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ of these vectors.

More generally, a point $\vec{\mathbf{w}}$ in three-dimensional space can be reached by traveling (forward or backward in time) from a point $\vec{\mathbf{u}}$ first in the direction of one vector $\vec{\mathbf{v}}_1$ and then in the direction of another vector $\vec{\mathbf{v}}_2$ if, and only if, the displacement $\vec{\mathbf{w}} - \vec{\mathbf{u}}$ is in the linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ of these vectors.